A Multi-level Factor Model: 
Asymptotic Theory and an Application

In Choi, Dukpa Kim, Yun Jung Kim, and Noh-Sun Kwark
Sogang Univ., Korea Univ., Sogang Univ., and Sogang Univ.
(incomplete)

Abstract

This paper considers a model with multi-level factors, where the global factors affect all individuals while the regional factors affect only those in one specific region. An innovative strategy to separately identify the global and regional factors is proposed. The asymptotic properties of the proposed estimation method are analyzed. Information criteria, which can consistently estimate the number of global factors as well as the numbers of regional factors, are also proposed. Some Monte Carlo simulation results are reported to assess the adequateness of our asymptotic theory.


1 Introduction

Models similar to the one of this paper are proposed in Kose et al. (2003) and Wang (2008). In particular, Wang studies exactly the same model as in this paper with different assumptions. But his asymptotic theory is applicable only when $M$ is infinite. Even in this case, it is not clear how an initial estimate of the global factor can be obtained.


2 The model and identification strategy

2.1 The model and assumptions

We are concerned with the multi-level factor model

$$x_{mit} = \gamma_{mi}^tG_t + \lambda_{mi}^tF_{mt} + e_{mit}, \quad (m = 1, \ldots, M; \ i = 1, \ldots, N_m; \ t = 1, \ldots, T), \quad (1)$$

where $m$ is the index for a region, $i$ that for each individual in region $m$, $t$ that for time, $G_t$ is an $s$ by 1 vector of unobserved, global factors that affect individuals in all the regions, $F_{mt}$ is an $r_m$ by 1 vector of unobserved, regional factors that affect individuals only in region $m$, $\gamma_{mi}$ and $\lambda_{mi}$ are unobserved factor loadings, and $e_{mit}$ is an idiosyncratic error. Note that each region is allowed to have a different number of individuals $N_m$. We assume that $s$ and $\{r_m\}$ are known and will discuss methods of estimating them in Section 4.

In vector notation, the model in (1) is written as

$$X_{mt} = \Gamma_m G_t + \Lambda_m F_{mt} + e_{mt}, \quad (m = 1, \ldots, M) \quad (2)$$

$$= [\Gamma_m \ A_m] \begin{bmatrix} G_t \\ F_{mt} \end{bmatrix} + e_{mt}$$

$$= \Theta_m K_{mt} + e_{mt}, \ \text{say,}$$
where $X_{mt} = \begin{bmatrix} x_{1mt} \\ \vdots \\ x_{N_mmt} \end{bmatrix}$, and $\Gamma_m$, $\Lambda_m$ and $e_{mt}$ are similarly defined.

Regarding the factors and idiosyncratic errors, we make the following assumption.

**Assumption 1**

(i) $\{G_t\}, \{F_{1t}\}, \ldots, \{F_{Mt}\}$ are zero-mean, stationary processes that satisfy the conditions for the law of large numbers and the central limit theorem.

(ii) $\{G_t\}, \{F_{1t}\}, \ldots, \{F_{Mt}\}$ are uncorrelated at all leads and lags. That is, $E(F_{ms}F_{nt}') = 0$ for all $s, t, \text{ and } m \neq n$; and $E(G_s'F_{mt}) = 0$ for all $m, n, t, \text{ and } s$.

(iii) $\{G_t\}, \{F_{1t}\}, \ldots, \{F_{Mt}\}$ satisfy conditions for the law of large numbers and the central limit theorem applied to their self- and cross-products.

The first and third parts of this assumption are of standard nature and does not require further elaborations. The second part implies that the global factors, regional factors and idiosyncratic errors are uncorrelated. This seems to be intuitively appealing and will be used for the identification strategy of this section. Note that the presence of the global factors make all of the observed data correlated unless the global factor loadings are equal to zero.

### 2.2 Identification strategy

Suppose that $M$ is finite. Then, identification of the spaces of the factors and factor loadings can be made by going through the following steps. Further details and asymptotic justifications of these steps will be given in Section 3.

**Step 1:** Select two regions and obtain an estimator of $G_t$, denoted as $\hat{G}_t^{(1)}$, by using the canonical correlation analysis. The superscript $(1)$ is employed to distinguish the current estimator from the other estimator of $G_t$ obtained in a later step. It will be shown later that $\hat{G}_t^{(1)} \xrightarrow{p} G_{tj}$ or $-G_{tj}$ ($j = 1, \ldots, s$) for every $t$ as $N_1, N_2, T \to \infty$ under proper conditions, where $G_{tj}$ denotes the $j$-th element of $G_t$. Equivalently, this can be written in vector notation as $\hat{G}_t^{(1)} \xrightarrow{p} DG_t$, where $D$ is a diagonal matrix of 1 and $-1$.  

3
Step 2: Rewrite model (2) as

\[
X_{mt} = \Gamma_m D^{-1} \hat{G}_t^{(1)} + \Lambda_m F_{mt} + e_{mt} - \Gamma_m D^{-1} (\hat{G}_t^{(1)} - DG_t) \\
= \Gamma_m^* \hat{G}_t^{(1)} + \Lambda_m F_{mt} + e_{mt}, \text{ say,} \quad (3)
\]

and, estimate \( \Lambda_m \) and \( F_{mt} \) by the principal component method. These estimators are denoted as \( \hat{\Lambda}_m^{(1)} \) and \( \hat{F}_{mt}^{(1)} \), respectively.

Step 3: Using \( \hat{\Lambda}_m^{(1)} \hat{F}_{mt}^{(1)} \), rewrite model (2) as

\[
X_{mt} - \hat{\Lambda}_m^{(1)} \hat{F}_{mt}^{(1)} = \Gamma_m G_t + e_{mt} - (\hat{\Lambda}_m^{(1)} \hat{F}_{mt}^{(1)} - \Lambda_m F_{mt}) \\
= \Gamma_m G_t + e_{mt}, \text{ say,} \quad (4)
\]

stack them as

\[
\begin{bmatrix}
X_{1t} - \hat{\Lambda}_1^{(1)} \hat{F}_{1t}^{(1)} \\
\vdots \\
X_{Mt} - \hat{\Lambda}_M^{(1)} \hat{F}_{Mt}^{(1)}
\end{bmatrix}
= 
\begin{bmatrix}
\Gamma_1 \\
\vdots \\
\Gamma_M
\end{bmatrix}
G_t + 
\begin{bmatrix}
e_{1t}^{(2)} \\
\vdots \\
e_{Mt}^{(2)}
\end{bmatrix}, \quad (5)
\]

and estimate \( \{\Gamma_m\} \) and \( G_t \) by the principal component method. The estimators are written as \( \hat{\Gamma}_m^{(1)} \) and \( \hat{G}_t^{(2)} \). The estimator \( \hat{G}_t^{(2)} \) uses information in the whole sample, whereas \( \hat{G}_t^{(1)} \) is based on the sample from only two regions.

Step 4: Using \( \hat{\Gamma}_m^{(1)} \) and \( \hat{G}_t^{(2)} \), rewrite model (2) as

\[
X_{mt} - \hat{\Gamma}_m^{(1)} \hat{G}_t^{(2)} = \Lambda_m F_{mt} + e_{mt} - (\hat{\Gamma}_m^{(1)} \hat{G}_t^{(2)} - \Gamma_m G_t) \\
= \Lambda_m F_{mt} + e_{mt}, \text{ say,} \quad (6)
\]

and estimate the spaces of \( \Lambda_m \) and \( F_{mt} \) by the principal component method. We denote these estimators as \( \hat{\Lambda}_m^{(2)} \) and \( \hat{F}_{mt}^{(2)} \), respectively. These estimators use \( \hat{G}_t^{(2)} \) which is based on the whole sample. In contrast, those in Step 2 use \( \hat{G}_t^{(1)} \) that are estimated by using the sample from only two regions.

3 Asymptotic theory for the multi-level factor model

This section details the estimation procedure for the multi-level factor model (1) and develops asymptotic theory that justifies the identification strategy of Subsection 2.2.
3.1 Step 1: Initial estimation of the global factor

Suppose that we chose two regions 1 and 2. In practice, it is advised to select two regions that yield the maximum sample mean of the eigenvalues from relation (7) below. Denote the PCE of $K_{mt}$ in model (2) as $\hat{K}_{mt}$ ($m = 1, 2$). Under proper assumptions, $\hat{K}_{mt}$ estimates a rotation of $K_{mt}$, $L_m K_{mt}$, consistently, where the matrix $L_m$ is $O_p(1)$ and defined in Bai (2003). Equivalently, we may write $\hat{K}_{mt} = L_m K_{mt} + o_p(1)$. Detailed conditions for this can be found in Bai (2003).

The global factor $G_t$ is present at both $K_{1t}$ and $K_{2t}$. Thus, the maximal correlation of arbitrary linear combinations of $K_{1t}$ and $K_{2t}$ should be equal to plus or minus one. In other words, the maximal correlation of arbitrary linear combinations of $K_{1t}$ and $K_{2t}$ carries information on the presence of the global factor at $K_{1t}$ and $K_{2t}$. This prompts us to use the canonical correlation analysis to estimate the global factor.

For the canonical correlation analysis, let $\hat{S}_{ab} = T^{-1} \sum_{t=1}^{T} \hat{K}_{at} \hat{K}_{bt}' (a, b = 1, 2)$. The generalized eigenvalues $\hat{\lambda}$ that satisfy the determinantal equation

$$
\begin{vmatrix}
\hat{S}_{12} \hat{S}_{22}^{-1} \hat{S}_{21} - \hat{\lambda} \hat{S}_{11}
\end{vmatrix} = 0
$$

provides the squared sample correlation coefficients between arbitrary linear combinations of $K_{1t}$ and $K_{2t}$, or $R^2$s resulting from regressing arbitrary linear combinations of $\hat{K}_{1t}$ on $\hat{K}_{2t}$ (see Chapter 12 of Anderson, 2003). We order the eigenvalues such that $\hat{\lambda}_1 \geq \cdots \geq \hat{\lambda}_{s+r_1}$. The coefficients for the linear combination of $\hat{K}_{1t}$ that correspond to $\hat{\lambda}_j$ are given by the eigenvector $\hat{p}_j$ that satisfies the equation

$$
\left( \hat{S}_{12} \hat{S}_{22}^{-1} \hat{S}_{21} - \hat{\lambda}_j \hat{S}_{11} \right) \hat{p}_j = 0
$$

with the restriction $\hat{p}_j' \hat{p}_j = 1$ ($j = 1, \ldots, s+r_1$) and $\hat{p}_j' \hat{p}_k = 0$ ($k = 1, \ldots, s+r_1-1$, $k < j$). In other words, the squared sample multiple correlation coefficient between $\hat{p}_j' \hat{K}_{1t}$ and $\hat{K}_{2t}$ is equal to $\hat{\lambda}_j$.

Since $\hat{S}_{ab} = L_a' T^{-1} \sum_{t=1}^{T} K_{at} K_{bt}' L_b + o_p(1) = L_a' S_{ab} L_b + o_p(1)$, where $S_{ab} = T^{-1} \sum_{t=1}^{T} K_{at} K_{bt}'$ ($a, b = 1, 2$), the determinantal equation (7) is equivalent to

$$
\begin{vmatrix}
S_{12} S_{22}^{-1} S_{21} - \hat{\lambda}_j S_{11} + o_p(1)
\end{vmatrix} = 0
$$

(8)
and the relevant equation for the eigenvector is

\[
\left( S_{12} S_{22}^{-1} S_{21} - \hat{\lambda}_j S_{11} + o_p(1) \right) \hat{q}_j = 0,
\]

where \( \hat{q}_j = L_1 \hat{p}_j \). This eigenvector cannot be computed in practice since \( L_1 \) is not known, but will be used later to derive an estimator of \( G_t \).

Under Assumption 1, the population equations corresponding to equations (8) and (9) are written as, respectively,

\[
\begin{align*}
\left( \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} - \lambda_j \Sigma_{11} \right) q_j &= 0, \\
\left( \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} - \lambda_j \Sigma_{11} \right) q_j &= 0,
\end{align*}
\]

where \( \Sigma_{ab} = E(K_{at} K_{bt}) (a, b = 1, 2) \) for any arbitrary \( t \). We order the population eigenvalues such that \( \lambda_1 \geq \cdots \geq \lambda_{s+r_1} \). As shown in Lemma A.1 in Appendix A, \( \lambda_1 = \cdots = \lambda_s = 1, \lambda_{s+1} = \cdots = \lambda_{s+r_1} = 0 \) and \( q_j = (0, \ldots, 1^{\text{th}}, \ldots, 0)' \) for \( j = 1, \ldots, s + r_1 \).

