Strategic discontinuity in simple and complicated games*

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Abstract

We study Harsanyi types which exhibit strategic discontinuity in simple/complicated games. Following the idea of Ely and Peski (2007), we say that a type t is $n-$critical, if there exists an $n \times n$ game and a sequence of types whose beliefs match those of t up to any finite order and whose interim rationalizable behaviors fail to converge to those of t. We show that every finite type is $3-$critical, every common prior assigns probability 1 to $3-$critical types, and moreover, $3-$critical types are generic in the universal type space under the strategic topology defined in Dekel, Fudenberg, and Morris (2006). However, for any integer $n \geq 2$, there exists an $n'-$critical type with $n' > n$ which is not $n-$critical. Consequently, the characterization of all critical types obtained by Ely and Peski (2007) necessarily involved complicated games. Finally, every type is in fact $\infty-$critical.

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1 Introduction

In an incomplete-information game, there is a payoff-relevant unknown variable θ, whose value may not be common knowledge. Since θ (partially) determines an agent's payoff, his belief about θ (i.e., the first-order belief) influences his decision. Since his opponents' decisions also (partially) determine his payoff, an agent's belief about his opponents' beliefs about θ (i.e., the second-order belief) also influences his decision. Similarly, we are led to consider the third-order beliefs, and so on, ad infinitum.

Harsanyi (1967/1968) proposes the notion of type spaces, which is a parsimonious way to represent players' hierarchies of beliefs. A (Harsanyi) type corresponds to a *coherent* hierarchy of higher-order beliefs. Mertens and Zamir (1985) introduce the *universal* type space, which is a special Harsanyi type space. They prove that the universal type space is homeomorphic to the set of all possible coherent higher-order beliefs. Further, any nonredundant Harsanyi type space can be embedded as a subspace in the universal type space. As a result, Mertens and Zamir (1985) show that Harsanyi's approach suffers no loss of generality.

Economic models are typically approximations of complicated situations in higher-order beliefs. For instance, we often assume that the game we analyze is commonly known among all players, while these assumptions might only be approximately satisfied in reality. However, these approximations are not innocuous if predictions of models based upon them are not always similar to those in situations being approximated. Rubinstein (1989) construct the e-mail game example to precisely highlight this point — strategic behaviors in the common knowledge scenario, can be very different from those in the scenario when players mutually know the game they are playing up to any arbitrarily high but finite levels.

A sequence of higher-order beliefs t_n convergs to a higher-order belief t in *product topology* if for any positive integer k, the k-th order belief of t_n converges to that of t. Product topology is the usual way to measure closeness of higher-order beliefs starting from Mertens and Zamir (1985). However, a compact way to summarize Rubinstein's e-mail game result is that t_n converging to t in product topology does not imply strategic behaviors of t_n converging to those t. Following Dekel, Fudenberg, and Morris (2006) and Ely and Peski (2007), we call this phenomenum, strategic discontinuity.

Strategic discontinuity has inspired two distinct lines of research. The first line of research studies alternative notion of proximity in beliefs which guarantees proximity of strategic behaviors (Monderer and Samet (1989, 1996), Kajii and Morris (1998), Dekel, Fudenberg, and Morris (2006) and Chen, Di Tillio, Faingold, and Xiong (2009)). The other line of research exploits strategic discontinuity to study robustness of predictions and equilibrium selections, which include the whole literature on global games (Carlsson and Damme (1993) and Monderer and Samet (1989)) and the recent comments on its approach (Weinstein and Yildiz (2007)). However, fundemental questions remain: When strategic discontinuity happens? How "pervasive" is the phenomenon of strategic discontinuity? To what extent should we be concerned about it?

Recently, Ely and Peski (2007) attempt to answer these questions. They define a (Harsanyi) type to be critical if there exists a sequence of types t_n whose hierarchies of beliefs match those of t up to any finite order and whose rationalizable actions does not converge to those of t in *some finite game*. A type is regular if it is not critical.[1] Ely and Peski (2007) offer a surprisingly concise characterization of *critical types*: a type is critical if and only if for some $p > 0$, it has common $p-$belief for some closed proper subset in the universal type space.[2] As a consequence, type spaces commonly used in the economics literature consist entirely of critical types. In particular, all finite types are critical, and almost all (i.e., with measure 1) types in a common prior type space are critical under the prior. Chen, Di Tillio, Faingold, and Xiong (2008) subsequently show that the set of critical types are indeed "large" in the sense that it is open and dense in the strategic topology defined in Dekel, Fudenberg, and Morris (2006).[3]

The definition of critical type is permissive because to be critical it suffices for a type to

[1] Equivalently, critical types are types around which the strategic topology defined in Dekel, Fudenberg, and Morris (2006) is strictly finer than the product topology on hierarchies of beliefs, while regular types are types around which these two topologies are equivalent. See Section 2 for the formal definitions.

[2] More precisely, Ely and Peski study the solution concept of interim independent rationalizability rather than the solution concept of interim correlated rationalizability used by DFM to define the strategic topology. Their analysis can however be applied to the latter case and here we state the version of Ely and Peski's result under interim correlated rationalizability. See Section 6 for more discussion about their differences.

[3] In contrast, Ely and Peski (2007) demonstrate that a straightforward consequence of their characterization is that the set of critical types is "small" in that it is contained in a meager set in the *product topology*.

exhibit strategic discontinuity in some finite game. For each type which has common $p-$belief of a closed proper subset in the universal type space, Ely and Peski (2007) construct a finite game to demonstrate its discontinuity. The game they construct is very complicated and in general involves a large number of actions. However, this is in sharp contrast to the simple (i.e., 2×2) game employed by Rubinstein (1989). Strategic discontinuity is clearly less a concern if it happens only in complicated and rarely seen games. This leads us to the main questions of this paper: what are the critical types that can display strategic continuity in simple games and how simple can these games be? Are there critical types which display strategic discontinuity only in complicated games?

In this paper, we classify games by the number of actions among which players can choose. For any positive integer n, we say a type is $n-$critical if it displays strategic discontinuity in some $n \times n$ game. A type is $n-$regular if it is not $n-$critical. As a consequence, a type is critical if and only if it is n-critical for some n. For instance, the common-knowledge type in Rubinstein's e-mail game is a $2-$critical type. An $n-$critical type can be viewed as a generalized version of this common-knowledge type in the sense that we can construct a $n \times n$ game to demonstrates its strategic discontinuity.

We first prove that every type which has common $p-$belief of a closed proper *interval* of first-order beliefs over some payoff parameter θ exhibits strategic discontinuity in a 2×3 game.[4] This enables us to strengthen the results in Ely and Peski (2007) and Chen, Di Tillio, Faingold, and Xiong (2008) to (1) all finite types are $3-$critical; (2) in a common prior type space, almost all types are $3-$critical under the prior; (3) $3-$critical types are generic in the sense that it contains a subset which is open and dense in the strategic topology. Thus, type spaces commonly used in the economic literature consist entirely of $3-$critical types. In other words, we can take any finite type or virtually any type in a common prior type space, and demonstrates its strategic discontinuity via a simple (i.e., 3×3) game which can be viewed as a general version of Rubinstein's e-mail game. Further, the set of types, which can be used to replicate Rubinstein's e-mail-game argument using a 3×3 game, is generic.

The prevalence of $3-$critical raises the question whether every critical type is in fact

[4] Interestingly, Ely and Peski use a 2×3 game to demonstrate that every type along the sequence defined in Rubinstein's e-mail game is critical. Our result shows that this phenomenon is not specific to these e-mail types.

3−critical, or more generally, whether there is some fixed integer n such that every critical type is n−critical. We answer this question by showing that for any integer n, there exists an n'−critical type with $n' > n$ which is n−regular. Its existence shows that indeed some types exhibit strategic discontinuity *only* in complicated games and thus the complicated games employed by Ely and Peski (2007) are necessary to characterize all critical types.

We finally turn to study ∞−critical types which exhibits strategic discontinuity in games with countably infinitely many actions. Our main finding is that actually every type, regular or critical, displays strategic discontinuity in some infinite game. The result shows that regular types can alternatively be viewed as ∞−critical types which is n−regular for every finite n. Our key step in proving this result is to show that around every type t, there is a sequence of types whose beliefs match those of t up to any finite order failing to converge to t in the uniform strategic topology defined and studied in Dekel, Fudenberg, and Morris (2006).[5]

The sequel of this paper is organized as follows. Section 2 provides notations and definitions. Section 3 presents the results for 3−critical types. Section 4 constructs the critical types which are n−regular and Section 5 deals with ∞−critical types. We conclude with a remark on the alternative notion of strategic discontinuity in terms of the solution concept of interim independent rationalizability (IIR) studied in Ely and Peski (2006).

2 Preliminaries

Throughout this paper, we fix a two-player set I and a set of payoff-relevant states Θ. Assume that Θ is a compact metric space. Given a player $i \in I$, we write $-i$ to designate the other player in I. For any arbitrary metric space Y, let $\Delta(Y)$ be the space of all probability measures on the Borel σ-algebra of Y endowed with the weak*-topology. All product spaces will be endowed with the product topology and subspaces with the relative topology. Every finite or countable set is endowed with the discrete topology and the cardinality of a finite set E is denoted by $|E|$. Moreover, let suppμ be the support of a measure μ. Finally, for

[5]Convergence in the uniform strategic topology requires that the speed of convergence of strategic behaviors is uniform over all finite games with a uniform payoff bound.

any $E \subseteq Y$ and $y \in Y$, let 1_E be the indicator function on E and $\delta_{\{y\}}$ be the Dirac measure on y.

2.1 Belief hierarchies and types

By a type space we mean a tuple $(T_i, \pi_i)_{i \in I}$ where T_i is the set of player i's types and π_i is a mapping which associate each type $t_i \in T_i$ with a belief $\pi_i(t_i) \in \Delta(\Theta \times T_{-i})$. We say that a type t is a finite type if there exists a type space $(T_i, \pi_i)_{i \in I}$ such that $t \in T_i$ and $T_i \cup T_{-i}$ is a finite set. For any $\mu \in \Delta(\Theta)$, denote by the type t^μ under which it is commonly known that both player has the first-order belief μ, i.e. t^μ is contained in the following type space: $T_1 = T_2 = \{t^\mu\}$ and $\pi_i(t^\mu)[(\theta, t^\mu)] = \mu(\theta)$, for all θ. For simplicity, we will write t^θ instead of $t^{\delta_{\{\theta\}}}$ for the type under which θ is commonly known.

Let $Y^0 = \Theta$ and $Y^1 = Y^0 \times \Delta(Y^0)$. Then, for $k \geq 2$ define recursively

$$Y^k = \left\{\left(\theta, \mu^1, ..., \mu^k\right) \in Y^0 \times \Delta\left(Y^0\right) \times \cdots \times \Delta\left(Y^{k-1}\right) : \text{marg}_{Y^{l-2}} \mu^l = \mu^{l-1}, \forall l = 2, ..., k\right\}.$$

Then, the Mertens-Zamir universal type space is defined as

$$\mathcal{T} = \left\{\left(\mu^1, \mu^2, ...\right) \in \times_{k=0}^{\infty} \Delta\left(Y^k\right) : \text{marg}_{Y^{l-2}} \mu^l = \mu^{l-1}, \forall l \geq 2\right\}.$$

For each $k \geq 1$, let $\pi^k : \mathcal{T} \to \Delta\left(Y^{k-1}\right)$ be the natural projection. For every player i and $k \geq 1$, let $\mathcal{T}_i$ and Y_i^k denote the copies of $\mathcal{T}$ and Y^k respectively, write $\pi_i^k : \mathcal{T}_i \to \Delta\left(Y_{-i}^{k-1}\right)$ for π^k, and define $\mathcal{T}_i^k = \pi_i^k(\mathcal{T}_i)$. An element $t_i \in \mathcal{T}_i$ is a type of player i. For simplicity, we will write t_i^k instead of $\pi_i^k(t_i)$ for the k^{th}-order belief of type t_i.[6] A sequence of types $t_{i,m}$ converges to t_i in product topology if for every k, $t_{i,m}^k \to t_i^k$ in the weak*-topology. Mertens and Zamir (1985) show that $\mathcal{T}_i$ (endowed with product topology) is a compact metric space and is homeomorphic to $\Delta(\Theta \times \mathcal{T}_{-i})$. Let π_i^* denote this homeomorphism. We will often abuse the notation to identify $t_i \in \mathcal{T}_i$ with its image under this homeomorphism and write $t_i[E]$ instead of $\pi_i^*(t_i)[E]$ for the probability t_i assigns to E.

[6]Note that $\mathcal{T}_i^k = \pi_i^k(\mathcal{T}_i)$ and hence when we write $t_i^k \in \mathcal{T}_i^k$ without specifying the type t_i, t_i^k should be understood as the k^{th}-order belief of some type $t_i \in \mathcal{T}_i$.

2.2 Bayesian game and ICR

Let $G = \langle A_i, g_i \rangle_{i \in I}$ be a game where A_i is a finite or a countably-infinite set of actions for player i and $g_i : A_i \times A_{-i} \times \Theta \to \Re$ is the payoff function. We will only deal with infinite games in Section 5. Following Bergemann and Morris (2009), we define the set of $\varepsilon-$interim correlated rationalizable (ICR) actions of type t_i as follows.