The sample eigenvalues and eigenvectors converge to their population counterparts (up to sign in case of the sample eigenvectors) as shown in the following lemma. The proof of this lemma follows from the standard theory of multivariate analysis (e.g., Anderson, 2003, Chapter 13) and is omitted.

**Lemma 1** Under Assumption 1, the following results hold, as \( N_1, N_2, T \to \infty \).

\( (i) \) \( \hat{\lambda}_j \xrightarrow{p} 1 \), for \( j = 1, \ldots, s \).

\( (ii) \) \( \hat{\lambda}_j \xrightarrow{p} 0 \), for \( j = s + 1, \ldots, s + r_1 \).

\( (iii) \) \( \hat{q}_j \xrightarrow{p} q_j \) or \( -q_j \), where \( q_j = (0, \ldots, 1^{\text{th}}, \ldots, 0)' \) for \( j = 1, \ldots, s + r_1 \).

Let \( \hat{G}^{(1)}_{tj} = \hat{p}_j' \hat{K}_{1t} \) (\( j = 1, \ldots, s \)). Then, we obtain by using Lemma 1

**Proposition 2** Assume Assumption 1. For every \( t \) and \( j \), as \( N_1, N_2, T \to \infty \),

\( (i) \) \( \hat{G}^{(1)}_{tj} \xrightarrow{p} G_{tj} \) or \( -G_{tj} \), for \( j = 1, \ldots, s \).

\( (ii) \) \( \hat{G}^{(1)}_{tj} \xrightarrow{p} F_{1tj} \) or \( -F_{1tj} \), for \( j = s + 1, \ldots, s + r_1 \).

\( (iii) \) \( \hat{G}^{(1)}_{tj} \pm G_{tj} = O_p \left( \frac{1}{\min\sqrt{T, \sqrt{N_1}}} \right) \), for \( j = 1, \ldots, s \).

\( (iv) \) \( \hat{G}^{(1)}_{tj} \pm F_{tj} = O_p \left( \frac{1}{\min\sqrt{T, \sqrt{N_1}}} \right) \), for \( j = s + 1, \ldots, s + r_1 \).
Part (i) of this proposition shows that the global factor \( \{G_t\} \) can consistently be estimated up to sign by \( \hat{G}_t^{(1)} \). Notice that \( \hat{G}_t^{(1)} \) estimates either \(+G_{tj}\) or \(-G_{tj}\) where the sign is common for all \( t \). In other words, the event where \( \hat{G}_t^{(1)} \) estimates \(+G_{tj}\) while \( \hat{G}_{sj} \) does \(-G_{sj}\) \((s \neq t)\) would not happen. Part (ii) of this proposition show the rate of consistency of \( \hat{G}_t^{(1)} \). This will be used for Proposition 3 in the next subsection.

3.2 Step 2: Initial estimation of the regional common components

This subsection studies model (3). The PCEs for model (3) can be derived by the conditional maximum likelihood estimation method as shown in Choi (2012). Therefore, this subsection starts from the conditional maximum likelihood estimation of model (3). To this end, assume \( e_{mt} | \{K_{mt}\} \sim iid \ N(0, I_{N_m}) \). Let \( X_m = \begin{bmatrix} X_{m1}' \\ \vdots \\ X_{mT}' \end{bmatrix} \), \( \hat{G}^{(1)} = \begin{bmatrix} \hat{G}_1^{(1)'} \\ \vdots \\ \hat{G}_T^{(1)'} \end{bmatrix} \) and \( F_m = \begin{bmatrix} F_{m1}' \\ \vdots \\ F_{mT}' \end{bmatrix} \). Then, treating \( \{\hat{G}_t^{(1)}\} \) as if they were \( \{G_t\} \) and ignoring the \( o_p(1) \) term \( \Gamma_m D^{-1}(\hat{G}_t^{(1)} - DG_t) \), the conditional log-likelihood function (multiplied by -2) for model (3) is:

\[
l(\{\Gamma_m^{(1)}\}, \{\Lambda_m\}, \{F_m\}) = T \sum_{m=1}^{M} N_m \ln(2\pi) + T \sum_{m=1}^{M} \ln |I_{N_m}| \\
+ \sum_{m=1}^{M} \text{tr} \left\{ (X_m - \hat{G}_m^{(1)} \Gamma_m^{*m} - F_m \Lambda_m')'(X_m - \hat{G}_m^{(1)} \Gamma_m^{*m} - F_m \Lambda_m') \right\}.
\]

For the conditional maximum likelihood estimation, we need to minimize

\[
\text{tr} \left\{ (X_m - \hat{G}_m^{(1)} \Gamma_m^{*m} - F_m \Lambda_m')(X_m - \hat{G}_m^{(1)} \Gamma_m^{*m} - F_m \Lambda_m') \right\}
\]

with respect to \( \Gamma_m^{*m} \), \( \Lambda_m \) and \( F_m \). Using the standard theory of multivariate linear regression, we obtain

\[
\hat{\Lambda}_m = X_m' (I - P_{\hat{G}_m^{(1)}}) F_m (F_m' (I - P_{\hat{G}_m^{(1)}}) F_m)^{-1} = X_m G_f G_f' (F_m' F_m)^{-1},
\]
with $X_m^G = (I - P_{G^{(1)}})X_m$ and $F_m^G = (I - P_{G^{(1)}})F_m$, and
\[
\hat{\Gamma}_m^* = X_m'(I - P_{F_m})\hat{G}^{(1)}(\hat{G}^{(1)'}(I - P_{F_m})\hat{G}^{(1)})^{-1} = X_m^{F'}\hat{G}F\left((\hat{G}^{F'}\hat{G}^F)^{-1}\right),
\]
where $X_m^F = (I - P_{F_m})X_m$ and $\hat{G}^F = (I - P_{F_m})\hat{G}^{(1)}$. Plugging $\hat{\Lambda}_m$ and $\hat{\Gamma}_m^*$ successively into the objective function (10), we obtain $tr\left\{X_m^{G'}(I - P_{F_m^G})X_m^G\right\}$. With the standardization $F_m^G = T \times I_r$, the PCE of $F_m^G$, denoted by $\hat{F}_m^{(1)}$, is obtained by maximizing $tr\left\{F_m^GX_m^GX_m^F\right\}$ with respect to $F_m^G$. It is $\sqrt{T}$ times the matrix consisting of the eigenvectors corresponding to the $r_m$ largest eigenvalues of the matrix $X_m^G X_m^G$. The PCE of $\Lambda_m$ is given by $\hat{\Lambda}_m^{(1)} = \frac{1}{T} X_m^G \hat{F}_m^{(1)}$.

Note that the PCE $\hat{F}_m^{(1)}$ estimates the space of $(I - P_{G^{(1)}})F_m$, not that of $F_m$. However, the PCE estimates a rotation of $F_m$ as will be discussed in more detail later.

The following proposition reports asymptotic results for the PCEs. This follows from Bai’s (2003) Theorems 1, 2 and 3 upon minor adaptations. Assumptions for the proposition are given in Bai (2003) and contained in Appendix B for the reader’s convenience. These assumptions are about the idiosyncratic errors, factors and factor loadings. The rest are basically the law of large numbers and the central limit theorem for the quantities involving the idiosyncratic errors, factor loadings and factors.

**Proposition 3** Assume Assumption 1. In addition, suppose that Assumptions B.1, B.2 and B.3 in Appendix B hold. Then, the following results hold for any $m$ as $N_m, T \to \infty$.

(i) (a) If $\frac{\sqrt{N_m}}{T} \to 0$, $\hat{F}_m^{(1)} - H_m^1 F_m^G = O_p(N_m^{-1/2})$ for each $t$, where $\hat{F}_m^{(1)}$ is the $t^{th}$ column of $\hat{F}_m^{(1)'}$ and $H_m^1$ is a random matrix.\(^1\)

(b) If $\liminf \frac{\sqrt{N_m}}{T} \geq \tau > 0$, $\hat{F}_m^{(1)} - H_m^1 F_m^G = O_p(T^{-1})$ for each $t$.

(ii) (a) If $\frac{\sqrt{T}}{N_m} \to 0$, $\hat{\lambda}_m^{(1)} - H_m^{-1} \lambda_m = O_p(T^{-1/2})$ for each $i$.

(b) If $\liminf \frac{\sqrt{T}}{N_m} \geq \tau > 0$, $\hat{\lambda}_m^{(1)} - H_m^{-1} \lambda_m = O_p(N_m^{-1})$ for each $i$.

(iii) Let $\hat{C}_m^{(1)} = \hat{\lambda}_m^{(1)'} \hat{F}_m^{(1)}$ and $C_m^{G} = X_m^{G} F_m^{G}$. Then, $\hat{C}_m^{(1)} - C_m^{G} = O_p(k_{N_m,T}^{-1})$.

where $k_{N_m,T} = \min\{\sqrt{N_m}, \sqrt{T}\}$.

\(^{1}\)See Bai’s (2003) Theorem 1 for the definition of $H_m^1$. 

8
Part (i) of this proposition shows that $\hat{F}_{mt}^{(1)} - H_m' F_{mt}^G \overset{p}{\rightarrow} 0$. Since $H_m = O_p(1)$, $F_m' \hat{G}^{(1)} = F_m' G D_s + F_m' (\hat{G}^{(1)} - GD_s) = O_p(T^{1/2}) + O_p(\kappa_{N_1 T}^{-1})$ and $(T^{-1} \hat{G}^{(1)} G^{(1)})^{-1} = O_p(1)$, we have

\[
\hat{F}_{mt}^{(1)} - H_m' F_{mt}^G = \hat{F}_{mt}^{(1)} - H_m' F_{mt} - H_m' F_m' (\hat{G}^{(1)} G^{(1)})^{-1} \hat{G}_t^{(1)} - \hat{F}_{mt}^{(1)} - H_m' F_{mt} + O_p(\kappa_{N_1 T}^{-1}),
\]

(11)

which implies that $\hat{F}_{mt}^{(1)} - H_m' F_{mt} \overset{p}{\rightarrow} 0$. That is, $\hat{F}_{mt}^{(1)}$ estimates a rotation of $F_{mt}$ consistently.

Part (ii) of this proposition shows that $H_m^{-1} \lambda mi$ can be estimated consistently by $\hat{\lambda}_{mi}^{(1)}$ at the rate of either $T^{1/2}$ or $N_m$.

Using the same arguments as for equation (11), we have

\[
\hat{C}^{(1)}_{mit} - C^{G(1)}_{mit} = \hat{C}_{mit}^{(1)} - C_{mit} + O_p(\kappa_{N_1 T}^{-1}).
\]

(12)

Thus, $\hat{C}_{mit}^{(1)}$ estimates the common components $C_{mit}$ consistently with the convergence rate $\kappa_{N_1 T}$ or $\kappa_{N_1 T}$ since $\hat{C}_{mit}^{(1)} - C^{G(1)}_{mit} = O_p(\kappa_{N_1 T}^{-1})$.

### 3.3 Step 3: Estimation of the spaces of the global factor and factor loadings

This subsection considers estimating the spaces of the global factor and factor loadings by using the estimators of the regional common components from Step 2. For this, write model (5) as

\[
Z_t = \Gamma G_t + e_t^{(2)},
\]

(13)

where, letting $N = \sum_{m=1}^M N_m$, $Z_t$ is an $N$ by 1 vector, $\Gamma$ is an an $N$ by $s$ matrix. Since Proposition 3 implies that $e_{mt}^{(2)} = e_{mt} + O_p(\kappa_{N_1 T}^{-1})$ for any $m$. When the model is stacked across regions, the relevant number of cross sectional units is $N$ and we may write $e_t^{(2)} = e_t + O_p(\delta_{NT}^{-1})$ with $e_t = [e_{1t}', ..., e_{Mt}']'$ and $\delta_{NT} = \min\{\sqrt{N}, \sqrt{T}\}$. Since model (13) is the typical factor model except the presence of the $O_p(\delta_{NT}^{-1})$ term, the PCEs of the $j$-th column of $\Gamma'$ ($j = 1, ..., N$) and $G_t$, denoted by $\hat{\Gamma}_j^{(1)}$ and $\hat{G}_t^{(2)}$, respectively, esti...
respectively, follow the asymptotic distributions given in Bai (2003) and are stated concretely as follows.