Given a type space $(T_i, \pi_i)_{i \in I}$, for any $i \in I$, $t_i \in T_i$, and $\varepsilon \geq 0$, we say that a correspondence $\left(\overline{R}_i\right)_{i \in I}$ with $\overline{R}_i : T_i \to 2^{A_i} \backslash \varnothing$ has *the ε-best-reply property* iff for every $i \in I$, every $t_i \in T_i$, and every $a_i \in \overline{R}_i(t_i)$, there exists a measurable function $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ such that

$$\text{supp}\sigma_{-i}(\theta, t_{-i}) \subseteq \overline{R}_{-i}(t_{-i}) \text{ for } t_i - \text{almost surely } (\theta, t_{-i});$$
$$\int_{\Theta \times \mathcal{T}_{-i}} \sum_{a_{-i} \in A_{-i}} \left[g_i(a_i, a_{-i}, \theta) - g_i(a_i', a_{-i}, \theta) \right] \sigma_{-i}(\theta, t_{-i})[a_{-i}]\, dt_i[(\theta, t_{-i})] \geq -\varepsilon, \forall a_i' \in A_i.$$

Clearly, if $\left(\overline{R}_{i,c}\right)_{i \in I}$ has the ε-best-reply property for all c in some index set C, then $\cup_{c \in C} \overline{R}_{i,c}$ (whose image is $\cup_{c \in C} \overline{R}_{i,c}(t_i)$ for every $t_i \in T_i$) also has the ε-best-reply property. We then define

$$R_i(t_i, G, \varepsilon) = \cup \left\{ \overline{R}_i(t_i) : \left(\overline{R}_i\right)_{i \in I} \text{ has the } \varepsilon\text{-best-reply property} \right\}. \tag{1}$$

When G is a finite game (i.e., when A_i is finite for every i), we can alternatively have the following recursive definition of ICR. Let $R_i^0(t_i, G, \varepsilon) = A_i$. For any integer $k \geq 1$, $a_i \in R_i^k(t_i, G, \varepsilon)$ iff there exists a measurable function $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ such that

$$\text{supp}\sigma_{-i}(\theta, t_{-i}) \subseteq R_{-i}^{k-1}(t_i, G, \varepsilon) \text{ for } \pi_i(t_i) - \text{almost surely } (\theta, t_{-i});$$
$$\int_{\Theta \times \mathcal{T}_{-i}} \sum_{a_{-i} \in A_{-i}} \left[g_i(a_i, a_{-i}, \theta) - g_i(a_i', a_{-i}, \theta) \right] \sigma_{-i}(\theta, t_{-i})[a_{-i}]\, d\pi_i(t_i)[(\theta, t_{-i})] \geq -\varepsilon, \forall a_i' \in A_i.$$

When G is a finite game, Dekel, Fudenberg, and Morris (2006, 2007) show that $R_i(t_i, G, \varepsilon) = \cap_{k=1}^{\infty} R_i^k(t_i, G, \varepsilon)$, and moreover, $R_i^k(t_i, G, \varepsilon) = R_i^k(s_i, G, \varepsilon)$ for any types t_i and s_i with $t_i^{k'} = s_i^{k'}$ for any $k' \leq k$ (and hence $R_i(t_i, G, \varepsilon) = R_i(s_i, G, \varepsilon)$ if $t_i^{k'} = s_i^{k'}$ for any k'). When G is infinite, while it is still true that $R_i(t_i, G, \varepsilon) \subseteq \cap_{k=1}^{\infty} R_i^k(t_i, G, \varepsilon)$, these two sets are not necessarily equal.[7] Finally, let $h_i(t_i, G, a_i) \equiv \min\{\varepsilon : a_i \in R_i(t_i, G, \varepsilon)\}$. The

[7] This approach is also adopted by Bergemann and Morris (2009) when they deal with "integer games" in their study of rationalizable implementation. Also when G is infinite, a recursive definition of $R_i(t_i, G, \varepsilon)$ which is equivalent to (1) may involve transfinite induction. See, for example, Lipman (1994).

strategic topology on $\mathcal{T}_i$ is the weakest topology such that for each G and a_i, the mapping $t_i \mapsto h_i(t_i, G, a_i)$ is continuous.[8]

We then define the following key notions in our paper.

Definition 1 *For $n = 1, 2, ..., \infty$, a type t_i is $n-$critical iff there is a $n \times n$ game G and a sequence of types $t_{i,m}$ such that $t_{i,m} \to t$ under the product topology but for each m, there is some $\varepsilon \geq 0$, $\delta > 0$, and action a_i in G, $a_i \in R_i(t_i, G, \varepsilon)$ and $a_i \notin R_i(t_{i,m}, G, \varepsilon + \delta)$. A type t_i is $n-$regular iff it is not $n-$critical.*

Following Ely and Peski (2007), we say a type is critical iff it is $n-$critical for some $n < \infty$ and a type is regular iff it is not critical.

2.3 $p-$beliefs and common $p-$beliefs

We follow Ely and Peski (2007) to define $p-$belief and common $p-$belief as follows. For any measurable set $E_{-i} \subseteq \mathcal{T}_{-i}$ and $p \in (0, 1]$, define

$$B_i^p(E_{-i}) = \{t_i \in \mathcal{T}_i : t_i[\Theta \times E_{-i}] \geq p\}.$$

For any product event $E = E_1 \times E_2$, we define $B_i^p(E) = E_i \cap B_i^p(E_{-i})$

$$B^p(E) = B_i^p(E) \times B_{-i}^p(E).$$

Consequently, $B^p(E) \subseteq E$.

Common $p-$belief in E occurs when both players $p-$believe in E, and both players p-believe in $B^p(E)$, and so on. This concept was introduced by Monderer and Samet (1989). Formally,

$$C^p(E) = \bigcap_{k \geq 1} [B^p]^k(E).$$

[8]The definition of strategic topology is different from the original definition in Dekel, Fudenberg, and Morris (2006). They are nonetheless equivalent as shown in Ely and Peski (2007).

Lemma 6 of Ely and Peski (2007) shows that for any product event E, $C^p(E) = C_i^p(E) \times C_{-i}^p(E)$ where

$$C_i^p(E) = E_i \cap C_{-i}^p(E) = B_i^p\left(\bigcap_{k\geq 0} [B^p]^k(E)\right).$$

For any measurable $E_i \subseteq \mathcal{T}_i$, we view it as a product event $E_i \times \mathcal{T}_{-i}$ and write $B^p(E_i) = B^p(E_i \times \mathcal{T}_{-i})$ and $C^p(E_i) = B^p(E_i \times \mathcal{T}_{-i})$.

2.4 Patching types

In this section, we formerly describe a novel technique of constructing types, which we call "patching types".

First, we say a type t is a countable type if there exists a type space $(T_i, \pi_i)_{i\in I}$ such that $t \in T_i$ and $T_i \cup T_{-i}$ is a finite or a countably infinite set. For simplicity, in this section, we illustrate patching types only on countable types. In fact, in this paper, we only using the patching-type technique on countable types. In fact, we can define $t_i \leftrightharpoons^n t_j$ even when t_i and t_j are not countable types. In Appendix A.1, we will formally demonstrate how to construct $t_i \leftrightharpoons^n t_j$ in general.

Second, given a countable type t_i, we define the $k-$step reachable types from t_i. Define the set of $1-$step reachable types from type t_i as

$$r(t_i) = \{t_{-i} \in T_{-i} : \pi_i(t_i)[t_{-i}] > 0\}.$$

For any $E \subset T_i$, define $r(E) = \cup_{t_i\in E} r(t_i)$. For any integer $k \geq 2$, let $r^k(t_i)$ denote the iteration of $r(\cdot)$ for k times. That is, $r^k(t_i)$ is the $k-$step reachable types from a type t_i.

We write $t_i \rightrightarrows t_{-i}$ if $t_{-i} \in r(t_i)$. For example, for the complete information type with θ, we have $t^\theta \rightrightarrows t^\theta \rightrightarrows \cdots$. Let t_i and t_j be two countable types. Let $t_i \leftrightharpoons^n t_j$ denote the type s_i such that (a) $s_i^k = t_i^k$ for all $k = 1, ..., n$ and (b) $r^n(t_i) = \{t_j\}$. In words, $t_i \leftrightharpoons^n t_j$ is the type whose beliefs agree with t_i up to order n and the only $n-$step reachable type from $t_i \leftrightharpoons^n t_j$ is t_j. By definition, $t_i \leftrightharpoons^n t_j$ converges to t_i in product topology as $n \to \infty$ and $R_i^k(t_i \leftrightharpoons^n t_j, G, \varepsilon) = R_i^k(t_i, G, \varepsilon)$ for all $k = 1, ..., n$.

For example, we can define $t^{\theta'} \leftrightharpoons^2 t^{\theta''}$ as follows. Let $T_1 = \{t_1', t_1''\}$ and $T_2 =$

$\{t'_2, t''_2\}$. Define the beliefs of types as follows: $\pi_1(t'_1)[(\theta', t'_2)] = 1$, $\pi_1(t'_2)[(\theta', t''_1)] = 1$, $\pi_i(t''_i)\left[\left(\theta'', t''_{-i}\right)\right] = 1$ for $i = 1, 2$. Then, t'_1 represent the type $t^{\theta'} \leftrightharpoons^2 t^{\theta''}$.

Next, we describe how to patch a countable set of countable types. For any three countable types $t(1)$, $t(2)$ and $t(3)$ and positive integers k_1 and k_2 given, we can also patch $t(1)$, $t(2)$ and $t(3)$ as follows. First, we construct a new type $t(1) \leftrightharpoons^{k_1} t(2)$ by patching $t(1)$ and $t(2)$ with order k_1. Second, we construct $t(1) \leftrightharpoons^{k_1} t(2) \leftrightharpoons^{k_2} t(3)$ by patching $t(1) \leftrightharpoons^{k_1} t(2)$ and $t(3)$ with order k_2. Similarly, given a countable set of countable types $t(1)$, $t(2)$,... and positive integers k_1, k_2,..., we can patch them by applying the operation defined above inductively and get

$$t(1) \leftrightharpoons^{k_1} t(2) \leftrightharpoons^{k_2} \cdots t(l) \leftrightharpoons^{k_l} \cdots.$$

For any $l \geq 1$, let $t[l] \equiv t(l) \leftrightharpoons^{k_l} t(l+1) \leftrightharpoons^{k_{l+1}} \cdots$. The following lemma can be proved by applying arguments in Lemma 4 in Appendix A.1 inductively.

Lemma 1 *The belief of $t[l]$ agrees with t^l up to order k_l. Moreover, for any l and l', $r^{\sum_{l''=l}^{l'} k_{l''}}(t[l]) = \{t[l+l']\}$.*

3 3−critical types

In this section, we consider the set of 3×3 games. We study strategic discontinuity in such simple games. In particular, we show that every finite type is 3−critical, and that every common-prior type space assigns probability 1 to 3−critical types. Furthermore, 3−critical types are generic in the universal type space under the strategic topology defined in Dekel, Fudenberg, and Morris (2006). Before dealing with 3−critical types, we briefly review the e-mail game argument.

3.1 E-mail game argument

To simplify our exposition, we consider the following modification of Rubinstein's e-mail game.

	a_2	b_2	a_2	b_2
a_1	$1,1$	$0,0$	$0,0$	$0,1$
b_1	$0,0$	$1,1$	$1,0$	$1,1$
	$\theta=\theta'$		$\theta=\theta''$	

.

When $\theta = \theta'$, the game is "meet in New York." When $\theta = \theta''$, each player has a strictly dominant strategy (i.e., b_1 for player 1 and b_2 for player 2.) Recall that $t^{\theta'} \leftrightharpoons^n t^{\theta''}$ converges to $t^{\theta'}$ in product topology as $n \to \infty$. However, it is straightforward to check that a_1 is rationalizable for $t^{\theta'}$, but a_1 is not $\frac{1}{2}-$rationalizable action for $t^{\theta'} \leftrightharpoons^n t^{\theta''}$ for any n. The intuition is that a_1 (or a_2) is not $\frac{1}{2}-$rationalizable for $t^{\theta''}$, and the usual infection argument (à la Rubinstein (1989), Carlsson and Damme (1993)) shows that a_1 is not $\frac{1}{2}-$rationalizable action for $t^{\theta'} \leftrightharpoons^n t^{\theta''}$ either. Therefore, $t^{\theta'}$ is a $2-$critical type.

3.2 Finite types

We now show that all finite types are $3-$critical. Fix an arbitrary finite type t of player 1. Since t is finite, $|T_1 \cup T_2| < \infty$ where $T_1 \times T_2$ is the smallest belief-closed subset containing t in the universal type spaces $\mathcal{T}_1 \times \mathcal{T}_2$. Thus, there is a parameter, say θ_0, and some open interval (y,z) such that $0 < y < z < 1$ and $t_i[\theta = \theta_0] \notin (y,z)$ for every $t_i \in T_i$ and $i = 1,2$.

Consider the following 2×3 games G parametrized by two positive numbers, x_1 and x_2.

	a_2	b_2	c_2	a_2	b_2	c_2
a_1	$0,-1$	$0,-1$	$1,0$	$0,-1$	$0,-1$	$1,0$
b_1	$1,-x_2$	$1,1$	$0,0$	$1,1$	$1,-x_1$	$0,0$
	$\theta=\theta_0$			$\theta\neq\theta_0$		

Observe that if players commonly know $\theta = \theta_0$, (a_1, c_2) and (b_1, b_2) are the two pure strategy Nash equilibria, and if they commonly know $\theta \neq \theta_0$, (a_1, c_2) and (b_1, a_2) are the two pure strategy Nash equilibria. Then, choose x_1 and x_2 such that $\frac{1}{1+x_2} = y$ and $\frac{x_1}{x_1+1} = z$.

Let s be a type of player 2. Observe that a_2 is strictly dominated by c_2 for s iff

$$s[\theta=\theta_0]\times(-x_2)+(1-s[\theta=\theta_0])\times 1<0,\ \text{i.e., } s[\theta=\theta_0]>\frac{1}{1+x_2}=y. \tag{2}$$

Similarly, action b_2 is strictly dominated by c_2 for s iff

$$s[\theta=\theta_0]\times 1+(1-s[\theta=\theta_0])\times(-x_1)<0,\ \text{i.e., } s[\theta=\theta_0]<\frac{x_1}{x_1+1}=z. \tag{3}$$

Let μ be a first-order belief with $\mu[\theta_0]=\alpha\in(y,z)$. Consider t^{μ}, the type with common knowledge of first-order belief being μ. Define a sequence of types $t_{1,m}\equiv t\leftrightarrows^{2m+1}t^{\mu}$. As argued above, $t_{1,m}$ converges to t in product topology. However, we will show that $t_{1,m}$ does not converge to t in behaviors. In particular, b_1 is rationalizable for t, but b_1 is not rationalizable for any $t_{1,m}$. The intuition is similar to the e-mail argument. To rationalize a_1 for player 1, player 2 should play c_2. To rationalize b_1 for player 1, player 2 should play either a_2 or b_2. By (2), (3) and $\mu[\theta_0]=\alpha\in(y,z)$, both a_2 and b_2 are strictly dominated for t^{μ}. Then, the usual infection argument (à la Rubinstein (1989), Carlsson and Damme (1993)) shows that b_1 is not rationalizable for $t_{1,m}\equiv t\leftrightarrows^{2m+1}t^{\mu}$.