**Proposition 4** The results in Proposition 3 hold as $N_1, \ldots, N_M, T \to \infty$, once $\hat{F}^{(1)}_{mt}, F^G_{mt}, \hat{\lambda}^{(1)}_{mi}, \lambda_{mi}, \hat{C}^{(1)}_{mit}, N_m$, and $\kappa_{N_mT}$ are replaced by $\hat{G}^{(2)}_t, G_t, \hat{\Gamma}^{(1)}_j, \Gamma_j, \hat{\Gamma}^{(1)}_j, \hat{G}^{(2)}_t$, $N$ and $\delta_{NT} = \min\{\sqrt{N}, \sqrt{T}\}$, respectively; and $H_m$ is appropriately redefined.

This proposition implies that the global factor space is consistently estimated by the PCE $\hat{G}^{(2)}_t$ at the rate either $\sqrt{N}$ or $T$. $\hat{G}^{(2)}_t$ estimates the global factor space faster than or at the same rate as the initial estimator $\hat{G}^{(1)}_t$ which has the the consistency rate $\kappa_{N_1T}$. The intuition behind this is that the PCE $\hat{G}^{(2)}_t$ use all the information in the whole sample, while $\hat{G}^{(1)}_t$ is based on the sample from region 1.

Moreover, the common components $\{\Gamma G_t\}$ are consistently estimated at the rate $\delta_{NT}$. This will be used for Step 4 in the next subsection.

### 3.4 Step 4: Estimation of the spaces of the regional factor and factor loadings

In this subsection, we consider estimating the spaces of the regional factor and factor loadings by using model (6) and the estimators of the common components from Step 3. Write model (6) as

$$Y_{mt} = \Lambda_m F_{mt} + e^{(3)}_{mt}. $$

Since $e^{(3)}_{mt} = e_{mt} + O_p(\delta^{-1}_{NT})$, we obtain the following results for the PCEs $\hat{F}^{(2)}_{mt}$ and $\hat{\lambda}^{(2)}_{mi}$.

**Proposition 5** The results in Proposition 3 hold as $N_m, T \to \infty$, once $\hat{F}^{(1)}_{mt}, F^G_{mt}, \hat{\lambda}^{(1)}_{mi}$, and $\hat{C}^{(1)}_{mit}$ are replaced by $\hat{F}^{(2)}_{mt}, F_{mt}, \hat{\lambda}^{(2)}_{mi}$, and $\hat{\lambda}^{(2)}_{mi}$, respectively; and $H_m$ is appropriately redefined.

This proposition implies that a rotation of $F^G_{mt}$ can be estimated consistently at the rate of either $\sqrt{N_m}$ or $T$. In line with the fact that the relation (11) along with
Proposition 3 implies that $F_{mt}^1 - H'_m F_{mt}$ is either $O_p(N_m^{-1/2}) + O_p(\kappa_{N_1T}^{-1})$ or $O_p(T^{-1}) + O_p(\kappa_{N_1T}^{-1})$, it can be shown that the PCE from Step 4 is such that $F_{mt}^2 - H'_m F_{mt}$ is either $O_p(N_m^{-1/2}) + O_p(\delta_{N_1T}^{-1})$ or $O_p(T^{-1}) + O_p(\delta_{N_1T}^{-1})$, a possibly faster rate that that from Step 2. This will be studied further via simulation in Section 5.

The relation (12) and Proposition 3 imply that the common components can be estimated at the rate of $N_m T$, the same rate implied by Proposition 5. However, the estimators from Step 2 involve the $O_p(\kappa_{N_1T}^{-1})$ term as shown in relation (12) that is expected to bring biases in finite samples, while the estimators from Step 4 have an $O_p(\delta_{N_1T}^{-1})$ term. Again, going through the steps outlined in Section 3 will improve the efficiency of the common components estimation. This will be confirmed by simulation in Section 5.

4 Inference on the number of factors

This section considers estimating the numbers of the global and regional factors. There are various procedures available for the selection of the number of static factors. Bai and Ng (2002) and Choi and Jeong (2012) propose several information criteria for the number of factors in approximate factor models. Onatski (2009, 2010) and Ahn and Horenstein (2009) suggest methods using ordered eigenvalues. When the number of static factors is determined, we may use the methods of Amengual and Watson (2007), Bai and Ng (2007) and Breitung and Pigorsch (2012) to determine the number of dynamic (or primitive) factors.

As a matter of notation, denote by $s; r_1, \ldots, r_M$ the true number of factors and by $u, k_1, \ldots, k_M$ the number of factors specified in the estimation. We introduce the following assumption.

**Assumption 2** $N_m/N \rightarrow \eta_m > 0$ for all $m$.

Assumption 2 says that the number of individuals in each region is of the same order of magnitude. This assumption combined with Assumption B.2 (iv) implies
that regional factors can be regarded as valid global factors, because $N^{-1} \Lambda_m' \Lambda_m \to \eta_m \Sigma_m > 0$. When one region is much smaller so that $N_m/N \to 0$, estimating insufficient number of factors in that region might not affect asymptotically the overall goodness of fit. Assumption 2 excludes such a case.

We consider the following information criterion

$$PC(u, k_1, \ldots, k_M) = V_{NT}(u, k_1, \ldots, k_M) + \sum_{m=1}^{M} r_m \cdot g_m(N, T) + s \cdot p(N, T)$$

where

$$V_{NT}(u, k_1, \ldots, k_M) = \frac{1}{NT} \sum_{m=1}^{M} \sum_{t=1}^{T} \hat{e}_{mt} \hat{e}_{mt}$$

and

$$\hat{e}_{mt} = X_{mt} - \hat{\Lambda}_m^{(2)} \hat{F}_{mt} - \hat{\Gamma}_m^{(1)} G^{(2)}_{t}.$$

Note that $\{\hat{e}_{mt}\}$ are using the estimators from Steps 3 and 4. The first term in $PC(u, k_1, \ldots, k_M)$ measures the goodness of fit while the second and third ones are penalty terms for the number of regional factors and global factors. The penalty terms $g_m(N, T)$ for $m = 1, \ldots, M$ and $p(N, T)$ may depend not only on $(N, T)$ but also on regional sample sizes $(N_1, \ldots, N_m)$, but we omit such dependence for notational simplicity.

In order to prove the consistency, it is required to find out the asymptotic behavior of the estimators in all four steps when the numbers of factors are misspecified. While this level of full analysis is beyond the scope of this paper, we deal with a simpler version by assuming that the total number of factors $s + r_m$ is known for region 1 and region 2. One justification for this assumption is that the total number of factors in each region is consistently estimable using information criteria such as BIC.

When $s + r_1$ and $s + r_2$ are known, one needs to estimate the Step 1 estimator once and for all. For the given $\hat{K}_1$ and $\hat{K}_2$, obtain the eigenvectors $\hat{p}_1, \ldots, \hat{p}_{s+r_1}$. In order to compute $PC(u, k_1, \ldots, k_M)$ for each specified value of $u$, one simply takes $\hat{K}_1[\hat{p}_1, \ldots, \hat{p}_u]$ as $\hat{G}^{(1)}$. Then, the asymptotic behavior of $\hat{G}^{(1)}$ is completely characterized by Proposition 2. If $u \leq s$, $\hat{G}^{(1)}$ consistently estimates (a rotation of) the first $u$
columns of the true global factor $G$. If $u > s$, the first $s$ columns of $\hat{G}^{(1)}$ consistently estimates (a rotation of) the true global factor $G$ and the rest $r_1$ columns the first $u - s$ columns of the true regional factor $F_1$.

The next proposition gives a set of sufficient conditions on the penalty terms for consistency.

**Proposition 6** Suppose that Assumptions 1, 2, B.1, B.2, and B.3 hold. Let $(\hat{s}, \hat{r}_1, ..., \hat{r}_M)$ be the minimizer of $PC(u, k_1, ..., k_M)$ over $0 \leq u \leq s^{\max}$ and $0 \leq k_m \leq k_m^{\max}$. Assume that each column of $\hat{G}^{(1)}$ satisfies the results in Proposition 2. Then,

$$\lim_{N_1, ..., N_M, T \to \infty} \Pr(\hat{s} = s, \hat{r}_1 = r_1, ..., \hat{r}_M = r_M) = 1$$

if (i) $g_m(N, T) \to 0$ for all $m$;
(ii) $\delta_{NT}^2 g_m(N, T) \to \infty$ for all $m$;
(iii) $\delta_{NT}^2 [p(N, T) - g_m(N_m, T)] \to \infty$; and
(iv) $\delta_{NT}^2 [\sum_{m=1}^M g_m(N, T) - p(N, T)] \to \infty$.

Note that the same information criterion is expected to work well even without the simplification assumption of known total number of factors in regions 1 and 2. If the Step 1 estimator is computed with misspecifying the total numbers of factors, that only deteriorates the fit for that particular combination of specified numbers of factors, making it easy to distinguish the true number of factors from the wrong ones.

Now, we consider some of the information criteria suggested in Bai and Ng (2002)
and Choi and Jeong (2012).

\[ IC_{p2} = \ln \left( V_{NT}(u, k_1, \ldots, k_M) \right) \]
\[ + \ln(\min\{N, T\}) \left( \frac{\sum_{m=1}^{M} k_m (N_m + T) + u(N + T)}{NT} \right) ; \]

\[ BIC = T \sum_{m=1}^{M} tr \left( \ln \left( \sum_{t=1}^{T} \hat{e}_{mt} e_{mt}/T \right) \right) \]
\[ + \ln(NT) \left[ \sum_{m=1}^{M} k_m (N_m + T) + u(N + T) + N \right] ; \]

\[ HQ_c = T \sum_{m=1}^{M} tr \left( \ln \left( \sum_{t=1}^{T} \hat{e}_{mt} e_{mt}/T \right) \right) \]
\[ + c \ln \ln(NT) \left[ \sum_{m=1}^{M} k_m (N_m + T) + u(N + T) + N \right], \]

where \( c \) is a constant of our choice. The first one is Bai and Ng’s \( IC_{p2} \), which shows good performance according to the simulation results of Choi and Jeong (2012). The second and third are \( BIC \) and Hannan and Quinn’s (1979) criterion suggested in Choi and Jeong, respectively. Choi and Jeong show that \( BIC \) and \( HQ_c \) perform well in finite samples. These criteria satisfy the four conditions in Proposition 6 if neither \( T \) nor \( N \) dominates the other, that is, \( N \) and \( T \) are of the same order of magnitude. Section 5 will examine the finite-sample performance of the information criteria of this section.
5 Simulation

In this section we evaluate finite-sample performance of our estimators of global and regional factors and the model selection criteria.

Consider the data-generating process (DGP)

\[
x_{mit} = \sum_{k=1}^{s} \gamma_{mik} g_{tk} + \sqrt{\theta_{m1}} \sum_{k=1}^{r_m} \lambda_{mik} f_{mtk} + \sqrt{\theta_{m2}} e_{mit} \quad (14)
\]

\(m = 1, \ldots, M; \ i = 1, \ldots, N_m; \ t = 1, \ldots, T)\)

\(\gamma_{mik} \sim iid N\left(0, \sigma_{\gamma}^2\right) ; \ \lambda_{ik} \sim iid N\left(0, \sigma_{\lambda}^2\right)\)

\(g_{tk} = \alpha g_{t-1,k} + v_{ik}; \ v_{tk} \sim iid N(0,1)\)

\(f_{mtk} = \phi_m f_{m,t-1,k} + w_{mtk}; \ w_{tk} \sim iid N(0,1)\)

\(e_{mit} = \rho_m e_{m,i,t-1} + e_{it} + \beta_m \sum_{1\leq|j|\leq8} e_{m,i-j,t} ; \ e_{mik} \sim iid N(0,1).\)

Since

\[
\text{Var} \left( \sum_{k=1}^{s} \gamma_{mik} g_{tk} \right) = \sum_{k=1}^{s} E(\gamma_{mik} g_{tk})^2 = \sum_{k=1}^{s} E(\gamma_{mik}^2) E(g_{tk}^2) = \frac{s \sigma_{\gamma}^2}{1 - \alpha^2},
\]

\[
\text{Var} \left( \sum_{k=1}^{r_m} \lambda_{mik} f_{mtk} \right) = \sum_{k=1}^{r_m} E(\lambda_{mik} f_{mtk})^2 = \sum_{k=1}^{r_m} E(\lambda_{mik}^2) E(f_{mtk}^2) = \frac{r_m \sigma_{\lambda}^2}{1 - \phi_m^2},
\]

and

\[
\text{Var} \ (e_{mit}) = \frac{1 + 16 \beta_m^2}{1 - \rho_m^2},
\]
setting \( \theta_{m1} = \left( \frac{s\sigma^2}{1-\alpha^2} \right) / \left\{ \frac{r_m\sigma^2}{1-\phi_m} \right\} \) and \( \theta_{m2} = \left( \frac{s\sigma^2}{1-\alpha^2} \right) / \left\{ \frac{1+16\phi_m^2}{1-\phi_m} \right\} \) make the three components of equation (14) have the same variance \( \frac{s\sigma^2}{1-\alpha^2} \).