Claim 1 *b_1 is $0-$rationalizable for t in G.*

Proof. Consider $S_i:T_i\to 2^{A_i}$ defined as $S_1(t_1)=\{b_1\}$ for $t_1\in T_1$, and for $t_2\in T_2$, $S_2(t_2)=\{a_2\}$ if $t_2[\theta=\theta_0]\leq y$ and $S_2(t_2)=\{b_2\}$ if $t_2[\theta=\theta_0]\geq z$. Then, $\left((S_i(t_i))_{t_i\in T_i}\right)_{i\in I}$ satisfies the $0-$best reply property. Hence, b_1 is $0-$rationalizable for t in $G(y,z)$.■

Claim 2 *b_1 is not $\gamma-$rationalizable for $t_{1,m}$ in $G(y,z)$ for any m, where*

$$\gamma=\min\left\{1,\ \frac{|\alpha\times(-x_2)+(1-\alpha)\times 1|}{2},\ \frac{|\alpha\times 1+(1-\alpha)\times(-x_1)|}{2}\right\}>0$$

Proof. Player 2's type t^{μ} gets 0 if he chooses c_2, regardless of his opponent's action. By choosing a_2, t^{μ} gets $-1<-\gamma<0$ if his opponent chooses a_1, and t^{μ} gets $[\alpha\times 1+(1-\alpha)\times(-x_1)]<-\gamma<0$ if his opponent chooses b_1. Also, by choosing b_2, t^{μ} gets $-1<-\gamma<0$ if his opponent chooses a_1, and t^{μ} gets $[\alpha\times 1+(1-\alpha)\times(-x_1)]<-\gamma<0$ if his opponent chooses b_1. Therefore, $R_2(t^{\mu},G,\gamma)=\{c_2\}$. Since $r^{2m+1}(t)=\{t^{\mu}\}$, $R_1(t_1,G,\gamma)=\{a_1\}$

for all $t_1 \in r^{2m}(t)$. Hence, $R_2(t_2, G, \gamma) = \{c_2\}$ for all $t_2 \in r^{2m-1}(t)$. Inductively, we have $R_1(t_1, G, \gamma) = \{a_1\}$.■

Since $t_{1,m}$ converges to t in product topology, we get the following theorem as a consequence of Claims 1 and 2.

Theorem 1 *Every finite type is* $3-$*critical.*

In fact, the argument above can be applied to a much broader class of types. Say a set $E \subset \mathcal{T}_i$ is a proper first-order interval set if there is $\theta_0 \in \Theta$ and $0 \leq y < z \leq 1$ such that $t_i[\theta_0] \notin (y, z)$ for all $t_i \in E$.

Theorem 2 *A type* t_i *is* $3-$*critical if* $t_i \in C_i^p(E_i)$ *for some* $p > 0$ *and some closed proper first-order interval set* E_i*.*

Proof. See Appendix A.2.■

3.3 Common-prior types

We now show that almost all types from types spaces with a common prior are $3-$critical. Let (T_i, π_i) be a type space. Following Ely and Peski (2007), we say that (T_i, π_i) is a common-prior type space if there exists a prior $\psi \in \Delta(T_i \times T_{-i})$ on (T_i, π_i) such that for any bounded measurable function $f : T_i \times T_{-i} \to \Re$ and any player i,

$$\int_{T_i \times T_{-i}} f(t_i, t_{-i})\, d\psi = \int_{T_i} \int_{T_{-i}} f(t_i, t_{-i})\, d\pi_i(t_i)[t_{-i}]\, d\psi_i[t_i],$$

where $\psi_i = \text{marg}_{T_i}\psi$.

The following theorem is our main result in this section.

Theorem 3 *Suppose that* ψ *is a common prior on a type space* (T_i, π_i)*. Then, for each player* i*,* t_i *has* $3-$*critical hierarchy* ψ_i-*almost surely.*

To prove this result, we need the following two lemmas. Lemma 3 is Lemma 11 in Ely and Peski (2007), which is a version of one-side of the critical-path lemma due to Kajii and Morris (1997). The proof of Lemma 2 can be found in Appendix A.3.

Lemma 2 *Let ψ^* be a common prior on the universal type space $(\mathcal{T}_i, \pi_i^*)$. Then, for any $\varepsilon > 0$, there are closed proper first-order interval sets E_i such that $\psi^*(E_1 \times E_2) \geq 1 - \varepsilon$.*

Lemma 3 (Ely and Peski 2007, Lemma 11) *Let ψ^* be a common prior on the universal type space $(\mathcal{T}_i, \pi_i^*)$. For any measurable sets $E_i \subseteq \mathcal{T}_i$*

$$\psi^*\left(C^{1/4}(E_1 \times E_2)\right) \geq \frac{3}{2}\psi^*(E_1 \times E_2) - \frac{1}{2}.$$

Proof of Theorem 3. Suppose that ψ is a common prior on a type space (T_i, π_i). Consider the canonical mapping from T_i to the universal type space $\mathcal{T}_i$. This induce a common prior ψ^* on $\mathcal{T}_i$. Suppose $\psi^*(F) = \varepsilon > 0$, where F is the set of 3−regular types, i.e., the statement of Theorem 3 is incorrect. By Lemma 2, there are closed proper first-order interval sets E_i and $\psi^*(E_1 \times E_2) \geq 1 - \frac{\varepsilon}{2}$. Next, Lemma 3 implies that

$$\psi^*\left(C^{1/4}(E_1 \times E_2)\right) \geq 1 - \frac{3}{4}\varepsilon.$$

By Theorem 2, the set $C^{1/4}(E_1 \times E_2)$ consists of only 3−critical types, hence,

$$\left(C^{1/4}(E_1 \times E_2)\right) \bigcap F = \varnothing.$$

Thus,

$$\begin{aligned}
\psi^*\left[\left(C^{1/4}(E_1 \times E_2)\right) \bigcup F\right] &= \psi^*\left(C^{1/4}(E_1 \times E_2)\right) + \psi^*(F) \\
&= 1 - \frac{3}{4}\varepsilon + \varepsilon \\
&= 1 + \frac{1}{4}\varepsilon \\
&> 1.
\end{aligned}$$

which is a contradiction.■

3.4 Genericity of 3−critical types

By the result in Ely and Peski (2007), we know that regular types and hence 3−regular types exist and form a residual set in the product topology. In sharp contrast to this result, we show that 3−critical types contain a set which is open and dense in the strategic topology.

To do this, we need the following notion of convergence. Let d denote the product metric on the universal type space. For any set $E_i \subset \mathcal{T}_i$, denote the ε-open ball containing E_i under the product topology by E_i^{ε}, i.e.,

$$E_i^{\varepsilon} \equiv \{t_i \in \mathcal{T}_i : d(t_i, t_i') < \varepsilon \text{ for some } t_i' \in E_i\}.$$

Definition 2 (convergence in common-p belief) *A sequence of types $(t_{i,m})_{m=1}^{\infty}$ converges to a type t_i in common $p-$belief if for any $\varepsilon > 0$, any $p \in (0,1]$ and any closed proper subset $E_i \subseteq \mathcal{T}_i$ such that $t_i \in C_i^p(E_i)$, there exists a positive integer N such that $u_i^n \in C_i^{p-\varepsilon}(E_i^{\varepsilon})$ for any $m \geq N$.*

We now formally state and prove this genericity result.

Theorem 4 *The strategic closure of 3−regular types consists no finite types. Thus, the set of 3−critical types contains a set which is open and dense in the strategic topology.*

To prove Theorem, we need the following proposition whose proof can be found in Chen, Di Tillio, Faingold, and Xiong (2008).

Proposition 1 *A sequence of types $t_{i,m} \to t_i$ under the strategic topology only if $t_{i,m} \to t_i$ in common $p-$belief.*

Proof of Theorem 4. Suppose instead that a sequence of 3−regular types $(t_{i,m})_{m=1}^{\infty}$ converges to a finite type t_i under the strategic topology. Since t_i is a finite type, $t_i \in C_i^p(E_i)$ for a finite set $E_i \subseteq \mathcal{T}_i$. Moreover, since $t_{i,m} \to t_i$ under the strategic topology, $t_{i,m} \to t_i$ in common $p-$belief by Proposition 1. Since E_i is finite, for sufficiently small $\varepsilon > 0$, the product closure of E_i^ε, denoted by $\overline{E_i^\varepsilon}$, is still a closed proper first-order interval set E_i, and moreover, $p - \varepsilon > 0$. Since $t_{i,m} \to t_i$ in common $p-$belief, there exists a positive integer N such that $t_{i,m} \in C_i^{p-\varepsilon}\left(\overline{E_i^\varepsilon}\right)$. Hence, $t_{i,m}$ is a $3-$critical type for all $m \geq N$ by Theorem 2, which contradicts to the assumption that $t_{i,m}$ is $3-$regular for all m. Since finite types are dense in the strategic topology by the results in Dekel, Fudenberg, and Morris (2006), the set of $3-$critical types contains a set which is open and dense in the strategic topology.■

4 Critical types which are $n-$regular

Our results in the previous section suggest that almost all types commonly used in the economics literature are $3-$critical types and the set of $3-$critical types is "large" in the strategic topology. This raises the question whether every critical type is in fact $3-$critical, or more generally, whether there some integer n, such that every critical type is $n-$critical. In this section, we prove the following theorem that for every integer $n \geq 2$, there exists a critical type which is $n-$regular. Hence, those complicated games used in the proof of Ely and Peski (2007) are necessary to characterize all critical types.

Theorem 5 *For every integer $n \geq 2$, there is a critical type which is $n-$regular.*

The proof of Theorem 5 is involved and is relegated to Appendix A.4. We only provide a sketch of our arguments here.

Let $n \geq 2$ be a fixed integer. By definition, a $n-$regular exhibits (strategic) continuity in all $n \times n$ games, i.e., $h_i(t_{i,m}, G, a_i) \to h_i(t_i, G, a_i)$ for any $a_i \in A_i$ and any $t_{i,m} \to t_i$ in product topology. However, consider an alternative notion: we say t_i exhibits continuity at 0 in game G, if $h_i(t_{i,m}, G, a_i) \to h_i(t_i, G, a_i)$ for any $a_i \in A_i$ such that $h_i(t_i, G, a_i) = 0$ and any $t_{i,m} \to t_i$ in product topology. By Lemma 4 of Ely and Peski (2007), a type exhibits continuity in all $n \times n$ games if it exhibits continuity at 0 in all $n^2 \times n^2$ games. Second, there is a countable set of games $\{G_l\}$ which is dense in the space of all $n^2 \times n^2$ games. Therefore, in order to prove t_i is $n-$regular, we only need to show t_i exhibits continuity at 0 in G_l for every l.

The rest of the proof crucially relies upon a notion called *minimal rationalizable type* studied in Ely and Peski (2007). A type t is a minimal rationalizable type in a game G if there is no type t' whose $0-$rationalizable set is a proper subset of the $0-$rationalizable set of t. Ely and Peski (2007) provide the following insight: if t is a minimal rationalizable type in G, then t exhibits continuity at 0 in G. The intuition is the following. Suppose $t_{i,m} \to t_i$ in product topology. First, product convergence implies upper-hemi strategic continuity (by Theorem 2 in Dekel, Fudenberg, and Morris (2006)). Second, if t_i is a a minimal rationalizable type, the lower-hemi strategic continuity cannot fail either.[9]

Our key observation is the following propositions which offer a easy way for us to construct minimal rationalizable types.

Proposition 2 *For any game $G \in \mathcal{G}^n$, there exists a finite type $\widehat{t_i}$ which is a minimal rationalizable type in G, and $R_i^{2^{n+1}}\left(\widehat{t_i}, G, 0\right) = R_i\left(\widehat{t_i}, G, 0\right)$.*

Proposition 3 *Given t_i, a minimal rationalizable type in G, such that $R_i^k(t_i, G, 0) = R_i(t_i, G, 0)$, if another s_i has the same $k-$th order belief as t_i, i.e., $s_i^k = t_i^k$, then s_i is a minimal rationalizable type in G.*

The proof of Proposition 2 is relegated to Appendix A.4.1. The proof of Proposition 3 is straightforward. With $s_i^k = t_i^k$, we have

$$R_i(s_i, G, 0) \subseteq R_i^k(s_i, G, 0) = R_i^k(t_i, G, 0) = R_i(t_i, G, 0). \tag{4}$$

Since t_i is a minimal rationalizable type in G, 4 implies $R_i(s_i, G, 0) = R_i(t_i, G, 0)$, and hence s_i is a minimal rationalizable type in G.

By these proposition, when we fix n^2, we can find a finite type $\widehat{t_i}$ whose rationalizable set is not only minimal but depends only upon the the $2^{n^2+1}th$-order belief of $\widehat{t_i}$. For each

[9] Upper-hemi continuity implies that for any sufficient small $\varepsilon > 0$ and some sufficient large n, $R_i(t_{i,m}, G, \varepsilon) \subseteq R_i(t_i, G, 0)$. Further, t_i being minimal raaionalizable implies that we cannot have $R_i(t_{i,m}, G, \varepsilon) \subsetneq R_i(t_i, G, 0)$. Hence, we have $R_i(t_{i,m}, G, \varepsilon) = R_i(t_i, G, 0)$, and lower-hemi continutity is satisfied.

G_l, let $t_{i,l}$ be the finite type for G_l given by the proposition. Then, fix $\theta_0 \in \Theta$ and define the $n-$regular types as

$$t_i^* \equiv t_{i,1} \leftrightharpoons^{k^*} t^{\theta_0} \leftrightharpoons^{1} t_{i,2} \leftrightharpoons^{k^*} t^{\theta_0} \leftrightharpoons^{1} \cdots \leftrightharpoons^{1} t_{i,l} \leftrightharpoons^{k^*} t^{\theta_0} \leftrightharpoons^{1} \cdots .$$

By Lemma 8 of Ely and Peski (2007), a type t_i exhibits continuity in G at 0 if for some k the only $k-$step reachable type from t_i is a minimal rationalizable type in G. This implies that t_i^* exhibits continuity in every G_l at 0 and thus is $n-$regular. Moreover, it is common 1-believed within type space containing t_i^* that every type reaches a type assigning probability 1 to θ_0 in at most 2^{n^2+1} steps, which is clearly a closed proper subset in the universal type space. Hence, t_i^* is critical, by Theorem 3 in Ely and Peski (2007).

5 $\infty-$critical types

In this section we show that when Θ contains at least two points, every type is $\infty-$critical. As an intermediate step, we show first that every type t_i is "uniform critical" in the sense that there exists a sequence of types $t_{i,m} \to t_i$ in product topology but $t_{i,m}$ fails to converge to t_i under the uniform strategic topology defined in Dekel, Fudenberg, and Morris (2006).[10]

Recall that $h_i(t_i, G, a_i) = \min\{\varepsilon : a_i \in R_i(t_i, G, \varepsilon)\}$. Let $\mathcal{G}[-1,1]$ be the collection of all finite games which assume payoffs in the interval $[-1,1]$. The uniform strategic topology is defined to be the metric topology under the metric

$$d^{us}(t_i, s_i) = \sup_{a_i \text{ in } G,\, G \in \mathcal{G}[0,1]} |h_i(s_i, G, a_i) - h_i(t_i, G, a_i)| .$$

We now formally state the proposition showing that every type is "uniform critical."

Proposition 4 *For every type $t_i \in \mathcal{T}_i$, there is a product convergent sequence $t_{i,m} \to t_i$ such that $t_{i,m}$ does not converge to t_i in the uniform strategic topology.*

The following theorem is our main result in this section.

[10] Whether this intermediate step (i.e., Proposition 4) is true was posted independently as an open question by Drew Fudenberg in his website in April 2009.

Theorem 6 *Every type is* $\infty-$*critical.*

Proof. Without loss of generality, consider a type t_1 of player 1 in $\mathcal{T}_1$. By Proposition 4 and its proof, there is an $\varepsilon \in (0,1)$ and a sequence of types $t_{1,m}$ converging to t_1 in the product topology and for every m, there is a game $G^m = \langle A_i^m, g_i^m \rangle_{i \in I} \in \mathcal{G}[-1,1]$ and an action a_1^m in A_1^m such that a_1^m is $0-$rationalizable for t_1 but a_1^m is not $\varepsilon-$rationalizable for $t_{1,m}$ in G^m.

Let $\alpha = \frac{\varepsilon}{8-4\varepsilon}$. Since $\varepsilon \in (0,1)$, $\alpha \in (0, 1/4)$. We now define a new game $G = \langle A_i, g_i \rangle_{i \in I} \in \mathcal{G}[-1,1]$ such that

$$
\begin{aligned}
A_i &= \{a^0\} \cup \left(\cup_{m=1}^{\infty} A_i^m\right); \ A_{-i} = \{a^0\} \cup \left(\cup_{m=1}^{\infty} A_{-i}^m\right); \\
g_i(a_i, a_{-i}, \theta) &= \begin{cases} \alpha\left(g_i^m(a_i, a_{-i}, \theta) + 1\right), & \text{if } (a_i, a_{-i}) \in A_i^m \times A_{-i}^m; \\ 0, & \text{if } a_i = a^0; \\ -1, & \text{otherwise.} \end{cases}
\end{aligned}
$$

$$
g_{-i}(a_i, a_{-i}, \theta) = \begin{cases} \alpha\left(g_i^m(a_i, a_{-i}, \theta) + 1\right), & \text{if } (a_i, a_{-i}) \in A_i^m \times A_{-i}^m; \\ 0, & \text{if } a_{-i} = a^0; \\ -1, & \text{otherwise.} \end{cases}
$$

Claim 3 $R_i(t_i, G, \gamma) \supseteq R_i(t_i, G^m, \gamma)$ *for any* m*, any* $\gamma \geq 0$*, any player* i*, and any* $t_i \in \mathcal{T}_i$*.*

Claim 4 *If* $a_i \in R_i(t_i, G, \gamma) \cap A_i^m$*, then* $a_i \in R_i(t_i, G^m, \delta)$ *for* $\delta = 2\left(\frac{\gamma+2\alpha}{1+2\alpha}\right) + \frac{\gamma}{\alpha}$*, for any player* i*, any* $t_i \in \mathcal{T}_i$*, and any* $1 > \gamma \geq 0$*.*

Theorem 6 then follows from the two claims above, whose proof is relegated to Appendix A.6.To see this, recall that $a_1^m \in R_1(t_1, G^m, 0)$ and $a_1^m \notin R_1(t_{1,m}, G^m, \varepsilon)$. By Claim 3, $a_1^m \in R_1(t_1, G, 0)$. Since $\delta \to \frac{4\alpha}{1+2\alpha} < \varepsilon$ (because $\alpha = \frac{\varepsilon}{8-4\varepsilon}$) as $\gamma \to 0$, there exists some $\gamma^* > 0$ such that $\delta^* = 2\left(\frac{\gamma^*+2\alpha}{1+2\alpha}\right) + \frac{\gamma^*}{\alpha} < \varepsilon$. Since $a_1^m \notin R_1(t_{1,m}, G^m, \varepsilon)$, by Claim 4 we have $a_1^m \notin R_1(t_1, G, \gamma^*)$. Note that γ^* is independent of m. Therefore, t_1 is $\infty-$critical.■

6 Concluding Remarks

Throughout the paper, we only consider the ICR. However, a competing solution concept exists, i.e., interim independent rationalizability (IIR) studied in Ely and Peski (2007). Nevertheless, most of our results remain true if we consider IIR instead of ICR. In Appendix A.7.2, we prove the following two results which offer a vivid distinction between ICR and IIR.

Proposition 5 *There exists a finite type which is* $2-$*regular under ICR.*

Proposition 6 *Any finite type is* $2-$*critical under IIR.*

A Appendix

A.1 Patching types

In this section, we formally define how we patch two types t_i and t_j to produce $t_i \leftrightharpoons^n t_j$. Say every type reaches itself in 0 step. For any types $\bar{t}_i$ and $\bar{s}_i$ and even number $n \geq 1$, say $\bar{t}_i$ *surely reaches* $\bar{s}_i$ *in* n *steps* iff there are set of types T^0, T^1,..., T^n, T^{n+1} such that (a) $T^0 = \{\bar{t}_i\}$, $T^{n+1} = \{\bar{s}_i\}$; (b) $T^k \subseteq \mathcal{T}_{-i}$ if k is odd and $T^k \subseteq \mathcal{T}_i$ if k is even, and moreover,

$$t_j \left[T^k\right] = 1 \text{ for all } t_j \in T^{k-1} \text{ and } k = 1, ..., n+1.$$

Similarly, we can define that $\bar{t}_i$ reaches $\bar{s}_{-i}$ in n steps for any $\bar{t}_i$ and $\bar{s}_{-i}$ and odd number $n \geq 1$. Clearly, $\bar{t}_i$ reaches $\bar{s}_j$ in n steps implies there is a set T_{-i} such that $\bar{t}_i [T_{-i}] = 1$ and t_{-i} reaches $\bar{s}_j$ in $n-1$ steps for any $t_{-i} \in T_{-i}$.

Lemma 4 (patching types) *For any types* $\bar{t}_i$ *and* $\mathcal{T}_i$ *and* $\bar{s}_{-i}$ *in* $\mathcal{T}_{-i}$ *(resp.* $\bar{s}_i \in \mathcal{T}_i$*) and any odd (resp. even) integer* n*, there is a type* $\bar{t}_i \leftrightharpoons^n \bar{s}_{-i}$ *(resp.* $\bar{t}_i \leftrightharpoons^n \bar{s}_i$*) such that (a) the beliefs of* $\bar{t}_i \leftrightharpoons^n \bar{s}_{-i}$ *(resp.* $\bar{t}_i \leftrightharpoons^n \bar{s}_i$*) agrees with the beliefs of* t_i *up to order* n*; (b)* $\bar{t}_i \leftrightharpoons^n \bar{s}_{-i}$ *(resp.* $\bar{t}_i \leftrightharpoons^n \bar{s}_i$*) surely reaches* $\bar{s}_{-i}$ *(resp.* $\bar{s}_i$*) in* n *steps.*

Proof Let $\left(T_j^1, \pi_j^1\right)$ be the type space containing $\bar{t}_i$, and $\left(T_j^2, \pi_j^2\right)$ be the type space containing $\bar{s}_{-i}$. To define $\bar{t}_i \leftrightharpoons^n \bar{s}_{-i}$, we first define a type space as follows. For any $j \in I$ and odd k, let $T_{-i}^{1,k}$ be an identical copy of T_{-i}^1 indexed by k, and similarly, for even k, let

$T_i^{1,k}$ be an identical copy of T_i^1 indexed by k. Thus, $t_i \in T_i^{1,k}$ is understood to be the type which corresponds to t_i in T_i^1. Moreover, for any $j \in I$ and $k \leq n-1$, let I_Θ be the identity mapping on Θ and $\varphi_{j,k} : T_j^{1,k} \to T_j^1$ be the identity embedding on T_j^1.

$$\text{Let } T_i = \left[\cup_{k=0}^{n-1} T_i^{1,k}\right] \bigcup T_i^2 \text{ and } T_{-i} = \left(\cup_{k=0}^{n-2} T_{-i}^{1,k}\right) \bigcup T_{-i}^2 \text{ and define for } j = i, -i,$$

$$\text{for } k \leq n-1,\ \vartheta_j^{1,k} : 2^{T_i} \to 2^{T_i} \text{ such that } \vartheta_j^{1,k}(E) = E \cap T_j^{1,k};$$

$$\vartheta_{-i}^{1,n} : 2^{T_{-i}} \to 2^{T_{-i}} \text{ such that } \vartheta_{-i}^{1,n}(E) = E \cap \{\overline{s}_{-i}\};$$

$$\varphi_{-i,n} : \{\varnothing, \{\overline{s}_{-i}\}\} \to T_{-i}^1 \text{ such that } \varphi_{-i,n}(\{\varnothing\}) = \varnothing \text{ and } \varphi_{-i,n}(\{\overline{s}_{-i}\}) = T_{-i}^1;$$

$$\pi_j(t_j) = \begin{cases} \pi_j^1\left[\varphi_{j,k}(t_j)\right] \circ \left[I_\Theta \times \left(\varphi_{-j,k+1} \circ \vartheta_{-j}^{1,k+1}\right)\right], & \text{if } t_j \in T_j^{1,k},\ 0 \leq k \leq n-1; \\ \pi_j^2(t_j), & \text{if } t_j \in T_j^2. \end{cases}$$

Let $\overline{t}_i \leftrightharpoons^n \overline{s}_{-i}$ be the type $\overline{t}_i \in T_i^{1,0}$. Then, The property (b) follows directly from our construction and the proof of property (b) is identical to the proof of Lemma 1 in Ely and Peski (2007). ■

A.2 Proof of Theorem 2

Theorem 2 *A type t_i is $3-$critical if $t_i \in C_i^p(E_i)$ for some $p > 0$ and some closed proper first-order interval set E_i.*

Proof. Let $\overline{t_i}$ be a type in $C_i^p(E_i)$ for some *some closed proper first-order interval set* E_i. It is without loss of generality to assume $i = 2$ and $p < \frac{1}{2}$, because $C_i^p(E_i) \subseteq C_i^{p'}(E_i)$ for $p \geq p'$. Since E_2 is a closed proper first-order interval set, there exists $\theta_0 \in \Theta$ and some open interval (y, z) such that $0 \leq y < z \leq 1$ and $t_i[\theta_0] \notin (y, z)$ for every $t_i \in E_2$.

Consider the following class of 2×3 games G parametrized by four positive variables, x_1, x_2, x_3 and x_4 to be determined later.

	a_2	b_2	c_2	a_2	b_2	c_2
a_1	$0, -x_3$	$0, -x_3$	$x_4, 0$	$0, -x_3$	$0, -x_3$	$x_4, 0$
b_1	$x_4, -x_2$	$x_4, 1$	$0, 0$	$x_4, 1$	$x_4, -x_1$	$0, 0$
	$\theta = \theta_0$			$\theta \neq \theta_0$		

Let s be a type of player 2. Observe that a_2 is strictly dominated by c_2 for player 2 iff $(1 - s[\theta_0]) - s[\theta_0] x_2 < 0$, i.e., $s[\theta_0] > \frac{1}{1+x_2}$. Similarly, action b_2 is strictly dominated by c_2 for player 2 iff $-(1 - s[\theta_0]) x_1 + s[\theta_0] < 0$, i.e. $s[\theta_0] < \frac{x_1}{x_1+1}$. Observe that (a_1, c_2) and (b_1, b_2) are the two pure strategy Nash equilibrium when players commonly know $\theta = \theta_0$; (a_1, c_2) and (b_1, a_2) are the two pure strategy Nash equilibrium when players commonly know $\theta \neq \theta_0$.