We consider three cases for the true number of global factors \( s \in \{1, 2, 3\} \). The true numbers of regional factors are randomly selected from \( \{1, 2, 3\} \). We set the number of regions \( M \in \{5, 20\} \). The cross-sectional sample sizes \( N_m \) are either 20 or randomly selected from \( \{50, 100, 150\} \), and the time-series sample size \( T \) is either 100 or 200. The parameters \( \sigma^2 \gamma \) and \( \sigma^2 \chi \) are assumed to be 1. For the autocorrelation parameters for the global and regional factors, we consider the following four pairs: \((\alpha, \phi_m) = (0.5, 0.5), (0.5, 0.85), (0.85, 0.5) \) and \((0.85, 0.85)\). The parameters \( \rho_m \) and \( \beta_m \) set the degrees of serial and cross-sectional correlations of the idiosyncratic errors, respectively. We consider the combinations \( \rho_m = 0, 0.5, 0.85 \) and \( \beta_m = 0, 0.2, 0.5 \). All the simulation results are based on 1,000 monte carlo replications.

5.1 Efficiency in estimating the global and regional factors

This subsection studies how closely the estimates of \( G_t \) and \( F_t \) are related to the true \( G_t \) and \( F_t \) by using the trace-ratio statistic. The trace-ratio statistic is generalized squared correlation coefficient used as a goodness-of-fit measure in multivariate analysis. Denote the true global factor and its estimate by \( G_t \) and \( \hat{G}_t \). Then the trace ratio is

\[
TR(\hat{G}) = \frac{tr(G'\hat{G}'(\hat{G}'\hat{G})^{-1}\hat{G}'G)}{tr(G'G)}.
\]

We compute the trace-ratio statistics for the estimates of the global and regional factors. In order to study how much efficiency is obtained by going through our estimation procedure, we compare the trace-ratio statistics for the estimates from each of four estimation step.

Tables 1-*** report the trace-ratio statistics for the estimates of global and regional factors. The rows labeled \( \hat{G}_t^{(1)} \) and \( \hat{G}_t^{(2)} \) report the average of the trace ratios of the initial and the final (step-3) estimates of the global factor \( G_t \) across 1,000 replications. The rows labeled \( \hat{F}_t^{(1)} \) and \( \hat{F}_t^{(2)} \) show the trace ratios of the initial (step-2) and
the final (step-4) estimates of the regional factors, averaged across regions and across 1,000 replications. The parameters \((\alpha, \phi_m)\) are set \((0.5, 0, 5)\) for Tables 1-3, \((0.5, 0, 85)\) for Tables 4-6, \((0.85, 0, 5)\) for Tables 7-9, and \((0.85, 0, 85)\) for Tables 10-12. The true number of global factors varies from 1 to 3 for each table. In each table, the rows show the results for different values \(M, N_m,\) and \(T,\) each column shows the results for different structures of the idiosyncratic errors. The true numbers of regional factors are randomly selected from \(\{1, 2, 3\}\) in all specifications.

The most notable observation from the simulation results is that our final estimates of the global factors perform very well – the trace ratios range from 0.7936 to 0.9995 and are over 0.9 in most cases, often over 0.95. As the number of global factors increases, the trace ratios of the estimates of the global factor worsens, the deteriorations being greater for the initial estimates. However, the efficiency gains from the initial to the final estimates rise. Thus the efficiency for the final estimates of the global factors does not fall much. For example, for parameter values \((M, N_m, T, \alpha, \beta_m, \rho_m, \phi_m) = (5, 10, 100, 0.5, 0.5, 0, 0),\) the efficiency gains from the initial to the final estimates of the global factors increase from 3.3\% when \(s = 1,\) to 8.3\% and 13.6\% for \(s = 2\) and 3, respectively. The trace ratios for \(\hat{G}^{(2)}_t\) are 0.9874, 0.9735, and 0.9592 for \(s = 1, 2, 3\) cases, respectively. Performances for the estimates of the regional factors are not affected much by the increase in the number of global factors.

5.2 Efficiency in estimating the variance of the global common components

This subsection studies how well our procedures estimate the variance ratio \(\frac{\text{Var}(\sum_{k=1}^{s} \gamma_{mk}g_{tk})}{\text{Var}(x_{mit})} (= \frac{1}{3})\).
5.3 Estimation of the number of factors

This subsection studies finite-sample performance of the model selection criteria introduced in Section 4. Table ** reports the root mean squared errors (RMSE) of the estimated number of factors for each sample size and set of parameter values.

When there are no serial and cross-sectional correlations in the idiosyncratic errors \((\rho, \beta) = (0, 0)\), \(IC_{\varphi^2}\) performs very well. \(BIC\) and \(HQ_4\) work well only for the global factors.

6 Applications
Appendix A: Proofs

Lemma A.1 \( \lambda_1 = \ldots = \lambda_s = 1, \lambda_{s+1} = \ldots = \lambda_{s+r_1} = 0 \) and \( q_{1j} = (0, \ldots, 1, \ldots, 0) \) (j = 1, \ldots, s + r_1).

Proof. Suppose that \( r_1 \leq r_2 \). Since \{\( G_t \), \( F_t \)\} and \{\( F_t \)\} are independent, \[
\begin{bmatrix}
\Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} - \lambda \Sigma_{11} \\
\Sigma_{GG} & 0 \\
0 & 0
\end{bmatrix} - \lambda
\begin{bmatrix}
\Sigma_{GG} & 0 \\
0 & \Sigma_{F_1}
\end{bmatrix}
\]
where \( \Sigma_{GG} = E(G_t G_t') \) for any arbitrary \( t \). Thus, the solutions of the determinantal equation \( \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} - \lambda \Sigma_{11} = 0 \) are \( \lambda = 1 \) (of multiplicity \( s \)) and \( \lambda = 0 \) (of multiplicity \( r_1 \)). If \( r_1 > r_2 \), we have \( \lambda_{s+1} = \ldots = \lambda_{s+r_2} = 0 \) for the same reason as above and \( \lambda_{s+r_2+1} = \ldots = \lambda_{s+r_1} \) by construction (cf. Anderson, 2003, Subsection 12.2). The eigenvectors are trivially \( q_{1j} = (0, \ldots, 1, \ldots, 0) \).

Proof of Proposition 2: (i) Since \( \hat{p}_j = L_1^{-1} \hat{q}_{1j} \) and \( \hat{K}_{1t} - L_1' K_{1t} \rightarrow^p 0 \) for each \( t \), we obtain \( \hat{G}_{tj}^{(1)} = \hat{p}_j' \hat{K}_{1t} \rightarrow^p q_{1j} L_1^{-1} L_1' \hat{K}_{1t} = \pm G_{tj} \) as required, where \( L_1 = p \lim_{N,T \to \infty} L_1 \) and \( G_{tj} \) is the \( j \)-th element of \( G_t \).

(ii) Assume without loss of generality that \( \hat{q}_{1j} \rightarrow^p q_j \). Write
\[
\hat{G}_{tj}^{(1)} - G_{tj} = \hat{q}_{1j}' L_1^{-1} \hat{K}_{1t} - q_{1j}' K_{1t} = (\hat{q}_{1j} - q_j)' L_1^{-1} \hat{K}_{1t} + q_{1j}' L_1^{-1} (\hat{K}_{1t} - K_{1t}). \tag{A.1}
\]

Since \( S_{11}^{-1} S_{12} S_{22}^{-1} S_{21} - \Sigma_{11}^{-1} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} = O_p(\frac{1}{\sqrt{T}}) \) under Assumption 1, \( \hat{q}_{1j} - q_j = O_p(\frac{1}{\sqrt{T}}) \) (see Anderson, 2003, Chapter 13, for related techniques). The second term in equation (A.1) is either \( O_p(\frac{1}{\sqrt{T}}) \) or \( O_p(\frac{1}{T}) \) (cf. Bai, 2003). Thus, \( \hat{G}_{tj}^{(1)} - G_{tj} = O_p(\frac{1}{\min\{\sqrt{T}, \sqrt{N}\}}) \) as required.

Proof of Proposition 3: (i) (a) Multiplying \( I - P_{\hat{G}^{(1)}} \) on both sides of model (3),
we obtain

\[ X_{mt}^G = \Lambda_m F_{mt}^G + e_{mt} - \epsilon_m' \hat{G}^{(1)} \left( \hat{G}^{(1)'} \hat{G}^{(1)} \right)^{-1} \hat{G}_t^{(1)} - \Gamma_mD^{-1}(\hat{G}_t^{(1)} - DG_t) + \Gamma_mD^{-1} \left( \hat{G}^{(1)} - GD \right)' \hat{G}_t^{(1)} \left( \hat{G}^{(1)'} \hat{G}^{(1)} \right)^{-1} \hat{G}_t^{(1)} \]

\[ = \Lambda_m F_{mt}^G + e_{mt} + \epsilon_{mtT}, \text{ say.} \]

The only differences between this model and the one in Bai (2003) are the presence of the term \( \epsilon_{mtT} \), which is \( Op \left( \frac{1}{\min\{N_1, T\}} \right) \), and the use of \( F_{mt}^G \) instead of \( F_{mt} \). Since \( \frac{1}{N_m} \epsilon_{ms} \epsilon_{mt} = O_p \left( \frac{1}{\min\{N_1, T\}} \right) \) for any \( m, s \) and \( t \), \( \epsilon_{mtT} \) has no effects on Lemmas A.1, A.2 and A.3 of Bai under Assumption B.3 (iv) except part (b) of Lemma A.2. The term \( \frac{1}{N_m} \epsilon_{ms} \epsilon_{mt} \) affects part (b) of Lemma A.2 such that the term there becomes \( Op \left( \frac{1}{\min\{N_m, T\}} \right) \). But because this does not affect the proof of Theorem 1 of Bai, it holds without any changes.

(b) The second terms of Lemmas B.1 and B.2 of Bai (2003) are affected by the presence of \( \epsilon_{mtT} \), but the main results of Lemmas B.1 and B.2 remain intact despite of it. Thus, Theorem 2 of Bai holds without any changes.

(c) This follows from parts (a) and (b) as in Bai (2003). ■

**Proof of Proposition 4:** This follows from Bai (2003) with some minor changes as in the proof of Proposition 3. The details are omitted. ■

**Proof of Proposition 5:** These are omitted since they are similar to those of Theorems 3 and 4. ■

Before we prove Proposition 6, we derive a few more related results.

**Orthogonal Reparameterization:** Suppose that \( H \) is a \( T \times r \) matrix of common factors and \( \Xi' \) is an \( r \times N \) matrix of factor loadings. For any given \( r \times k \) matrix \( D_k \) with \( k \leq r \), let \( \tilde{D}_k \) be an \( r \times (r - k) \) matrix such that \( D_k' \tilde{D}_k = 0_{k \times (r-k)} \). Then, we define an \( r \times r \) rotation matrix \( R = [D_k, \tilde{D}_k - D_k C] \) where \( C = (D_k'H'HD_k)^{-1}D_k'H'H\tilde{D}_k \). The rotation of the common factors \( H \) by \( R \) yields another set of \( r \) common factors. We call the first \( k \) columns and the rest \( r - k \) columns of the rotated common factors...
by $H_k$ and $H_{r-k}$, respectively. That is,

$$HR = [HD_k, H(\tilde{D}_k - D_kC)] = [H_k, H_{r-k}].$$

Since the inverse of the rotation matrix $R$ is given by

$$R^{-1} = \begin{bmatrix} (D'_kD_k)^{-1}D'_k + C(\tilde{D}'_k\tilde{D}_k)^{-1}\tilde{D}'_k \\ (D'_kD_k)^{-1}D'_k \end{bmatrix},$$

we define the loading matrix corresponding to $[H_k, H_{r-k}]$ by

$$\begin{bmatrix} \Xi'_k \\ \Xi'_{r-k} \end{bmatrix} = R^{-1}\Xi' = \begin{bmatrix} (D'_kD_k)^{-1}D'_k\Xi' + C(\tilde{D}'_k\tilde{D}_k)^{-1}\tilde{D}'_k\Xi' \\ (D'_kD_k)^{-1}D'_k\Xi' \end{bmatrix}.$$  

Then, we can write

$$H\Xi' = HRR^{-1}\Xi' = [H_k, H_{r-k}]\begin{bmatrix} \Xi'_k \\ \Xi'_{r-k} \end{bmatrix} = H_k\Xi'_k + H_{r-k}\Xi'_{r-k}. \quad (A.2)$$

Note that $H'_kH_{r-k} = D'_kH'H(\tilde{D}_k - D_kC) = 0$ by construction. For that reason, we call $(A.2)$ as the orthogonal reparameterization of the common components $H\Xi'$ with respect to $D_k$. The purpose of the orthogonal reparameterization is to separate those common components that are consistently estimated from those that are not. In addition, those two groups are orthogonal.