Then, choose x_1 and x_2 such that $\frac{1}{1+x_2} = y$ and $\frac{x_1}{x_1+1} = z$. Let μ be a first-order belief such that $\mu[\theta_0] = \alpha \in (y, z)$. Define $t_{2,m} \equiv \left[\overline{t_2} \leftrightarrows^{2m} t^{\mu}\right]$ for any positive integer m, i.e., we patch t^* to $\overline{t_2}$ at the $(2m+1)^{th}$-order. By Lemma 4, $t_{2,m} \to \overline{t_2}$ in product topology. However, we will show that $t_{2,m}$ does not converge to $\overline{t_2}$ under the strategic topology.

Define

$$\gamma = \min\left\{\frac{|\alpha \times (-x_2) + (1-\alpha) \times 1|}{2}, \frac{|\alpha \times 1 + (1-\alpha) \times (-x_1)|}{2}\right\} > 0. \tag{5}$$

Moreover, we choose x_3 and x_4 so that

$$\frac{\gamma}{1-p} > x_3 > \gamma, \tag{6}$$

$$\frac{\gamma}{1-2p} > x_4 > \gamma. \tag{7}$$

Define

$$\gamma' = \frac{\gamma + \max\{(1-p) x_3, (1-2p) x_4\}}{2}. \tag{8}$$

By (6) and (7), we have $(1-p) x_3 < \gamma$ and $(1-2p) x_4 < \gamma$. As consequences, we have

$$0 < \gamma' < \gamma, \tag{9}$$

$$(1-p) x_3 < \gamma', \tag{10}$$

$$(1-2p) x_4 < \gamma'. \tag{11}$$

We now show by the two steps below that either a_2 or b_2 is $\gamma'-$ICR for $\overline{t_2}$, but neither a_2 nor b_2 is not $\gamma-$ICR for any $t_{2,m}$, hence, $t_{2,m}$ does not converge to $\overline{t_2}$ under the strategic topology.

Step 1 $\{a_2, b_2\} \cap R_2\left(\overline{t_2}, G, \gamma'\right) \neq \varnothing$

We show that $\{a_2, b_2\} \cap R_2(t_2, G, \gamma') \neq \varnothing$ for any $t_2 \in C_2^p(E_2)$ and in particular for $t_2 = \overline{t_2}$.

Define $\overline{R}_1(t_1) = \{b_1\}$ for all $t_1 \in C_1^p(E_2)$. Recall that for every $t_2 \in C_2^p(E_2)$, $t_2 \in E_2$ and hence $t_2[\theta_0] \notin (y, z)$. For $t_2 \in C_2^p(E_2)$, define

$$\overline{R}_2(t_2) = \begin{cases} \{a_2\}, & \text{if } t_2[\theta_0] \leq y \text{ and } t_2 \in C_2^p(E_2); \\ \{b_2\}, & \text{if } t_2[\theta_0] \geq z \text{ and } t_2 \in C_2^p(E_2). \end{cases} \tag{12}$$

For any other t_i, $\overline{R}_i(t_i)$ $(\subset R_2(t_i, G, \gamma'))$ is arbitrarily selected. We now verify that $\overline{R}$ has the γ'−best reply property.

First, consider player 1's type $t_1 \in C_1^p(E_2)$ and hence $t_1[C_2^p(E_2)] \geq p$. Suppose he believes $t_2 \in C_2^p(E_2)$ plays the action in $\overline{R}_2(t_2)$ defined in (12), he gets at least $p \times x_4$ by playing b_1, while he gets at most $(1-p) \times x_4$ by playing a_1. Since $p \times x_4 - (1-p) \times x_4 = -(1-2p)x_4 > -\gamma'$ by (11), b_1 is the unique γ'−best-reply for t_1.

Second, consider player 2's type t_2 such that $t_2[\theta_0] \geq z$ and $t_2 \in C_2^p(E_2)$. Suppose he believes that player 1 plays b_1 if $t_1 \in C_1^p(E_2)$. Since $t_2[\theta_0] \geq z > y$, then a_2 is dominated by c_2. Further, t_2 gets a payoff of 0 by playing c_2. However, he gets at least $p \times ((1 - t_2[\theta_0]) \times (-x_1) + t_2[\theta_0] \times 1) - (1-p) \times x_3$ by playing b_2, and furthermore,

$$\begin{aligned} & p \times ((1 - t_2[\theta_0]) \times (-x_1) + t_2[\theta_0] \times 1) - (1-p) \times x_3 \\ \geq\ & p \times ((1-y) \times (-x_1) + y \times 1) - (1-p) \times x_3 \\ =\ & -(1-p) \times x_3 \\ >\ & -\gamma' \end{aligned}$$

where the first inequality follows because $t_2[\theta_0] \geq y$; the equality follows because $x_1 = \frac{y}{1-y}$; the last inequality follows from (10). Therefore, b_2 is a γ'−best-reply for $t_2 \in C_2^p(E_2)$ with $t_2[\theta_0] \geq z$. Similarly, for type $t_2 \in C_2^p(E_2)$ such that $t_2[\theta_0] \leq y$, a_2 is a γ'−best-reply for t_2.

Step 2 $\{a_2, b_2\} \cap R_2(t_{2,m}, G, \gamma) = \varnothing$ *for any* m

Since $t^\mu[\theta_0] = \alpha \in (y, z)$, both a_2 and b_2 are γ−dominated by c_2 for player 2 by the definitions of γ and x_3. More specifically, t^μ gets 0 if he chooses c_2, regardless of his opponent's action. By choosing a_2, t^μ gets $-x_3 < -\gamma < 0$ (by 6) if his opponent chooses a_1, and t^μ gets $[\alpha \times 1 + (1-\alpha) \times (-x_1)] < -\gamma < 0$ (by 5) if his opponent chooses b_1. Also, by choosing b_2, t^μ gets $-x_3 < -\gamma < 0$ (by 6) if his opponent chooses a_1, and t^μ gets $[\alpha \times 1 + (1-\alpha) \times (-x_1)] < -\gamma < 0$ (by 5) if his opponent chooses b_1. Therefore, neither of a_2 and b_2 is $\gamma-$rationalizable for t^μ and $R_2(t^\mu, G, \gamma) = \{c_2\}$. Hence, $a_2, b_2 \notin R_2(t_{2,m}, G, \gamma)$ by the usual infection argument: since $x_3 > \gamma$ and $x_4 > \gamma$, given that player 2 chooses c_2, player 1 has a unique $\gamma-$best reply a_1; given player 1 playing a_1, player 2 has a unique $\gamma-$best reply c_2.■

A.3 Proof of Lemma 2

Lemma 2 *Let ψ^* be a common prior on the universal type space $(\mathcal{T}_i, \pi_i^*)$. Then, for any $\varepsilon > 0$, there are closed proper first-order interval sets E_i such that $\psi^*(E_1 \times E_2) \geq 1 - \varepsilon$.*

Proof. Let S be the support of ψ^*, i.e. the minimal closed set with probability 1 under ψ^*. Let S_i be the projection of S on $\mathcal{T}_i$. Pick an arbitrary $\theta_0 \in \Theta$. For any $n \geq 1$, let

$$S_i^n \equiv \left\{ t_i \in S_i : t_i \in S_i, t_i[\theta_0] \in \left(\frac{1}{n+1}, \frac{1}{n} \right) \right\}.$$

Since $\{S_i^n\}$ are mutually disjoint, $\psi^*[S_i^n] < \varepsilon/4$ for some n. Recall that π_i^1 is the projection mapping from $\mathcal{T}_i$ to the space of first-order beliefs $\mathcal{T}_i^1$. Since $\mathcal{T}_1 \times \mathcal{T}_2$ is a metric space, there is some closed set $E_i \subset S_i \backslash S_i^n$ such that $\psi^*[E_i] > \psi^*[S_i \backslash S_i^n] - \varepsilon/4$. By construction, E_i is a closed proper first-order interval set. Moreover, since $\psi^*[S_i] = 1$ and $\psi^*[S_i^n] < \varepsilon/4$, we have $\psi^*[S_i \backslash S_i^n] > 1 - \varepsilon/4$, and hence, $\psi^*[E_i] > 1 - \varepsilon/2$. Hence, $\psi^*(E_1 \times E_2) \geq 1 - \varepsilon$.■

A.4 Proof of Theorem 5

Before we prove Theorem 5, we first provide a proof for Proposition 2.

A.4.1 Proof of Proposition 2

In proving this proposition, we say a type t_i is *k-minimal for A'_i ($\subseteq A_i$)* if $R_i^k(t_i, G, 0) = A'_i$ and there is no $t'_i \in \mathcal{T}_i$ such that $R_i^{k'}(t'_i, G, 0) \subseteq A'_i$ for some $k' < k$, which also implies $R_i^{k-1}(t'_i, G, 0) \subseteq R_i^{k'}(t'_i, G, 0) \subseteq A'_i$.

Proposition 2 *For any game $G \in G^n$, there exists a finite type t_i^* which is a minimal rationalizable type in G, and $R_i^{2^{n+1}}(t_i^*, G, 0) = R_i(t_i^*, G, 0)$.*

Proof. We divide the proof into two steps.

Step 1 *For any finite type t_i and integer $k > 1$, if t_i is k-minimal for $R_i^k(t_i, G, 0)$, then there exists $t_{-i} \in T_{-i}$ such that $t_i[t_{-i}] > 0$ and t_{-i} is $(k-1)$-minimal for $R_{-i}^{k-1}(t_{-i}, G, 0)$.*

Suppose that every t_{-i} with $t_i[t_{-i}] > 0$ is not $(k-1)$-minimal for $R^{k-1}(t_{-i}, G, 0)$. Then, for every t_{-i} with $t_i[t_{-i}] > 0$, there exists a type $s_{-i}^{t_{-i}}$ such that $R_{-i}^{k-2}\left(s_{-i}^{t_{-i}}, G, 0\right) \subseteq R_{-i}^{k-1}(t_{-i}, G, 0)$. Consider a new type t'_i defined as follows.

$$t'_i\left[\left(\theta, s_{-i}^{t_{-i}}\right)\right] = t_i[(\theta, t_{-i})] \text{ for all } \theta \in \Theta \text{ and } t_{-i} \text{ with } t_i[t_{-i}] > 0.$$

Then, $R_i^{k-1}(t'_i, G, 0) \subseteq R_i^k(t_i, G, 0)$, which contradicts to t_i being k-minimal for $R^k(t_i, G, 0)$. The completes our proof.■

Step 2 *There exists a finite type t_i^* which is a minimal rationalizable type in G, and $R_i^{2^{n+1}}(t_i^*, G, 0) = R_i(t_i^*, G, 0)$.*

Pick a minimal rationalizable type t''_i in G. Note that there exists a finite type s_i such that $R_i(s_i, G, 0) = R_i(t''_i, G, 0)$.[11] Consider the number

$$k^* = \min\left\{k \geq 0 : \begin{array}{c} t_i \text{ is a finite type and} \\ R_i^k(t_i, G, 0) = R_i(t_i, G, 0) = A'_i \end{array}\right\}. \tag{13}$$

Suppose that t_i^* is one finite type achieving the minimum in (13), i.e., $R_i^{k^*}(t_i^*, G, 0) =$

[11] Recall that finite types are dense in the universal type space under product topology, and that product convergence implies upper-hemi strategic convergence. Hence, for some $\varepsilon > 0$, we can find a finite type s_i such that $R_i(s_i, G, \varepsilon) \subseteq R_i(t''_i, G, 0)$, which implies $R_i(s_i, G, 0) \subseteq R_i(s_i, G, \varepsilon) \subseteq R_i(t''_i, G, 0)$. Since t''_i is a minimal rationalizable type, we have $R_i(s_i, G, 0) = R_i(t''_i, G, 0)$.

$R_i(t_i^*, G, 0) = R_i(t_i'', G, 0)$. Note that t_i^* is also a minimal rationalizable type in G, and t_i^* is K-minimal for $R_i^K(t_i^*, G, 0)$.

Let $T_1^* \times T_2^*$ be the smallest belief-closed subset containing t_i^* in the universal type spaces $\mathcal{T}_1 \times \mathcal{T}_2$. We will prove that $k^* \leq 2^{n+1}$, which implies $R_i^{2^{n+1}}(t_i^*, G, 0) = R_i(t_i^*, G, 0)$. Suppose $k^* > 2^{n+1}$. Without loss of generality suppose k^* is even.

Since t_i^* is k^*-minimal for $R_i^{k^*}(t_i^*, G, 0)$, we can apply step 1 $k^* - 1$ times and construct a sequence of types t^{k^*} $(= t_i^*)$, t^{k^*-1},..., t^1 such that for every k, $t^k\left[t^{k-1}\right] > 0$ and

$$t^k \text{ is } k\text{-minimal for } R_i^{k^*-k}\left(t^k, G, 0\right) \text{ (resp. } R_{-i}^{k^*-k}\left(t^k, G, 0\right)\text{) if } k \text{ is even (resp. odd).} \quad (14)$$

Since G is a $n \times n$ game, A_i has $2^n - 1$ distinct nonempty subsets. Since $k^* > 2^{n+1}$, there are at least 2^n types in the finite sequence t^{k^*}, t^{k^*-2},..., t^2. Hence, there exist two even integers k, k' with $k < k'$ such that

$$R_i^{k^*-k}\left(t^{k^*-k}, G, 0\right) = R_i^{k^*-k'}\left(t^{k^*-k'}, G, 0\right).$$

Thus,$t^{k^*-k'}$ cannot be $k^* - k'$-minimal for $R_i^{k^*-k'}\left(t^{k^*-k'}, G, 0\right)$, which is a contradiction to (14). Hence, $R_i^{2^{n+1}}(t_i^*, G, 0) = R_i(t_i^*, G, 0)$.■

A.4.2 Proof of Theorem 5

Theorem 5 For every integer $n \geq 2$, there is a critical type which is $n-$regular.

Proof. We divide the proof into four steps.

Step 1 *A type t_i is $n-$regular if it is continuous in any $G \in G^{n^2}$ at 0.*

To prove this, we need the following lemma from Ely and Peski (2007).