For $k > r$, let $D_k^+$ be a $k \times r$ matrix such that $D_kD_k^+ = I_r$. Then, the relevant reparameterization of the common components is given by

$$H\Xi' = HD_kD_k^+\Xi' = H_k\Xi'_k \quad (A.3)$$

where $H_k = HD_k$ and $\Xi'_k = D_k^+\Xi'$. In this case $p\lim T^{-1}H'_kH_k$ is rank deficient.

**Supplementary Results:** Suppose that the true model is given by

$$X = K\Theta' + e = GT' + FA' + e,$$

where $Z$ is $T \times N$, $G$ is $T \times s$, and $F$ is $T \times r$. Assume that $T^{-1}G'F = O_p(T^{-1/2})$.  

21
Condition 1 The common factor estimator $\hat{G}$ is such that

$$c_{NT}^2 \left( \frac{1}{T} \sum_{t=1}^{T} \left\| \hat{G}_t - G_t \right\|^2 \right) = O_p(1) \quad \text{(A.5)}$$

and

$$(T^{-1} \hat{G}' \hat{G})^{-1} = O_p(1). \quad \text{(A.6)}$$

Condition 2 The common factor estimator $\hat{F}$ is such that

$$\frac{1}{T} \text{tr} \left[ (F - \hat{F})'(F - \hat{F}) \right] = O_p(\delta_{NT}^2)$$

and

$$\frac{1}{NT} \text{tr} \left[ (\hat{\Lambda}' - \Lambda')' \hat{F}' \hat{F} (\hat{\Lambda}' - \Lambda') \right] = O_p(\delta_{NT}^2).$$

Let $\hat{F}$ be the eigenvectors of $(NT)^{-1} \hat{M}_GXX' \hat{M}_G$ corresponding to the $k$ largest eigenvalues with the usual normalization $T^{-1}\hat{F}' \hat{F} = I$ and $\hat{\Lambda}' = T^{-1}\hat{F}' \hat{M}_G X$, and let $\hat{F} = N^{-1} \hat{M}_G X \hat{\Lambda}'$ and $\hat{\Lambda}' = (\hat{F}' \hat{F})^{-1} \hat{F}' \hat{M}_G X$. Then, $\hat{F} = \hat{F} V_{NT}$ and $\hat{\Lambda}' = V_{NT}^{-1} \hat{\Lambda}'$ where $V_{NT}$ is a diagonal matrix of the eigenvalues.

While each column of $\hat{F}$ is an eigenvector of $(NT)^{-1} \hat{M}_GXX' \hat{M}_G$ corresponding to a strictly positive eigenvalue, each column of $\hat{G}$ is an eigenvector of the same matrix corresponding to zero eigenvalue. Hence, we have

$$\hat{G}' \hat{F} = 0. \quad \text{(A.7)}$$

The next lemma shows that $\hat{F}$ satisfies Condition 2 after an appropriate reparameterization.

Lemma A.2 Suppose that $\hat{G}$ satisfies Condition 1. Then, we have the following results.

(i) There exists an $r \times k$ matrix $D_k$ such that

$$\frac{1}{T} \sum_{t=1}^{T} \left\| \hat{F}_t - D_k F_t \right\|^2 = O_p(\delta_{NT}^2) + O_p(c_{NT}^2),$$
where $F_t$ is the $t^{th}$ row of $F$ and $\|D_k\| = O_p(1)$.

(ii) If $r > k$, write

$$FN' = F_k \Lambda'_k + F_{r-k} \Lambda'_{r-k},$$

by applying the orthogonal reparameterization in (A.2) to $FN'$ with respect to $D_k$. If $r \leq k$, write $F_k = FD_k$ and $\Lambda'_k = D_k^+ \Lambda'$ by (A.3). Then, it follows that

$$\frac{1}{NT} tr \left[ (\hat{\Lambda'} - \Lambda'_k)' \hat{F}' \hat{F} (\hat{\Lambda'} - \Lambda'_k) \right] = O_p(\delta_{NT}^2 + O_p(c_{NT}^2),$$

where $\hat{\Lambda'} = (\hat{F}' \hat{F})^{-1} \hat{F} \hat{M}_G \hat{X}$.

Now, we estimate common factors from $\hat{X} = \hat{F} \hat{X}'$ where $\hat{F}$ and $\hat{X}'$ satisfy Condition 2. Let $\hat{G}^{(2)}$ collect the eigenvectors of $(NT)^{-1} \hat{X} \hat{X}'$ corresponding to the $k$ largest eigenvalues with the usual normalization $T^{-1} \hat{G}^{(2)} \hat{G}^{(2)*} = I$ and $\hat{\Gamma}' = T^{-1} \hat{G}^{(2)*} \hat{X}$, and let $\hat{G}^{(2)} = N^{-1} \hat{X} \hat{\Gamma}'$ and $\hat{\Gamma}' = (\hat{G}^{(2)*} \hat{G}^{(2)})^{-1} \hat{G}^{(2)} \hat{X}$.

**Lemma A.3** Suppose that $\hat{F}$ and $\hat{\Lambda}'$ satisfy Condition 2. Then, we have the following results.

(i) There exists an $s \times u$ matrix $D_u$ such that

$$\delta_{NT}^2 \left( \frac{1}{T} \sum_{t=1}^{T} \left\| \hat{G}_t^{(2)} - D_u^* G_t \right\|^2 \right) = O_p(1),$$

where $G_t$ is the $t^{th}$ row of $G$ and $\|D_u\| = O_p(1)$.

(ii) If $s > u$, write

$$GT' = G_u^* \Gamma_u' + G_{s-u} \Gamma_{s-u},$$

by applying the orthogonal reparameterization in (A.2) to $GT'$ with respect to $D_u$. If $s \leq u$, write $G_u = GD_u$ and $\Gamma_u = D_u^+ \Gamma'$ by (A.3). Then, it follows that

$$\frac{1}{NT} tr \left[ (\hat{\Gamma}' - \Gamma_u')' \hat{G}^{(2)*} \hat{G}^{(2)} (\hat{\Gamma}' - \Gamma_u') \right] = O_p(\delta_{NT}^2).$$

where $\hat{\Gamma}' = (\hat{G}^{(2)*} \hat{G}^{(2)})^{-1} \hat{G}^{(2)*} \hat{X}$. 
The proofs for Lemmas A.2 and A.3 are postponed after the proof of Proposition 6.

**Proof of Proposition 6:** First note that

\[
PC(u, k_1, ..., k_M) - PC(s, r_1, ..., r_M) = \sum_{m=1}^{M} \frac{N_m}{N} \left[ V_m(u, k_1, ..., k_M) - V_m(s, r_1, ..., r_M) \right] + \sum_{m=1}^{M} (k_m - r_m) \cdot g_m(N, T) + (u - s) \cdot p(N, T).
\]

In the above equation, we are using the fact that

\[
V_{NT}(u, k_1, ..., k_M) = \sum_{m=1}^{M} \left( \frac{N_m}{N} \right) V_m(u, k_1, ..., k_M)
\]

and

\[
V_m(u, k_1, ..., k_M) = \frac{1}{NmT} \sum_{t=1}^{T} \hat{e}_{mt} \hat{e}_{mt}.
\]

\(V_m(u, k_1, ..., k_M)\) measures the goodness of fit in region \(m\) only. For each region \(m\), we may write

\[
V_m(u, k_1, ..., k_M) = \frac{1}{NmT} tr \left[ \left( X_m - \hat{G}^{(2)} \hat{\Gamma}^{(1)r}_m \right)' M \hat{\Gamma}^{(2)} \left( X_m - \hat{G}^{(2)} \hat{\Gamma}^{(1)r}_m \right) \right].
\]

Since \(\hat{\Gamma}^{(1)}\) is obtained from \(\hat{K}_1\) and \(\hat{K}_2\) with correctly specified \(s + r_1\) and \(s + r_2\), the asymptotic behavior of \(\hat{\Gamma}^{(1)}\) is as described in Proposition 2. If \(u \leq s\), \(\hat{G}^{(1)}\) consistently estimates \(GD_u\) for some \(D_u\). If \(u > s\), partition \(\hat{G}^{(1)} = [\hat{G}^{(1)}_s, \hat{G}^{(1)}_{u-s}]\) where \(\hat{G}^{(1)}_s\) gets the first \(s\) columns of \(\hat{G}^{(1)}\) and \(\hat{G}^{(1)}_{u-s}\) the rest. Then, \(\hat{G}^{(1)}_s\) consistently estimates \(GD_s\) for some square matrix \(D_s\) and \(\hat{G}^{(1)}_{u-s}\) consistently estimates \(F_1 D_{u-s}\) for some matrix \(D_{u-s}\).

When \(u \leq s\), we obtain by applying the orthogonal reparameterization to \(GT'\) with respect to \(D_u\)

\[
GT'_m = G_{u} \Gamma'_{m,u} + G_{s-u} \Gamma'_{m,s-u}
\]
and write the model as

\[ X_m = K_m \Theta'_m + e_m \]
\[ = G \Gamma'_m + F_m \Lambda'_m + e_m \]
\[ = G_u \Gamma'_{m,u} + F_{m,u} \Lambda'_{m,u} + e_m \]

where \( G_u = GD_u, F_{m,u} = [G_{s-u}, F_m] \) and \( \Lambda_{m,u} = [\Gamma_{m,s-u}, \Lambda_m] \). Thus, \( G_u \) collects \( u \) common factors and \( F_{m,u} \) has \( r_m = r_m + (s - u) \) common factors.

When \( u > s \), we obtain by applying the orthogonal reparameterization to \( F_1 \Lambda'_1 \) with respect to \( D_{u-s} \)

\[ F_1 \Lambda'_1 = F_{1,u-s} \Lambda'_{u-s} + F_{1,u} \Lambda'_u, \]

and write the model as, for region 1,

\[ X_1 = G \Gamma'_1 + F_1 \Lambda'_1 + e_1 \]
\[ = [GD_s, F_{1,u-s}] \left( D_{s}^{-1} \Gamma'_1 \right)_{(u-s) \times N_m} + F_{1,u} \Lambda'_u + e_1 \]
\[ = G_u \Gamma'_{1,u} + F_{1,u} \Lambda'_{1,u} + e_1, \]

and for the other regions as

\[ X_m = G \Gamma'_m + F_m \Lambda'_m + e_m \]
\[ = [GD_s, F_{1,u-s}] \left( D_{s}^{-1} \Gamma'_m \right)_{(u-s) \times N_m} + F_m \Lambda'_m + e_m \]
\[ = G_u \Gamma'_{m,u} + F_{m,u} \Lambda'_{m,u} + e_m, \]

where \( F_{m,u} = F_m \) and \( \Lambda'_{m,u} = \Lambda'_m \) for \( m = 2, \ldots, M \). Hence, \( G_u \) collects \( u \) common factors and \( F_{m,u} \) has \( \tilde{r}_m \) factors where \( \tilde{r}_1 = r_1 + (s - u) \) and \( \tilde{r}_m = r_m \).

The model is now written in such a way that there are \( u \) global factors \( G_u \) and \( \tilde{r}_m \) regional factors \( F_{m,u} \). A useful decomposition to be used is given by

\[ PC(u, k_1, \ldots, k_M) - PC(s, r_1, \ldots, r_M) \]
\[ = [PC(u, k_1, \ldots, k_M) - PC(u, \tilde{r}_1, \ldots, \tilde{r}_M)] \]
\[ + [PC(u, \tilde{r}_1, \ldots, \tilde{r}_M) - PC(s, r_1, \ldots, r_M)]. \]
Since \( \hat{G}^{(1)} \) satisfies Condition 1 with respect to \( G_u \), Lemma A.2 characterizes the asymptotic behavior of \( \hat{F}_m^{(1)} \) and \( \hat{\lambda}_m^{(1)} \) in Step 2.

Note that

\[
PC(u, k_1, \ldots, k_M) - PC(u, \tilde{r}_1, \ldots, \tilde{r}_M) = \sum \frac{N_m}{N} [V_m(u, k_1, \ldots, k_M) - V_m(u, \tilde{r}_1, \ldots, \tilde{r}_M)] + \sum (k_m - r_m) \cdot g_m(N, T)
\]

(1) Suppose that \( k_m \geq \tilde{r}_m \) for all \( m \) in Step 2. Then, \( \hat{F}_m^{(1)} \) and \( \hat{\lambda}_m^{(1)} \) consistently estimate \( F_{m,uk} = F_{m,u}D_{k_m} \) and \( \Lambda'_{m,uk} = D_{k_m}^+ \Lambda'_{m,u} \) for some \( D_{k_m} \) in the sense that they satisfy the results in Lemma A.2 with respect to \( F_{m,uk} \) and \( \Lambda'_{m,uk} \). Thus, for all \( m \),

\[
\frac{1}{T} tr \left[ (\hat{F}_m^{(1)} - F_{m,uk})'(\hat{F}_m^{(1)} - F_{m,uk}) \right] = O_p(\kappa_{N_mT}^{-2})
\]

and

\[
\frac{1}{N_mT} tr \left[ (\hat{\lambda}_{m}^{(1)})' - \Lambda'_{m,uk} \right)'\hat{F}_m^{(1)}(\hat{\lambda}_{m}^{(1)})' - \Lambda'_{m,uk} \right] = O_p(\kappa_{N_mT}^{-2}).
\]

Stack \( \hat{F}_m^{(1)} \) and \( \hat{\lambda}_m^{(1)} \) as

(\ref{eq:A.8})

and the true common components as

(\ref{eq:A.9})

26
Then, we can show \( \hat{F} \) and \( \hat{\Lambda} \) satisfy Condition 2 with respect to \( F \) and \( \Lambda' \).

\[
\frac{1}{T} tr \left[ (\hat{F} - F)'(\hat{F} - F) \right] \\
= \frac{1}{T} tr \left[ (\hat{F} - F)'(\hat{F} - F) \right] \\
= \sum_{m=1}^{M} \frac{N_m}{N} tr \left[ (\hat{F}_m(1) - F_{m,uk})'(\hat{F}_m(1) - F_{m,uk}) \right] \\
= \sum_{m=1}^{M} \frac{N_m}{N} O_p(\kappa_{N_mT}^{-2}) \\
= O_p(\delta_{NT}^{-2}),
\]

and

\[
\frac{1}{NT} tr \left[ (\hat{\Lambda}' - \Lambda')'(\hat{F}')(\hat{\Lambda}' - \Lambda') \right] \\
= \sum_{m=1}^{M} \frac{1}{NT} tr \left[ (\hat{\Lambda}_m(1)' - \Lambda_{m,uk}')'(\hat{F}_m(1))'(\hat{\Lambda}_m(1)' - \Lambda_{m,uk}') \right] \\
= \sum_{m=1}^{M} \frac{N_m}{N} O_p(\kappa_{N_mT}^{-2}) \\
= O_p(\delta_{NT}^{-2}).
\]

Hence, \( \hat{G}^{(2)} \) and \( [\hat{\Gamma}(1)'_1, \ldots, \hat{\Gamma}(1)'_M] \), which are obtained from \( \hat{X} = [\hat{X}_1, \ldots, \hat{X}_M] \) with \( \hat{X}_m = X_m - \hat{F}_m(1)\hat{\Lambda}_m(1)' \), will satisfy Lemma A.3. Finally, \( \hat{F}_m^{(2)} \) and \( \hat{\Lambda}_m^{(2)}' \), which are obtained from \( X_m - \hat{G}^{(2)}\hat{\Gamma}_m(1)' \), will also satisfy Lemma A.3 because \( \hat{G}^{(2)} \) and \( \hat{\Gamma}_m(1)' \) satisfy Condition 2. Once the asymptotic properties of \( \hat{F}_m^{(2)} \) and \( \hat{\Lambda}_m^{(2)}' \) are given by Lemma A.3, we can directly apply Corollary 2 in Bai and Ng (2002) to obtain

\[
\frac{N_m}{N} [V_m(u, k_1, \ldots, k_M) - V_m(u, \bar{r}_1, \ldots, \bar{r}_M)] = O_p(\delta_{NT}^{-2}),
\]

for all \( m \). This implies that

\[
\delta_{NT}^2 [PC(u, k_1, \ldots, k_M) - PC(u, \bar{r}_1, \ldots, \bar{r}_M)] \\
= O_p(1) + \delta_{NT}^2 \sum_{m=1}^{M} (k_m - \bar{r}_m) \cdot g_m(N, T) \to \infty \quad (A.10)
\]

27
by the condition (ii) in the proposition.

(2) Suppose that there is at least one region under-fitting the number of factors in Step 2, say region \( m \), so that \( k_m < \tilde{r}_m \). \( \hat{F}_m^{(1)} \) will satisfy Lemma A.2 (i) with respect to some matrix \( D_{k_m} \). Using the orthogonal reparameterization of \( F_{m,u} \Lambda_{m,u}' \) with respect to \( D_{k_m} \), the model for region \( m \) is given by

\[
X_m = G_{u} \Gamma_{m,u}' + F_{m,u} \Lambda_{m,u}' + e_m
\]

\[
= G_{u} \Gamma_{m,u}' + F_{m,uk} \Lambda_{m,uk}' + F_{m,ur-k} \Lambda_{m,ur-k}' + e_m
\]

\[
= [G_{u}, F_{m,ur-k}'] \left( \begin{array}{c} \Gamma_{m,u}' \\ \Lambda_{m,ur-k}' \end{array} \right) + F_{m,uk} \Lambda_{m,uk}' + e_m
\]

\[
= G_{uk} \Gamma_{m,uk}' + F_{m,uk} \Lambda_{m,uk}' + e_m
\]

where \( G_{uk} \) has \( u + \tilde{r}_m - k_m \) common factors, \( F_{m,uk} = F_{m,u} D_{k_m} \) has \( k_m \) common factors and \( F_{m,ur-k} \) collects the remaining \( \tilde{r}_m - k_m \) common factors. \( \hat{F}_m^{(1)} \) and \( \hat{\Lambda}_{m}' \) also satisfy the results in Lemma A.2 (ii) with respect to \( F_{m,uk} \) and \( \Lambda_{m,uk}' \). The \( \tilde{r}_m - k_m \) common factors collected in \( F_{m,ur-k} \) carry over into Step 3 as a part of \( G_{uk} \). For Step 3, we can stack \( \hat{F}_m^{(1)} \) and \( \hat{\Lambda}_{m}' \) as well as \( F_{m,uk} \) and \( \Lambda_{m,uk}' \) to define \( \hat{F}', \hat{\Lambda}', F, \) and \( \Lambda' \), similarly to (A.8) and (A.9). Then, Condition 2 will be met by \( \hat{F} \) and \( \hat{\Lambda} \) with respect to \( F \) and \( \Lambda' \), and \( \hat{G}^{(2)} \) will satisfy Lemma A.3 with respect to \( G_{uk} \). This means that the number of factors appearing in Step 3 will increase by \( \tilde{r}_m - k_m \). However, we estimate only \( u \) factors by \( \hat{G}^{(2)} \) in Step 3. In region \( m \), we can write

\[
X_m = G_{uk} \Gamma_{m,uk}' + F_{m,uk} \Lambda_{m,uk}' + e_m
\]

\[
= G_{uku} \Gamma_{m,uku}' + G_{uk-u} \Gamma_{m,uk-u}' + F_{m,uk} \Lambda_{m,uk}' + e_m
\]

where \( G_{uku} \) and \( \Gamma_{m,uku}' \) are obtained by applying the orthogonal reparameterization with respect to a relevant rotation matrix. \( G_{uk-u} \) is consistently estimated by \( \hat{G}^{(2)} \) and \( \Gamma_{m,uk-u}' \) by \( \hat{\Gamma}_{m,uk-u}' \), \( G_{uk-u} \) collects \( \tilde{r}_m - k_m \) common factors. Hence, region \( m \) gets \( \tilde{r}_m (= \tilde{r}_m - k_m + k_m) \) factors in Step 4 and \( \hat{F}_m^{(2)} \) will estimate only \( k_m < \tilde{r}_m \) common factors again. Then, using the result in Bai and Ng (2002), we get

\[
V_m(u, k_1, ..., k_M) - V_m(u, \tilde{r}_1, ..., \tilde{r}_M) \rightarrow \tau_m > 0.
\]
This implies that
\[ PC(u, k_1, \ldots, k_M) - PC(u, \bar{r}_1, \ldots, \bar{r}_M) \to \tau_m > 0. \] (A.11)

(3) The case where we estimate \( \bar{r}_m \) factors for region \( m \) means that we are estimating the true number of factors. Hence,
\[
\frac{N_m}{N} [V_m(u, \bar{r}_1, \ldots, \bar{r}_M) - V_m(s, r_1, \ldots, r_M)] = O_p(\delta_{NT}^{-2}).
\]

This implies that
\[
\delta_{NT}^2 [PC(u, \bar{r}_1, \ldots, \bar{r}_M) - PC(s, r_1, \ldots, r_M)]
= O_p(1) + \delta_{NT}^2 \sum_{m=1}^{M} (\bar{r}_m - r_m) \cdot g_m(N, T) + \delta_{NT}^2 (u - s) \cdot p(N, T).
\]

When \( u > s \), we have \( \bar{r}_1 = r_1 + (s - u) \) and \( \bar{r}_m = r_m \) for \( m = 2, \ldots, M \). Thus it follows that
\[
\delta_{NT}^2 \sum_{m=1}^{M} (\bar{r}_m - r_m) \cdot g_m(N, T) + \delta_{NT}^2 (u - s) \cdot p(N, T)
= \delta_{NT}^2 (u - s) [p(N, T) - g_1(N, T)] \to \infty
\] (A.12)

by the condition (iii) in the proposition. When \( u < s \), \( \bar{r}_m = r_m + (s - u) \). Thus,
\[
\delta_{NT}^2 \sum_{m=1}^{M} (\bar{r}_m - r_m) \cdot g_m(N, T) + \delta_{NT}^2 (u - s) \cdot p(N, T)
= \delta_{NT}^2 (s - u) \left[ \sum_{m=1}^{M} g_m(N, T) - p(N, T) \right] \to \infty
\] (A.13)

by the condition (iv) in the proposition.

Combining (A.10)~(A.13) completes the proof.
Proof of Lemma A.2: (i) Write
\[
\hat{F} = \frac{1}{NT} \hat{M}_G XX' \hat{M}_G \hat{F}
\]
\[
= \frac{1}{NT} \hat{M}_G (K\Theta' + e)(K\Theta' + e)' \hat{M}_G \hat{F}
\]
\[
= \frac{1}{NT} \hat{M}_G (K\Theta'K' + eK' + K\Theta'e' + ee') \hat{M}_G \hat{F}
\]
\[
= \hat{M}_G K \left( \frac{1}{N} \Theta' \Theta \right) \left( \frac{1}{T} K' \hat{F} \right) + r
\]
with
\[
r = \frac{1}{NT} \hat{M}_G (eK' + K\Theta'e' + ee') \hat{M}_G \hat{F}.
\]
Let
\[
\left( \frac{1}{N} \Theta' \Theta \right) \left( \frac{1}{T} K' \hat{F} \right) = \left( \frac{L_k}{D_k} \right),
\]
where \(||D_k|| = O_p(1)\) and \(||L_k|| = O_p(1)\), since \(||N^{-1} \Theta' \Theta|| = O_p(1)\) and \(||T^{-1} K' \hat{F}|| = O_p(1)\). Using this partition,
\[
\hat{F} = \hat{M}_G(G,F) \left( \frac{L_k}{D_k} \right) + r
\]
\[
= \hat{M}_G GL_k + \hat{M}_G FD_k + r
\]
\[
= FD_k + \hat{M}_G GL_k - \hat{P}_G FD_k + r. \quad \text{(A.14)}
\]
The second term in (A.14) is such that
\[
\frac{1}{T} tr \left[ L_k' G' \hat{M}_G GL_k \right] = tr \left[ L_k' \left( \frac{1}{T} (G - \hat{G})' \hat{M}_G (G - \hat{G}) \right) L_k \right]
\]
\[
\leq tr \left[ \frac{1}{T} (G - \hat{G})' (G - \hat{G}) \right] tr [L_k L_k']
\]
\[
= O_p(c_{NT}^{-2}). \quad \text{(A.15)}
\]
The third term in (A.14) is such that
\[
\frac{1}{T} tr \left[ D_k' F' \hat{P}_G FD_k \right] = tr \left[ D_k' \left( \frac{1}{T} F' \hat{G} \right) \left( \frac{1}{T} \hat{G} \hat{G}' \right)^{-1} \left( \frac{1}{T} \hat{G}' \hat{F} \right) D_k \right]
\]
\[
= O_p(c_{NT}^{-2}), \quad \text{(A.16)}
\]
which holds because \( \|D_k\| = O_p(1) \), \((T^{-1}\hat{G}'\hat{G})^{-1} = O_p(1)\), and

\[
\frac{1}{T}\hat{G}'F_k = \frac{1}{T}(\hat{G} - G)'F + \frac{1}{T}G'F = O_p(c_{NT}^{-1}) + O_p(T^{-1/2}).
\]