Lemma 5 (Ely and Peski 2007, Lemma 4) *For each game $G = \langle A_i, g_i \rangle \in \mathcal{G}^n$ and each $\varepsilon \geq 0$, there is a game $G' = \langle (A_i' = A_i \times A_{-i}), g_i' \rangle \in \mathcal{G}^{n^2}$, such that for any t_i and $\varepsilon' \geq 0$,*

$$a_i \in R_i(t_i, G, \varepsilon + \varepsilon') \text{ if and only if } (a_i, a_{-i}) \in R_i(t_i, G', \varepsilon') \text{ for any } a_{-i} \in A_{-i}.$$

Suppose that t_i is not $n-$regular, i.e., there exists a game $G = \langle A_i, g_i \rangle$ and a sequence of types $t_{i,m}$ converging to t_i such that $h_i(t_{i,m}, G, a_i)$ does not converge to $h_i(t_i, G, a_i)$ for some $a_i \in A_i$. That is, $a_i \in R_i(t_i, G, \varepsilon)$ and $a_i \notin R_i(t_{i,m}, G, \varepsilon + \gamma)$ for some $\varepsilon \geq 0$, $\gamma > 0$ and sufficiently large m.[12] By Lemma 5, there exists a game $G' \in \mathcal{G}^{n^2}$, such that $a_i' \in R_i(t_i, G', 0)$ and $a_i' \notin R_i(t_{i,m}, G', \gamma)$ for sufficiently large m, i.e., $h_i(t_i, G, a_i') = 0$ and $h_i(t_{i,m}, G, a_i') \geq \gamma > 0$ for sufficiently large m, contradicting to the fact that t_i is continuous in any $G \in \mathcal{G}^{n^2}$ at 0.■

Step 2 *Given a finite game G and a countable type t_i, suppose $r^k(t_i) = \{t_j^*\}$ for some positive integer k and some minimal rationalizable type t_j^* in G. Then, t_i is continuous in G at 0.*

Fix a game G. Following Ely and Peski (2007), we define the following set for any finite game $G = \langle A_i, g_i \rangle_{i \in I}$ and $\varepsilon > 0$.

$$\mathcal{A}_i^{\varepsilon} = \{A_i' \subseteq A_i : R_i(t_i, G, \varepsilon) = A_i' \text{ for some } t_i \in \mathcal{T}_i\}.$$

Note that $\mathcal{A}_i^{\varepsilon}$ is increasing in the following sense. For $\varepsilon'' < \varepsilon'$ and $A_i'' \in \mathcal{A}_i^{\varepsilon''}$, there exists $A_i' \in \mathcal{A}_i^{\varepsilon'}$ such that $A_i'' \subseteq A_i'$. Since A_i is a finite set, A_i has finitely many subsets. Hence, there exists $\varepsilon^* > 0$ such that $\mathcal{A}_i^0 = \mathcal{A}_i^{\varepsilon}$ for all $\varepsilon \in [0, \varepsilon^*]$.

Let A_i^* be a minimal element of $\mathcal{A}_i^0$. Define

$$U^{A_i^*} = \{t_i : R_i(t_i, G, 0) = A_i^*\}.^{13} \tag{15}$$

To prove this step, we need the following lemma from Ely and Peski (2007).

Lemma 6 (Ely and Peski 2007, Lemma 8) *Consider the closed set $E_i = \mathcal{T}_i \backslash U^{A_i^*}$. For any $p < \frac{\varepsilon^*}{6}$, any player j, any $k \geq 0$, any $t_j \notin B_j^p\left([B^p]^k(E_i)\right)$ (where $B_j^p\left([B^p]^0(E_i)\right) \equiv E_i$), any sequence $t_{j,m} \to t_j$ under the product topology, and any action $a_j \in A_j$ such that $a_j \in R_j(t_j, G, 0)$, there is a positive integer m^* such that $a_j \in R_j(t_{j,m}, G, 6p)$ for any $m \geq m^*$.*

We are now ready to prove Step 2. Consider $k > 0$, any sequence $t_j^n \to t_j$ under the product topology, and any action $a_j \in A_j$ such that $a_j \in R_j(t_j, G, 0)$. With t_i^* being a

[12] Product convergence implies upper strategic convergence. Hence, only lower strategic convergence is violated.

minimal rationalizable type in G, we let $A_i^* = R_i(t_i^*, G, 0)$, $U^{A_i^*}$ be defined as in (15), and $E_i = \mathcal{T}_i \backslash U^A$. Clearly, $t_i^* \in U^{A_i^*}$ and $t_i^* \notin E_i$. Since $r^k(t_j) = \{t_i^*\}$, $t_j \notin B_j^p\left([B^p]^{k-1}(E_i)\right)$, for any $p > 0$. Hence, by Lemma 6, for any $p \in \left(0, \frac{\varepsilon^*}{6}\right)$, there is a positive integer m^* such that $a_j \in R_j(t_{j,m}, G, 6p)$ for any $m \geq m^*$, i.e., $h_j(t_{j,m}, G, a_j) \leq 6p$. Therefore, $h_j(t_{j,m}, G, a_j) \to h_j(t_j, G, a_j) = 0$, i.e. t_i is continuous in G at 0.■

Step 3 *There exists an $n-$regular type.*

Take a countable dense set $\overline{\mathcal{G}} = \left\{G^l\right\}_{l=1}^{\infty}$ in $\mathcal{G}^{n^2}$ under the supmetric d_g where

$$d_g(G, G') = \sup_{j \in \{i,-i\},\ (\theta, a_i, a_{-i}) \in \Theta \times A_i \times A_{-i}} \left|g_j(\theta, a_i, a_{-i}) - g'_j(\theta, a_i, a_{-i})\right|$$
$$\text{for } G = \langle A_j, g_j \rangle, G' = \left\langle A_j, g'_j \right\rangle \in \mathcal{G}^{n^2}.$$

Let $k^* = 2^{n+1}$. By Proposition 2, we can find a countable types $\{t_{i,l}\}_{l=1}^{\infty}$ such that $t_{i,l}$ is a finite minimal rationalizable type in G^l, and $R_i^{k^*}(t_{i,l}, G, 0) = R_i(t_{i,l}, G, 0)$.

Now define

$$t_i^* \equiv t_{i,1} \leftrightharpoons^{k^*} t^{\theta_0} \leftrightharpoons^1 t_{i,2} \leftrightharpoons^{k^*} t^{\theta_0} \leftrightharpoons^1 \cdots \leftrightharpoons^1 t_{i,l} \leftrightharpoons^{k^*} t^{\theta_0} \leftrightharpoons^1 \cdots. \qquad (\blacklozenge)$$

To simplify our notation, for any $l \geq 1$, let

$$t_i[l] \equiv t_{i,l} \leftrightharpoons^{k^*} t^{\theta_0} \leftrightharpoons^1 t_{i,l+1} \leftrightharpoons^{k^*} t^{\theta_0} \leftrightharpoons^1 \cdots;$$
$$t_i[l, 0] \equiv t^{\theta_0} \leftrightharpoons^1 t_{i,l+1} \leftrightharpoons^{k^*} t^{\theta_0} \leftrightharpoons^1 \cdots.$$

We then show t_i^* is $n-$regular. First, the type $t_i[l]$ is a minimal rationalizable type in G^l. By Lemma 1, the beliefs of $t_i[l]$ agree with those of type $t_{i,l}$ up to order k^*. Hence, $R_i^{k^*}(t_i[l], G, 0) = R_i^{k^*}(t_{i,l}, G, 0)$. Since $R_i^{k^*}\left(t_{i,l}, G^l, 0\right) = R_i\left(t_{i,l}, G^l, 0\right)$, we have $R_i\left(t_i[l], G^l, 0\right) \subseteq R_i\left(t_{i,l}, G^l, 0\right)$. Moreover, since $t_{i,l}$ is a minimal rationalizable type in G^l, we have $R_i\left(t_i[l], G^l, 0\right) = R_i\left(t_{i,l}, G^l, 0\right)$, i.e., the type $t_i[l]$ is also a minimal rationalizable type in G^l.

Second, t_i^* is continuous in any $G^l \in \overline{\mathcal{G}}$ at 0. By Lemma 1, $r^{(l-1)(k^*+1)}(t_i^*) = \{t_i[l]\}$. Moreover, since the type $t_i[l]$ is a minimal rationalizable type in G^l, by step 2, t_i^* is continuous in any $G^l \in \overline{\mathcal{G}}$ at 0.

Third, t_i^* is continuous in any $G \in \mathcal{G}^{n^2}$ at 0. For any $G \in \mathcal{G}^{n^2}$, $a_i \in R_i\left(t_i^*, G^l, 0\right)$ and $t_{i,m} \to t_i^*$ in product topology, we have

$$\begin{aligned}
& \lim_{m\to\infty} \left|h_i\left(t_{i,m}, G, a_i\right) - h_i\left(t_i^*, G, a_i\right)\right| \\
\leq\ & \lim_{m\to\infty} \left|h_i\left(t_{i,m}, G^l, a_i\right) - h_i\left(t_i^*, G^l, a_i\right)\right| \\
& + \lim_{m\to\infty} \left|h_i\left(t_{i,m}, G, a_i\right) - h_i\left(t_{i,m}, G^l, a_i\right)\right| + \left|h_i\left(t_i^*, G, a_i\right) - h_i\left(t_i^*, G^l, a_i\right)\right| \\
\leq\ & \lim_{m\to\infty} \left|h_i\left(t_{i,m}, G^l, a_i\right) - h_i\left(t_i^*, G^l, a_i\right)\right| + 4d_g\left(G, G^l\right) \\
=\ & 4d_g\left(G, G^l\right).
\end{aligned}$$

where the last equality follows because t_i^* is continuous in G^l at 0, the first inequality follows from triangle inequality, the second inequality follows from the following inequality which can be easily check.

$$\left|h_i\left(t_i, G, a_i\right) - h_i\left(t_i, G^l, a_i\right)\right| \leq 2d_g\left(G, G^l\right) \text{ for any } t_i.$$

Note that $4d_g\left(G, G^l\right)$ can be arbitrarily close to 0, because $\left\{G^l\right\}_{l=1}^{\infty}$ is dense in $\mathcal{G}^{n^2}$ under the metric d_g. Hence, we have $\lim_{m\to\infty} \left|h_i\left(t_{i,m}, G, a_i\right) - h_i\left(t_i^*, G, a_i\right)\right| = 0$, i.e., t_i^* is continuous in any $G \in \mathcal{G}^{n^2}$ at 0. Therefore, by step 1, t_i^* is $n-$regular.■

Step 4 *The $n-$regular type t_i^* is critical.*

Let $\left(T_j^*, \pi_j^*\right)$ be the type space defined in Appendix A.1 which contains t_i^*. Define $E_i^1 = B_i^1\left(\left\{t_{-i} \in \mathcal{T}_{-i} : t_{-i}\left[\theta_0\right] = 1\right\}\right)$ and

$$E_i^m = \left(B_i^1 B_{-i}^1\right)^{m-1} E_i^1, \forall m \geq 2.$$

Clearly, E_j^m is a closed set under the product topology. Hence, $E_i \equiv \cup_{m=1}^{\frac{k^*+1}{2}} E_i^m$ is closed. Since t^{θ_0} assigns probability 1 to θ_0, and $r^{k^*+(k^*+1)l}\left(t_i^*\right) = \left\{t_i\left[l, 0\right]\right\}$ for any integer $l \geq 1$, $T_i^* \subset E_i$. Therefore, $t_i^* \in C_i^1\left(E_i\right)$. The last step is to show that E_i is a proper subset of the universal type space. This is because for any $\theta \neq \theta_0$, $t_i^\theta \notin E_i$ since $t_i \notin E_i^m$ for each m. Therefore, t_i^* is critical by (Ely and Peski, 2007, Theorem 3).■

A.5 Proof of Proposition 4

Proposition 4 For every type $t_i \in T_i$, there is a product convergent sequence $t_{i,m} \to t_i$ such that $t_{i,m}$ does not converge to t_i in the uniform strategic topology.

The proof of Proposition 4 requires the following two lemmas.

Lemma 7 *Let θ_0 and θ_1 be two distinct parameters in Θ. For every $n \geq 1$ and $i \in I$, there exists a finite game $G = (A_j, g_j)_{j \in I}$ with some a_i^* and $(\overline{a}_j)_{j \in I}$ such that*

1. *$g_j(\overline{a}_j, a_{-j}, \theta) = 1$ for any θ and a_{-j};*
2. *$a_i^* \in R_i(t_i, G, 0)$ and $a_i^* \notin R_i(s_i, G, 1/2)$ for any type t_i, s_i such that $r^n(t_i) = \{t^{\theta_0}\}$ and $r^n(t_i) = \{t^{\theta_0}\}$.*

To prove this lemma, we need the following result.

Lemma 8 (agumented games) *Let $G' = \left(A'_j, g'_j\right)_{j \in I}$ be a game with payoffs bounded by 1 and an action profile $(\overline{a}_j)_{j \in I}$ such that $g'_j(\overline{a}_j, a_{-j}, \theta) = 1$ for any (a_{-j}, θ). Let $G = (A_j, g_j)_{j \in I}$ such that $A_i = \{\overline{b}_i, b_i^*\} \times A'_i$ and $A_{-i} = A'_{-i}$ and the payoff function is defined for any $((b, c_i), a_{-i}, \theta) \in A_i \times A_{-i} \times \Theta$ as*

$$g_j((b, c_i), a_{-i}, \theta) = \min\{g''_j(b, a_{-i}), g'_j(c_i, a_{-i}, \theta)\}$$

where $g''_j : \{\overline{b}_i, b_i^*\} \times A_{-i}$ is defined using the matrix

	$a_{-i} = a_{-i}^*$	$a_{-i} \neq a_{-i}$
b_i^*	$1, 1$	$0, 1$
$\overline{b}_i$	$1, 1$	$1, 1$

where a_{-i}^* is some action in A_{-i}. Then, $R_{-i}(t_{-i}, G, \gamma) = R_{-i}(t_{-i}, G', \gamma)$ for any $\gamma \geq 0$ any type t_{-i}.