The last term in (A.14) is such that

\[
\frac{1}{T}tr [r'r] = \frac{1}{NT}tr \left[ \tilde{F}'(e\Theta K' + K\Theta'e' + ee')(e\Theta K' + K\Theta'e' + ee')\tilde{F} \right]
= O_p(\delta_{NT}^{-2}) \tag{A.17}
\]

from Bai and Ng (2002). Combining (A.15)~(A.17) gives

\[
\frac{1}{T} \sum_{t=1}^{T} \left\| \tilde{F}_t - D_k'F_t \right\|^2 \\
= \frac{1}{T}tr \left[ (\hat{F} - FD_k)'(\hat{F} - FD_k) \right] \\
\leq \frac{4}{T}tr \left[ L_k'G'M_GGL_k \right] + \frac{4}{T}tr \left[ D_k'F'\hat{P}_GFD_k \right] + \frac{4}{T}tr [r'r] \\
= O_p(\delta_{NT}^{-2}) + O_p(c_{NT}^{-2}).
\]

(ii) Write

\[
\hat{F}\hat{\Lambda}' = \hat{F}(\hat{F}'\hat{F})^{-1}\hat{F}'\hat{M}_GX \\
= \hat{P}_F GT' + \hat{P}_F FA' + \hat{P}_F e \\
= \hat{F}\hat{\Lambda}_k' + \hat{P}_F GT' + \hat{P}_F (F_k - \hat{F})\Lambda_k' + \hat{P}_F F_{r-k}\Lambda_{r-k}' + \hat{P}_F e
\]

where \( \hat{P}_F F_{r-k}\Lambda_{r-k}' \) exsits only when \( r > k \). Hence,

\[
\hat{F}(\hat{\Lambda}' - \Lambda_k') = \hat{P}_F GT' + \hat{P}_F (F_k - \hat{F})\Lambda_k' + \hat{P}_F F_{r-k}\Lambda_{r-k}' + \hat{P}_F e \tag{A.18}
\]

For the first term in (A.18), recall that \( \hat{F}'\hat{G} = 0 \). Then, it follows that

\[
\frac{1}{NT}tr \left[ \Gamma G'\hat{P}_F GT' \right] = \frac{1}{NT}tr \left[ \Gamma(G - \hat{G})'\hat{F}_F(G - \hat{G})\Gamma \right] \\
\leq tr \left[ \left( \frac{1}{T}(G - \hat{G})'(G - \hat{G}) \right) \left( \frac{1}{N}\Gamma\Gamma \right) \right] \\
= O_p(c_{NT}^{-2}). \tag{A.19}
\]
For the second term in (A.18), under Condition 1,

\[
\frac{1}{NT} \text{tr} \left[ \Lambda_k(F_k - \hat{F})\hat{P}_F(F_k - \hat{F})\Lambda_k \right] \\
\leq \text{tr} \left[ \left( \frac{1}{T}(F_k - \hat{F})'(F_k - \hat{F}) \right) \left( \frac{1}{N} \Lambda_k' \Lambda_k \right) \right] = O_p(c_{NT}^2). \tag{A.20}
\]

For the third term in (A.18), which exists only when \( r > k \),

\[
\frac{1}{NT} \text{tr} \left[ \Lambda_{r-k} F_{r-k}' \hat{P}_F F_{r-k} \Lambda_{r-k}' \right] \\
\leq \text{tr} \left[ \left( \frac{1}{T} F_{r-k}' \hat{P}_F F_{r-k} \right) \left( \frac{1}{N} \Lambda_{r-k}' \Lambda_{r-k} \right) \right] = O_p(\delta_{NT}^{-2}) + O_p(c_{NT}^{-2}), \tag{A.21}
\]

which holds because

\[
\frac{1}{T} F_{r-k}' \hat{P}_F F_{r-k} = \frac{1}{T} F_{r-k}' \hat{F} \left( \frac{1}{T} \hat{F}' \hat{F} \right)^{-1} \frac{1}{T} \hat{F}' F_{r-k},
\]

\((T^{-1} \hat{F}' \hat{F})^{-1} = O_p(1)\), and under Condition 1,

\[
\frac{1}{T} \hat{F}' F_{r-k} = \frac{1}{T} (\hat{F} - F_k)' F_{r-k} = O_p(\delta_{NT}^{-1}) + O_p(c_{NT}^{-1}).
\]

For the last term in (A.18), from Bai and Ng (2002),

\[
\frac{1}{NT} \text{tr} \left[ e' \hat{P}_F e \right] = O_p(\delta_{NT}^{-2}). \tag{A.22}
\]

Combining (A.19)~(A.22) gives that

\[
\frac{1}{NT} \text{tr} \left[ \left( \Lambda' - \Lambda_k' \right) \hat{F}' \hat{F} \left( \Lambda' - \Lambda_k' \right) \right] = O_p(\delta_{NT}^{-2}) + O_p(c_{NT}^{-2}). \tag*{\blacksquare}
\]

**Proof of Lemma A.3:** (i) Write

\[
\bar{X} = GT' + FN' - \hat{F} \hat{\Lambda}' + e \\
= GT' + (F - \hat{F}) \Lambda' + \hat{F} (\Lambda' - \hat{\Lambda}') + e \\
= \begin{bmatrix} G & F - \hat{F} \end{bmatrix} \begin{bmatrix} \Gamma' \\ \Lambda' \\ \Lambda' - \hat{\Lambda}' \end{bmatrix} + e \\
= GT' + e
\]

32
and then
\[
\hat{G}^{(2)} = \frac{1}{NT} \hat{X} \hat{X}' \hat{G}^{(2)} = \frac{1}{NT} \tilde{G} \tilde{T} \tilde{G}' \hat{G}^{(2)} + r^{(2)}
\]
\[
= \frac{1}{NT} G \tilde{T} \tilde{G}' \hat{G}^{(2)} + \frac{1}{NT} (F - \hat{F}) \Lambda \tilde{T} \tilde{G}' \hat{G}^{(2)} + r^{(2)}
\]
\[
= GD_u + (F - \hat{F}) L_u + \frac{1}{NT} \hat{F} (\Lambda' - \hat{\Lambda}') \tilde{G}' \hat{G}^{(2)} + r^{(2)}
\]
with
\[
D_u = \left( \frac{1}{N} \Gamma' T \right) \left( \frac{1}{T} G' \hat{G}^{(2)} \right)
\]
\[
L_u = \left( \frac{1}{N} \Lambda' \tilde{T} \right) \left( \frac{1}{T} \tilde{G}' \hat{G}^{(2)} \right)
\]
\[
r^{(2)} = \frac{1}{NT} (\tilde{G} \tilde{T} e' + e \tilde{G}' + e e') \hat{G}^{(2)}.
\]
Note that \( || D_u || = O_p(1) \) and \( || L_u || = O_p(1) \). Thus,
\[
\frac{1}{T} \text{tr} \left[ L_u' (F - \hat{F})' (F - \hat{F}) L_u \right] = O_p(\delta^{-2}_{NT})
\]
\[
\frac{1}{N^2 T^3} \text{tr} \left[ \hat{G}^{(2)' \hat{G}^{(2)}} (\Lambda' - \hat{\Lambda}')' \hat{F}' \hat{F} (\Lambda' - \hat{\Lambda}') \tilde{G}' \hat{G}^{(2)} \right]
\]
\[
\leq \frac{1}{T} \text{tr} \left[ (\Lambda' - \hat{\Lambda}')' \hat{F}' \hat{F} (\Lambda' - \hat{\Lambda}') \right] \text{tr} \left[ \left( \frac{1}{T} \tilde{G}' \hat{G}^{(2)} \right) \left( \frac{1}{N} \tilde{T} \tilde{T} \right) \left( \frac{1}{T} \tilde{G}' \hat{G}^{(2)} \right) \right]
\]
\[
= O_p(\delta^{-2}_{NT})
\]
and
\[
\frac{1}{T} \text{tr} \left[ r^{(2)' r^{(2)}} \right] = O_p(\delta^{-2}_{NT}).
\]
Therefore,
\[
\frac{1}{T} \sum_{t=1}^{T} \left\| \hat{G}_t^{(2)} - D'_u G_t \right\|^2 = \frac{1}{T} \text{tr} \left[ (\hat{G}^{(2)} - GD_u)' (\hat{G}^{(2)} - GD_u) \right]
\]
\[
= O_p(\delta^{-2}_{NT}).
\]
(ii) Since $\hat{\Gamma}' = (\hat{G}^{(2)})^{-1}(\hat{G}^{(2)})^\dagger \bar{X}$ and $\bar{X} = G_u \Gamma'_u + G_{s-u} \Gamma'_{s-u} + (F - \hat{F})\Lambda' + \hat{F}(\Lambda' - \hat{\Lambda}') + e$, write similarly to (A.18)

$$\hat{G}^{(2)}(\hat{\Gamma}' - \Gamma'_u) = \hat{P}'_G (G_u - \hat{G}^{(2)})\Gamma'_u + \hat{P}'_G G_{s-u} \Gamma'_{s-u}$$
$$\hspace{1cm}+ \hat{P}'_G (F - \hat{F})\Lambda' + \hat{P}'_G \hat{F}(\Lambda' - \hat{\Lambda}') + \hat{P}'_G e.$$

The result in (ii) follows from

$$\frac{1}{NT} tr \left[ \Gamma_u (G_u - \hat{G}^{(2)})' \hat{P}'_G (G_u - \hat{G}^{(2)}) \Gamma'_u \right] \leq tr \left[ \frac{1}{T} (G_u - \hat{G}^{(2)})' (G_u - \hat{G}^{(2)}) \right] tr \left[ \frac{1}{N} \Gamma'_u \Gamma_u \right]$$
$$\hspace{2cm} = O_p(\delta^{-2}_{NT}),$$

$$\frac{1}{NT} tr \left[ \Gamma_{s-u} G_{s-u}' \hat{P}'_G G_{s-u} \Gamma'_{s-u} \right] \leq tr \left[ \frac{1}{T} G_{s-u}' \hat{P}'_G G_{s-u} \right] tr \left[ \frac{1}{N} \Gamma'_{s-u} \Gamma_{s-u} \right]$$
$$\hspace{2cm} = O_p(\delta^{-2}_{NT}),$$

$$\frac{1}{NT} tr \left[ \Lambda (F - \hat{F})' \hat{P}'_G (F - \hat{F}) \Lambda' \right] \leq tr \left[ \frac{1}{T} (F - \hat{F})' (F - \hat{F}) \right] tr \left[ \frac{1}{T} \Lambda' \Lambda \right]$$
$$\hspace{2cm} = O_p(\delta^{-2}_{NT}),$$

$$\frac{1}{NT} tr \left[ (\Lambda - \hat{\Lambda}) \hat{F}' \hat{P}'_G \hat{F} (\Lambda - \hat{\Lambda})' \right] \leq \frac{1}{NT} tr \left[ (\Lambda - \hat{\Lambda}) \hat{F}' \hat{F} (\Lambda - \hat{\Lambda})' \right]$$
$$\hspace{2cm} = O_p(\delta^{-2}_{NT}),$$

and

$$\frac{1}{NT} tr \left[ e' \hat{P}'_G e \right] = O_p(\delta^{-1}_{NT}). \blacksquare$$
Appendix B: Assumptions

In the following, suppose that $M$ is a finite, positive constant.

**Assumption B.1**

(i) $E(e_{mit}) = 0$ for every $m, i$ and $t$; $\sup_{m,t} E|e_{mit}|^2 \leq M$.

(ii) Let $E(e_{ms}e_{mt}/N_m) = \gamma_{m,N_m}(s,t)$. Then, $\sup_{m,N_m} |\gamma_{m,N_m}(s,s)| \leq M$ and $\sup_{m,N_m} \sum_{s=1}^T |\gamma_{m,N_m}(s,t)| \leq M$ for every $t \leq T$.

(iii) Let $E(e_{mit}e_{mt}) = \tau_{i,j,mt}$ with $\sup_t |\tau_{i,j,mt}| \leq \tau_{i,j,m}$ for some $\tau_{i,j,m}$. In addition, $\sup_{m,N_m} \sum_{j=1}^{N_m} |\tau_{i,j,m}| \leq M$ for every $i \leq N_m$.