Proof First, we prove that $R_{-i}(t_{-i}, G', \gamma) \subseteq R_{-i}(t_{-i}, G, \gamma)$. Define $\overline{R}_{-i}(t_{-i}) = R_{-i}(t_{-i}, G', \gamma)$ and $\overline{R}_i(t_i) = \{\overline{b}_i\} \times R_i(t_i, G', \gamma)$ for all $t_i \in \mathcal{T}_i$ and $t_{-i} \in \mathcal{T}_{-i}$ Then, $\overline{R}(\cdot)$ has the $\gamma-$ best reply property in G. Hence, $R_{-i}(t_{-i}, G', \gamma) \subseteq R_{-i}(t_{-i}, G, \gamma)$.

Second, we prove that $R_{-i}(t_{-i}, G', \gamma) \supseteq R_{-i}(t_{-i}, G, \gamma)$. Define $\widehat{R}_{-i}(t_{-i}) = R_{-i}(t_{-i}, G, \gamma)$ and $\widehat{R}_i(t_i) = \{c_i \in A_i : (\overline{b}_i, c_i) \in R_i(t_i, G, \gamma)\}$. Then, $\overline{R}(\cdot)$ has the $\gamma-$ best reply property in G'. Hence, $R_{-i}(t_{-i}, G, \gamma) \subseteq R_{-i}(t_{-i}, G', \gamma)$. ■

We now prove Lemma 7.

Proof of Lemma 7 We prove this claim by induction on n. Consider first $n = 0$. Then, $t_i = t^{\theta_0}$ and $s_i = t^{\theta_1}$. Define $G = (A_j, g_j)_{j \in I}$ by the following payoff matrix

	$\overline{a}_{-i}$	
a_i^*	$1, 1$	$0, 1$
$\overline{a}_i$	$1, 1$	$1, 1$
	$\theta = \theta_0$	$\theta \neq \theta_0$

We now verify that G satisfies the claim of the lemma. Observe that by choosing a_i^*, player i with type t^{θ_0} gets the payoff $t^{\theta_0}[\theta_0] = 1$ and player i with type t^{θ_1} gets 0. Since player i always gets the payoff 1 by choosing $\overline{a}_i$ and player $-i$ has only one action, we conclude that $a_i^* \in R_i(t^{\theta_0}, G, 0)$ and $a_i^* \notin R_i(t^{\theta_1}, G, 1/2)$.

Assume our claim holds for some non-negative integer n and we now proceed to prove the case for $n+1$. By the induction hypothesis there is a game $G' = (A'_j, g'_j)_{j \in I}$ with actions a_{-i}^* and $(\overline{a}_j)_{j \in I}$ such that properties (1) and (2) hold. We now define $G = (A_j, g_j)_{j \in I}$ as follows to prove the claim for $n+1$. Let $A_i = \{b_i^*, \overline{b}_i\} \times A'_i$ and $A_{-i} = A'_{-i}$ and define the payoff function for any $j \in I$ and any $((b, c_i), a_{-i}, \theta) \in A_i \times A_{-i} \times \Theta$ as

$$g_j((b, c_i), a_{-i}, \theta) = \min\{g''_j(b, a_{-i}), g'_j(c_i, a_{-i}, \theta)\}$$

where $g''_j : \{\overline{b}_i, b_i^*\} \times A_{-i}$ is defined using the following matrix

	$a_{-i} = a_{-i}^*$	$a_{-i} \neq a_{-i}$
b_i^*	$1, 1$	$0, 1$
$\overline{b}_i$	$1, 1$	$1, 1$

We now prove our claim for the case $n+1$. Since $r^n(t_i) = \{t^{\theta_0}\}$, there is a set $T_{-i}^{t_i}$ such that $t_i[T_{-i}^{t_i}] = 1$ and $r^{n-1}(t_{-i}) = \{t^{\theta_0}\}$ for every $t_{-i} \in T_{-i}^{t_i}$. Similarly, since $r^n(s_{-i}) = \{t^{\theta_1}\}$, there is a set $T_i^{s_{-i}}$ such that $s_{-i}[T_i^{s_{-i}}] = 1$ and $r^{n-1}(s_i) = \{t^{\theta_1}\}$ for every $t_{-i} \in T_{-i}^{s_{-i}}$. Hence, $a_{-i}^* \in R_i(t_{-i}, G', 0)$ for every $t_{-i} \in T_{-i}^{t_i}$ and $a_{-i}^* \notin R_{-i}(t_{-i}, G', 1/2)$ for every $t_{-i} \in T_{-i}^{s_{-i}}$. Moreover, by Lemma 8, $a_{-i}^* \in R_i(t_{-i}, G, 0)$ for every $t_{-i} \in T_{-i}^{t_i}$ and $a_{-i}^* \notin R_{-i}(t_{-i}, G, 1/2)$ for every $t_{-i} \in T_{-i}^{s_i}$

We verify that G satisfies properties (1) and (2).

G satisfies property (1): Let $\overline{a}_i = \left(\overline{b}_i, \overline{c}_i\right)$ and $\overline{a}_{-i} = \overline{c}_{-i}$.

G satisfies property (2): Let $a_i^* = (b_i^*, \overline{c}_i)$. First, since $a_{-i}^* \in R_i\left(t_{-i}, G, 0\right)$ for every $t_{-i} \in T_{-i}^{t_i}$ and $t_i\left[T_{-i}^{t_i}\right] = 1$, $\sigma_{-i}\left(\theta, t_{-i}\right) = \delta_{a_{-i}^*}$ for any θ and any type $t_{-i} \in T_{-i}^{t_i}$ defines a valid conjecture for t_i, and moreover, a_i^* is a $0-$best reply to σ_{-i} for t_i. Therefore, $a_i^* \in R_i\left(t_i, G, 0\right)$.

Second, since $a_{-i}^* \notin R_{-i}\left(t_{-i}, G, 1/2\right)$ for every $t_{-i} \in T_{-i}^{s_{-i}}$ and $s_i\left[T_{-i}^{s_{-i}}\right] = 1$, a_i^* is not an $1/2-$best reply to any conjecture valid for s_i. Therefore, $a_i^* \notin R_i\left(s_i, G, 1/2\right)$. ■

Proof of Proposition 4 Both $t_i \leftrightharpoons^n t^{\theta_0}$ and $t_i \leftrightharpoons^n t^{\theta_1}$ converge to t_i in product topology. By Lemma 7 and the triangle inequality,

$$1/2 \leq d^{us}\left(t_i \leftrightharpoons^n t^{\theta_0}, t_i \leftrightharpoons^n t^{\theta_1}\right) \leq d^{us}\left(t_i, t_i \leftrightharpoons^n t^{\theta_0}\right) + d^{us}\left(t_i, t_i \leftrightharpoons^n t^{\theta_1}\right), \forall n$$

Hence, there is either a subsequence of $\left\{t_i \leftrightharpoons^n t^{\theta_0}\right\}_n$ or a subsequence of $\left\{t_i \leftrightharpoons^n t^{\theta_1}\right\}_n$ such that our claim holds. ■

A.6 Proofs of Claim 3 and 4

Claim 3 $R_i\left(t_i, G, \gamma\right) \supseteq R_i\left(t_i, G^m, \gamma\right)$ *for any* m, *any* $\gamma \geq 0$, *any player* i, *and any* $t_i \in \mathcal{T}_i$.

Proof. Observe that for each m, $\left(\overline{R}_i\right)_{i \in I}$ with $\overline{R}_i\left(t_i\right) = R_i\left(t_i, G^m, \gamma\right)$ for every $t_i \in \mathcal{T}_i$ has the $\gamma-$best reply property. Hence, the claim in step 1 follows.■

Claim 4 *If* $a_i \in R_i\left(t_i, G, \gamma\right) \cap A_i^m$ *and* $a_i \neq a^0$, *then* $a_i \in R_i\left(t_i, G^m, \delta\right)$ *for* $\delta = 2\left(\frac{\gamma+2\alpha}{1+2\alpha}\right) + \frac{\gamma}{\alpha}$, *for any player* i, *any* $t_i \in \mathcal{T}_i$, *and any* $1 > \gamma \geq 0$.

Proof. We prove this claim by showing that for each m, $\left(\overline{R}_i^m\right)_{i \in I}$ with

$$\overline{R}_i^m\left(t_i\right) \equiv R_i\left(t_i, G, \gamma\right) \cap A_i^m$$

satisfies $\delta-$best reply property in G^m. By Claim 3, $\overline{R}_i^m\left(t_i\right)$ is nonempty. Suppose that $a_i \in \overline{R}_i^m\left(t_i\right)$. Then, there exists a valid conjecture σ_{-i} in G such that a_i is a $\gamma-$ best reply for t_i under σ_{-i}. Let

$$E^{\sigma_{-i}} = \left\{\left(\theta, t_{-i}\right) : \sigma_{-i}\left(\theta, t_{-i}\right)\left[A_{-i}^m\right] > 0\right\}.$$

Let $\beta_m = \int E^{\sigma_{-i}} dt_i$. Then, the expected payoff of choosing a_i for t_i under σ_{-i} is at most

$2\alpha\beta_m + (1-\beta_m)(-1)$. Since the expected payoff of choosing a^0 is always 0, we then have $2\alpha\beta_m + (1-\beta_m)(-1) + \gamma \geq 0$, or equivalently $1-\beta_m \leq \frac{\gamma+2\alpha}{1+2\alpha}$.

We now define a new conjecture σ'_{-i} for type t_i: for every $(\theta, t_{-i}) \in \Theta \times \mathcal{T}_{-i}$, let $a_{-i}(\theta, t_{-i})$ be an arbitrary action in $R_{-i}(t_{-i}, G^m, \gamma)$. Note that $R_{-i}(t_{-i}, G^m, \gamma) \neq \varnothing$ since G^m is a finite game, and moreover, $a_{-i}(\theta, t_{-i}) \in R_{-i}(t_{-i}, G, \gamma)$ by Claim 3. Let

$$\sigma'_{-i}(\theta, t_{-i}) \equiv \begin{cases} \frac{\sigma_{-i}(\theta,t_{-i})}{\sigma_{-i}(\theta,t_{-i})\left[A^m_{-i}\right]}, & \text{if } \sigma_{-i}(\theta, t_{-i})\left[A^m_{-i}\right] > 0; \\ \delta_{\{a_{-i}(\theta,t_{-i})\}} & \text{if } \sigma_{-i}(\theta, t_{-i})\left[A^m_{-i}\right] = 0. \end{cases}$$

Then, $\sigma'_{-i}(\theta, t_{-i})[a_{-i}] > 0$ only for $a_{-i} \in \overline{R}^m_{-i}(t_{-i})$. Moreover, for any $a'_i \in A^m_i$,

$$\begin{aligned}
& \int_{\Theta\times\mathcal{T}_{-i}} \sum_{a_{-i}\in A^m_{-i}} \left[g^m_i(a_i, a_{-i}, \theta) - g^m_i(a'_i, a_{-i}, \theta)\right] \sigma'_{-i}(\theta, t_{-i})[a_{-i}]\, dt_i[(\theta, t_{-i})] \\
= & \int_{E^{\sigma_{-i}}} \sum_{a_{-i}\in A^m_{-i}} \left[g^m_i(a_i, a_{-i}, \theta) - g^m_i(a'_i, a_{-i}, \theta)\right] \frac{\sigma_{-i}(\theta, t_{-i})[a_{-i}]}{\sigma_{-i}(\theta, t_{-i})\left[A^m_{-i}\right]} dt_i[(\theta, t_{-i})] \\
& + \int_{[\Theta\times\mathcal{T}_{-i}]\setminus E^{\sigma_{-i}}} \left[g^m_i(a_i, a_{-i}(\theta, t_{-i}), \theta) - g^m_i(a'_i, a_{-i}(\theta, t_{-i}), \theta)\right] dt_i[(\theta, t_{-i})] \\
\geq & \frac{1}{\alpha} \int_{E^{\sigma_{-i}}} \sum_{a_{-i}\in A^m_{-i}} \left[g_i(a_i, a_{-i}, \theta) - g_i(a'_i, a_{-i}, \theta)\right] \sigma_{-i}(\theta, t_{-i})[a_{-i}]\, dt_i[(\theta, t_{-i})] - 2(1-\beta_m) \\
\geq & -\frac{\gamma}{\alpha} - 2(1-\beta_m) \geq -\delta.
\end{aligned}$$

where the last inequality follows because $1-\beta_m \leq \frac{\gamma+2\alpha}{1+2\alpha}$ and $\delta = \frac{\gamma}{\alpha} + 2\left(\frac{\gamma+2\alpha}{1+2\alpha}\right)$. This proves Claim 4.■

A.7 ICR and IIR

A.7.1 A finite type which is $2-$regular

Let $\Theta = \{0,1\}$ and consider the following type space: $T_1 = \{t_1, t'_1\}$, $T_2 = \{t_2, t'_2\}$, $\pi_1(t_1) = \delta_{(0,t_2)}$, $\pi_2(t_2) = \delta_{\left(1,t'_1\right)}$, $\pi_1(t'_1) = \delta_{\left(1,t'_2\right)}$, and $\pi_2(t'_2) = \delta_{(0,t_1)}$. We show that t_1 is a $2-$regular type. Let $\{t_{1,m}\}$ be a sequence of types such that $t_{1,m} \to t_1$ under product topology. Let $G = \langle A_i, g_i \rangle_{i\in I}$ be a 2×2 game and $\gamma \geq 0$. Let $A_i = \{a_i, b_i\}$. We claim that for every $a_1 \in R_1(t_i, G, \gamma)$ and $\varepsilon > 0$, $a_1 \in R_1(t_{1,m}, G, \gamma+\varepsilon)$ for sufficiently large m.

For each θ, let G_θ denote the complete information game with common knowledge of θ. First, suppose that in G_θ for each θ, every action of each player is a $\gamma-$best reply. Then, the claim holds because $R_i^1(t_i, G, \gamma) = A_i$ for all t_i and i. Hence, $R_i^k(t_i, G, \gamma) = A_i$ for every $k \geq 1$. Second, suppose that for some θ, some action a_i in G_θ is not a $\gamma-$best reply. Assume that for $\theta = 1$, b_2 is not a $\gamma-$best reply for player 2 and other cases are similar. Then, $R_2(t_2, G, \gamma) = \{a_2\}$. Since $R_2(\cdot, G, \gamma)$ is upper hemicontinuous, there is an product open neighborhood U of t_2 such that $R_2(s_2, G, \gamma) = \{a_2\}$ for all $s_2 \in U$. Since $t_{1,m} \to t_1$ in product topology, $t_{1,m}$ must assign large probabilities on U for sufficient large m. Hence, for any action $a_1 \in R_1(t_1, G, \gamma)$, we have $a_1 \in R_1(t_{1,m}, G, \gamma + \varepsilon)$ for sufficient large m. Thus, t_1 is a $2-$regular type.

A.7.2 Finite types are 2-critical under IIR

We first define IIR. Fix a type space $(T_i, \pi_i)_{i\in I}$. Let $\widetilde{R}_i^0(t_i, G, \varepsilon) = A_i$. For any integer $k \geq 1$, $a_i \in \widetilde{R}_i^k(t_i, G, \varepsilon)$ iff there exists a measurable function $\sigma_{-i} : T_{-i} \to \Delta(A_{-i})$ such that

$$\text{supp}\sigma_{-i}(t_{-i}) \subseteq R_{-i}^{k-1}(t_i, G, \varepsilon) \text{ for } t_i - \text{almost surely } (\theta, t_{-i});$$
$$\int_{\Theta\times\mathcal{T}_{-i}} \sum_{a_{-i}\in A_{-i}} [g_i(a_i, a_{-i}, \theta) - g_i(a_i', a_{-i}, \theta)]\,\sigma_{-i}(t_{-i})[a_{-i}]\,dt_i[(\theta, t_{-i})] \geq -\varepsilon, \forall a_i' \in A_i.$$

Then $\widetilde{R}_i(t_i, G, \varepsilon) = \bigcap_{k=1}^{\infty} \widetilde{R}_i^k(t_i, G, \varepsilon)$ the ε-IIR set for t_i in G. The strategic topology with IIR is defined similarly as the case with ICR.

We now show that any finite type must display strategic discontinuity in some 2×2 games. Consider the following game G. Let $\Theta = \{0, 1\}$.

	a_2	b_2	a_2	b_2
a_1	$0,0$	$1,1$	$0,0$	$1,-x_2$
b_1	$\lambda,0$	$0,-x_1$	$\lambda,0$	$0,1$
	$\theta=0$		$\theta=1$	

where x_1, x_2, $\lambda > 0$ are three numbers which will be determined later. Consider an arbitrary finite type t. Since t is finite, $|T_1 \times T_2| < \infty$ where $T_1 \times T_2$ is the smallest belief-closed subset in the universal type spaces $\mathcal{T}_1 \times \mathcal{T}_2$ containing t.

Step 1 *The calibration of λ.*

For any $t_1 \in T_1$, define $\Lambda(t_1)$ as the set of positive numbers such that $\alpha \in \Lambda(t_1)$ iff $t_1[E] \cdot \alpha = 1 - t_1[E]$ for some $E \subset T_2$ and $t_1[E] \notin \{0, 1\}$. Since T_1 is finite, $\cup_{t_1 \in T_1} \Lambda(t_1)$ is also a finite set. Pick any positive $\lambda \notin \cup_{t_1 \in T_1} \Lambda(t_1)$. Hence, for any $t_1 \in T_1$, $E \subset T_2$, we have[14]

$$t_1[E] \cdot \lambda \neq 1 - t_1[E], \forall t_1 \in T_1, E \subset T_2. \tag{16}$$

Since $|T_1 \times T_2| < \infty$, there is a $\gamma_1 > 0$, such that

$$\min_{t_1 \in T_1, E \subset T_2} |t_1[E] \cdot \lambda - (1 - t_1[E])| > \gamma_1. \tag{17}$$

Step 2 *The calibration of x_1 and x_2.*

Let $p = \Pr(\theta = 1)$. Observe that b_2 is dominated by a_2 for player 2 iff $(1-p) - px_2 < 0$ (when player 1 chooses a_1), i.e., $p > \frac{1}{1+x_2}$ and $-(1-p)x_1 + p < 0$ (when player 1 chooses b_1), i.e. $p < \frac{x_1}{x_1+1}$. Observe that (b_1, a_2) and (a_1, b_2) are the two pure strategy NEs at $\theta = 0$ and there is no pure strategy NE at $\theta = 1$.

Since t is finite, there exists some closed interval $[x, z] \subset [0, 1]$ such that $t_i[\theta = 1] \notin [x, z]$ for every $t_i \in T_i$ and $i = 1, 2$. Choose α_1 and α_2 such that $\frac{1}{1+\alpha_2} = x$ and $\frac{\alpha_1}{\alpha_1+1} = z$. We are now ready to define x_1 and x_2. For any $t_2 \in T_2$, we first define

$$\Phi(t_2) = \left\{ \begin{array}{c} t_2[\{\theta = 0\} \times F] - \alpha_2 \cdot t_2[\{\theta = 1\} \times F] - \\ \alpha_1 \cdot t_2[\{\theta = 0\} \times (T_1 \backslash F)] + t_2[\{\theta = 1\} \times (T_1 \backslash F)] : F \subset T_1 \end{array} \right\}$$

Since T_2 is a finite set, $\cup_{t_2 \in T_2} \Phi(t_2)$ is also a finite set.

Consider $\zeta = \min\{|h| : h \in \cup_{t_2 \in T_2} \Phi(t_2) \text{ and } h \neq 0\}$. We can choose a sufficiently small $\varepsilon > 0$, such that $\varepsilon < \zeta$ and

$$x = \frac{1}{1+\alpha_2} < \frac{1}{1+\alpha_2-\varepsilon} < \frac{\alpha_1-\varepsilon}{\alpha_1-\varepsilon+1} < \frac{\alpha_1}{\alpha_1+1} = z.$$

Then, let $x_1 = \alpha_1 - \varepsilon$ and $x_2 = \alpha_2 - \varepsilon$. Hence,

$$x = \frac{1}{1+\alpha_2} < \frac{1}{1+x_2} < \frac{x_1}{x_1+1} < \frac{\alpha_1}{\alpha_1+1} = z.$$

[14] If $t_1[E] \in \{0, 1\}$, we always have $t_1[E] \cdot \lambda \neq 1 - t_1[E]$.

Since $\zeta > \varepsilon > 0$, by the definitions of ζ, x_1, and x_2, for any $t_2 \in T_2$ and $F \subset T_1$,

$$\left| \begin{array}{c} t_2 [\{\theta = 0\} \times F] - x_2 \cdot t_2 [\{\theta = 1\} \times F] \\ -x_1 \cdot t_2 [\{\theta = 0\} \times (T_1 \backslash F)] + t_2 [\{\theta = 1\} \times (T_1 \backslash F)] \end{array} \right| \neq 0.$$

Since $|T_1 \times T_2| < \infty$, there is a $\gamma_2 > 0$ such that

$$\min_{t_2 \in T_2,\, E \subset T_1} \left| \begin{array}{c} t_2 [\{\theta = 0\} \times F] - x_2 \cdot t_2 [\{\theta = 1\} \times F] \\ -x_1 \cdot t_2 [\{\theta = 0\} \times (T_1 \backslash F)] + t_2 [\{\theta = 1\} \times (T_1 \backslash F)] \end{array} \right| > \gamma_2. \tag{18}$$

Step 3 *The construction of a product convergent sequence $t_{1,m}$.*

Let μ be a first-order belief such that $\mu(\theta = 1) = \beta \in (x, z)$. Note that b_2 is strictly dominated by a_2 for a type t^μ player 2. Define $t_{1,m} \equiv t \leftrightharpoons^m t^\mu$ for any positive integer m. By Lemma 4, $t_{1,m} \to t$ in product topology.

Step 4 $\widetilde{R}_1(t, G, 0) = \{a_1, b_1\}$.

Let $\overline{R}_1(t_1) = \{a_1, b_1\}$ for all $t_1 \in T_1$ and $\overline{R}_2(t_2) = \{a_2, b_2\}$ for all $t_2 \in T_2$. We now verify that $\overline{R}$ has the $0-$best reply property. Clearly, a_1 is $0-$best reply to b_2 and b_1 is a best-reply to a_2 regardless of their belief on θ. Moreover, since $t_2[\theta = 1] \notin [x, z]$, we have two cases, i) $t_2[\theta = 1] < x$; ii) $t_2[\theta = 1] > z$. In case i), b_2 is a best reply to a_1 and a_2 is a best reply to b_1. In case ii), b_2 is a best reply to b_1 and a_2 is a best reply to a_1.

Step 5 *For every m and $\gamma = \min\left\{\gamma_1, \gamma_2, \left|\frac{(1-\beta)-\beta x_2}{2}\right|, \left|\frac{-(1-\beta)x_1+\beta}{2}\right|\right\} > 0$, there is a unique $\gamma-$IIR action for type $t_{1,m}$ in G.*

Recall that $t_{1,m} = t \leftrightharpoons^m t^\mu$. Clearly, t^μ has a unique $\gamma-$rationalizable action, a_2. He gets payoff 0 by choosing a_2. However, suppose he chooses b_2. Then, he gets $(1-\beta) - \beta x_2 < -\gamma < 0$ if his opponent choose a_1, and he gets $-(1-\beta)x_1 + \beta < -\gamma < 0$ if his opponent choose b_1.

We will show all types in $r^k(t_{1,m})$, with $k < (2m+1)$, have a unique γ-rationalizable action. In particular $t_{1,m}$ has a unique $\gamma-$rationalizable action. Hence, either a_1 or b_1 is not $\gamma-$rationalizable for $t_{1,m}$, i.e., $t_{1,m}$ does not converge to t strategically.

Consider any $t_1 \in T_1 \cap r^k(t_{1,m})$, with $k < (2m+1)$ being even. By induction hypothesis, all types on the support of t_1 have a unique $\gamma-$rationalizable action. Let E be the set of types on the support of t_1, whose unique $\gamma-$rationalizable action is a_2. Then, by choosing a_1, type t_1 gets the payoff, $1 - t_1[E]$; by choosing b_1, type t_1 gets the payoff, $t_1[E] \cdot \lambda$. By (16) and (17), $\lambda \cdot t_1[E] \neq 1 - t_1[E]$ and $|\lambda \cdot t_1[E] - (1 - t_1[E])| > \gamma_1 \geq \gamma$. Hence, either a_1 or b_1 is the unique $\gamma-$rationalizable action for type t_1.

Consider any $t_2 \in T_2 \cap r^k(t_{1,m})$, with $k < (2m+1)$ being odd. By induction hypothesis, all types on the support of t_2 have a unique $\gamma-$rationalizable action. Let F be the set of types on the support of t_2, whose unique $\gamma-$rationalizable action is a_1. Then, by choosing a_2, type t_2 get the payoff 0; by choosing b_2, type t_2 get the payoff, $t_2[\{\theta=0\} \times F] - x_2 \cdot t_2[\{\theta=1\} \times F] - x_1 \cdot t_2[\{\theta=0\} \times (T_1 \backslash F)] + t_2[\{\theta=1\} \times (T_1 \backslash F)]$. By (18),

$$\left| \begin{array}{c} t_2[\{\theta=0\} \times F] - x_2 \cdot t_2[\{\theta=1\} \times F] \\ -x_1 \cdot t_2[\{\theta=0\} \times (T_1 \backslash F)] + t_2[\{\theta=1\} \times (T_1 \backslash F)] \end{array} \right| > \gamma_2 \geq \gamma.$$

Hence, either a_2 or b_2 is the unique $\gamma-$rationalizable action for type t_2.

References

Bergemann, D., and S. Morris (2009): "Rationalizable Implementation," mimeo.

Carlsson, H., and E. V. Damme (1993): "Global Games and Equilibrium Selection," *Econometrica*, 61, 989–1018.

Chen, Y.-C., A. Di Tillio, E. Faingold, and S. Xiong (2008): "Genericity of Critical Types," mimeo.

——— (2009): "Uniform Topologies on Types," mimeo.

Dekel, E., D. Fudenberg, and S. Morris (2006): "Topologies on Types," *Theoretical Economics*, 1, 275–309.

——— (2007): "Interim Correlated Rationalizability," *Theoretical Economics*, 2, 15–40.

Ely, J., and M. Peski (2006): "Hierarchies of Belief and Interim Rationalizability," *Theoretical Economics*, 1, 19–65.

——— (2007): "Critical Types," mimeo.

Harsanyi, J. (1967/1968): "Games with incomplete information played by "Bayesian" players, I-III," *Management Science*, 14, 159–182, 320–334, 486–502.

Kajii, A., and S. Morris (1997): "The Robustness of Equilibria to Incomplete Information," *Econometrica*, 65, 1283–1309.

——— (1998): "Payoff Continuity in Incomplete Information Games," *Journal of Economic Theory*, 82, 267–276.

Lipman, B. (1994): "A Note on the Implications of Common Knowledge of Rationality," *Games and Economic Behavior*, 6, 114–129.

Mertens, J.-F., and S. Zamir (1985): "Formulation of Bayesian Analysis for Games with Incomplete Information," *International Journal of Game Theory*, 14, 1–29.

Monderer, D., and D. Samet (1989): "Approximating Common Knowledge with Common Beliefs," *Games and Economic Behavior*, 1, 170–190.

——— (1996): "Proximity of Information in Games with Common Beliefs," *Mathematics of Operations Research*, 21, 707–725.

Rubinstein, A. (1989): "The Electronic Mail Game: the Strategic Behavior under Almost Common Knowledge," *American Economic Review*, 79, 385–391.

Weinstein, J., and M. Yildiz (2007): "A Structure Theorem for Rationalizability with Application to Robust Predictions of Refinements," *Econometrica*, 75, 365–400.