(iv) Let $E(e_{mit}e_{mjt}) = \tau_{i,j,mts}$. Then, $\sup_{m,N_m,T} (N_mT)^{-1} \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} \sum_{t=1}^T \sum_{s=1}^T |\tau_{i,j,mts}| \leq M$.

(v) $\sup_{t,s,m,N_m} E\left|N_m^{-1/2} \sum_{i=1}^{N_m} e_{mis}e_{mit} - E(e_{mis}e_{mit})\right|^4 \leq M$.

**Assumption B.2** Let $K'_{mt} = (G'_t, F'_{mt})'$.

(i) $\sup_{m,t} E\|K_{mt}\|^4 < \infty$.

(ii) $\left\| \frac{G'_t}{T} - \Sigma_{GG} \right\| \rightarrow^p 0$ and $\sup_{m} \left\| \frac{F'_m}{T} - \Sigma_{F_m} \right\| \rightarrow^p 0$, where $\Sigma_{GG}$ and $\Sigma_{F_m}$ are positive-definite matrices for every $m$.

(iii) $\sup_{m,i} |\gamma_{mi}| \leq \bar{\gamma} < \infty$ and $\sup_{m,i} |\lambda_{mi}| \leq \bar{\lambda} < \infty$, where $\bar{\gamma}$ and $\bar{\lambda}$ are constants.

(iv) $\sup_{m} \left\| \frac{\Gamma'_m}{N_m} - \Sigma_{\Gamma_m} \right\| \rightarrow 0$ and $\sup_{m} \left\| \frac{\Lambda'_m}{N_m} - \Sigma_{\Lambda_m} \right\| \rightarrow 0$, where $\Sigma_{\Gamma_m}$ and $\Sigma_{\Lambda_m}$ are positive-definite matrices.

(v) The eigenvalues of the matrix $\Sigma_{\Gamma_m} \Sigma_{GG}$ are distinct for every $m$, and so are the eigenvalues of the matrix $\Sigma_{\Lambda_m} \Sigma_{F_m}$.

**Assumption B.3** Let $K'_{mt} = (G'_t, F'_{mt})'$.

(i) $\sup_{m,N_m,T} E\left( \frac{1}{N_m} \sum_{i=1}^{N_m} \left| \frac{1}{\sqrt{T}} \sum_{t=1}^T K_{mt}e_{mit} \right| \right) \leq M$.

(ii) $\sup_{t,m,N_m,T} E\left\| \frac{1}{\sqrt{N_mT}} \sum_{s=1}^T \sum_{k=1}^{N_m} K_{ms} [e_{mks}e_{mkt} - E(e_{mks}e_{mkt})] \right\|^2 \leq M$.

(iii) $\sup_{m,N_m,T} E\left\| \frac{1}{\sqrt{N_mT}} \sum_{k=1}^{N_m} \sum_{t=1}^T K_{mt} \chi_{mk}e_{mkt} \right\|^2 \leq M$.
(iv) For any \( t \) and \( i \), as \( T, N_m \to \infty \),

\[
\left( \frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} \lambda_{mit} e_{mit} \quad \frac{1}{T} \sum_{s=1}^{T} K_{ms} e_{mis} \right) \xrightarrow{d} N \left( 0, \begin{pmatrix} \Psi_{mt} & 0 \\ 0 & \Phi_{mi} \end{pmatrix} \right),
\]

where \( \Psi_{mt} = \lim_{N_m \to \infty} \frac{1}{N_m} \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} \lambda_{mi} \lambda_{mj}' E(e_{mit} e_{mjt}) \) and

\( \Phi_{mi} = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} E(K_{mt} K_{ms}' e_{mit} e_{mis}). \)
Table 1: S=1, $\alpha = 0.5, \phi = 0.5$

<table>
<thead>
<tr>
<th>S</th>
<th>M</th>
<th>Nm</th>
<th>T</th>
<th>(\rho)</th>
<th>0</th>
<th>0.5</th>
<th>0.85</th>
<th>0</th>
<th>0.5</th>
<th>0.85</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\beta)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>100</td>
<td>(G^{(1)})</td>
<td>0.9558</td>
<td>0.9555</td>
<td>0.9511</td>
<td>0.9586</td>
<td>0.9582</td>
<td>0.9565</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{G}^{(2)})</td>
<td>0.9874</td>
<td>0.9869</td>
<td>0.9827</td>
<td>0.9878</td>
<td>0.9874</td>
<td>0.9848</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{F}^{(1)})</td>
<td>0.8971</td>
<td>0.8960</td>
<td>0.8730</td>
<td>0.7541</td>
<td>0.7566</td>
<td>0.7488</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{F}^{(2)})</td>
<td>0.9004</td>
<td>0.8994</td>
<td>0.8770</td>
<td>0.7561</td>
<td>0.7584</td>
<td>0.7515</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>200</td>
<td>(G^{(1)})</td>
<td>0.9561</td>
<td>0.9564</td>
<td>0.9541</td>
<td>0.9615</td>
<td>0.9602</td>
<td>0.9612</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{G}^{(2)})</td>
<td>0.9879</td>
<td>0.9877</td>
<td>0.9864</td>
<td>0.9888</td>
<td>0.9884</td>
<td>0.9875</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{F}^{(1)})</td>
<td>0.9051</td>
<td>0.9049</td>
<td>0.8953</td>
<td>0.7588</td>
<td>0.7596</td>
<td>0.7553</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{F}^{(2)})</td>
<td>0.9081</td>
<td>0.9079</td>
<td>0.8989</td>
<td>0.7604</td>
<td>0.7616</td>
<td>0.7574</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>100</td>
<td>(G^{(1)})</td>
<td>0.9911</td>
<td>0.9908</td>
<td>0.9883</td>
<td>0.9901</td>
<td>0.9897</td>
<td>0.9874</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{G}^{(2)})</td>
<td>0.9971</td>
<td>0.9969</td>
<td>0.9951</td>
<td>0.9969</td>
<td>0.9966</td>
<td>0.9948</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{F}^{(1)})</td>
<td>0.9508</td>
<td>0.9502</td>
<td>0.9344</td>
<td>0.9146</td>
<td>0.9127</td>
<td>0.8820</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{F}^{(2)})</td>
<td>0.9511</td>
<td>0.9505</td>
<td>0.9349</td>
<td>0.9149</td>
<td>0.9131</td>
<td>0.8826</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>200</td>
<td>(G^{(1)})</td>
<td>0.9912</td>
<td>0.9912</td>
<td>0.9903</td>
<td>0.9908</td>
<td>0.9907</td>
<td>0.9897</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{G}^{(2)})</td>
<td>0.9973</td>
<td>0.9972</td>
<td>0.9967</td>
<td>0.9972</td>
<td>0.9971</td>
<td>0.9965</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{F}^{(1)})</td>
<td>0.9590</td>
<td>0.9588</td>
<td>0.9538</td>
<td>0.9277</td>
<td>0.9265</td>
<td>0.9116</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{F}^{(2)})</td>
<td>0.9592</td>
<td>0.9590</td>
<td>0.9541</td>
<td>0.9280</td>
<td>0.9268</td>
<td>0.9119</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>100</td>
<td>(G^{(1)})</td>
<td>0.9645</td>
<td>0.9650</td>
<td>0.9623</td>
<td>0.9786</td>
<td>0.9787</td>
<td>0.9779</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{G}^{(2)})</td>
<td>0.9959</td>
<td>0.9955</td>
<td>0.9924</td>
<td>0.9972</td>
<td>0.9970</td>
<td>0.9959</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{F}^{(1)})</td>
<td>0.8679</td>
<td>0.8665</td>
<td>0.8295</td>
<td>0.6910</td>
<td>0.6976</td>
<td>0.7029</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{F}^{(2)})</td>
<td>0.8721</td>
<td>0.8708</td>
<td>0.8349</td>
<td>0.6924</td>
<td>0.6991</td>
<td>0.7046</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>200</td>
<td>(G^{(1)})</td>
<td>0.9653</td>
<td>0.9651</td>
<td>0.9642</td>
<td>0.9788</td>
<td>0.9786</td>
<td>0.9782</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{G}^{(2)})</td>
<td>0.9963</td>
<td>0.9961</td>
<td>0.9951</td>
<td>0.9974</td>
<td>0.9973</td>
<td>0.9969</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{F}^{(1)})</td>
<td>0.8774</td>
<td>0.8767</td>
<td>0.8605</td>
<td>0.6893</td>
<td>0.6932</td>
<td>0.6965</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{F}^{(2)})</td>
<td>0.8812</td>
<td>0.8807</td>
<td>0.8653</td>
<td>0.6907</td>
<td>0.6946</td>
<td>0.6980</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>100</td>
<td>(G^{(1)})</td>
<td>0.9939</td>
<td>0.9939</td>
<td>0.9927</td>
<td>0.9943</td>
<td>0.9942</td>
<td>0.9931</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{G}^{(2)})</td>
<td>0.9995</td>
<td>0.9993</td>
<td>0.9981</td>
<td>0.9995</td>
<td>0.9993</td>
<td>0.9982</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{F}^{(1)})</td>
<td>0.9580</td>
<td>0.9574</td>
<td>0.9430</td>
<td>0.9359</td>
<td>0.9335</td>
<td>0.9034</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{F}^{(2)})</td>
<td>0.9582</td>
<td>0.9576</td>
<td>0.9434</td>
<td>0.9361</td>
<td>0.9337</td>
<td>0.9038</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>200</td>
<td>(G^{(1)})</td>
<td>0.9940</td>
<td>0.9940</td>
<td>0.9938</td>
<td>0.9946</td>
<td>0.9945</td>
<td>0.9941</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{G}^{(2)})</td>
<td>0.9995</td>
<td>0.9995</td>
<td>0.9992</td>
<td>0.9995</td>
<td>0.9995</td>
<td>0.9992</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{F}^{(1)})</td>
<td>0.9662</td>
<td>0.9660</td>
<td>0.9615</td>
<td>0.9475</td>
<td>0.9462</td>
<td>0.9339</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\tilde{F}^{(2)})</td>
<td>0.9664</td>
<td>0.9662</td>
<td>0.9617</td>
<td>0.9477</td>
<td>0.9464</td>
<td>0.9341</td>
</tr>
<tr>
<td>S</td>
<td>M</td>
<td>Nm</td>
<td>T</td>
<td>$\rho$</td>
<td>$\beta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>-----------</td>
<td>----------</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td>0.5</td>
<td>0.85</td>
<td>0</td>
<td>0.5</td>
<td>0.85</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.8991</td>
<td>0.8979</td>
<td>0.8643</td>
<td>0.7293</td>
<td>0.7355</td>
<td>0.7414</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{G}^{(1)}$</td>
<td>0.9735</td>
<td>0.9720</td>
<td>0.9530</td>
<td>0.9238</td>
<td>0.9197</td>
<td>0.8941</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.8735</td>
<td>0.8707</td>
<td>0.8392</td>
<td>0.6768</td>
<td>0.6872</td>
<td>0.6933</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(1)}$</td>
<td>0.8829</td>
<td>0.8804</td>
<td>0.8526</td>
<td>0.7044</td>
<td>0.7123</td>
<td>0.7117</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>200</td>
<td>$G^{(1)}$</td>
<td>0.9009</td>
<td>0.8989</td>
<td>0.8878</td>
<td>0.7265</td>
<td>0.7262</td>
<td>0.7222</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{G}^{(1)}$</td>
<td>0.9752</td>
<td>0.9745</td>
<td>0.9688</td>
<td>0.9371</td>
<td>0.9340</td>
<td>0.9204</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.8909</td>
<td>0.8898</td>
<td>0.8769</td>
<td>0.6973</td>
<td>0.6997</td>
<td>0.7021</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(1)}$</td>
<td>0.8990</td>
<td>0.8983</td>
<td>0.8870</td>
<td>0.7246</td>
<td>0.7252</td>
<td>0.7238</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.9816</td>
<td>0.9809</td>
<td>0.9693</td>
<td>0.9703</td>
<td>0.9688</td>
<td>0.9427</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{G}^{(1)}$</td>
<td>0.9944</td>
<td>0.9938</td>
<td>0.9869</td>
<td>0.9931</td>
<td>0.9920</td>
<td>0.9782</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.9345</td>
<td>0.9334</td>
<td>0.9167</td>
<td>0.8971</td>
<td>0.8948</td>
<td>0.8595</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(1)}$</td>
<td>0.9353</td>
<td>0.9342</td>
<td>0.9182</td>
<td>0.8986</td>
<td>0.8966</td>
<td>0.8638</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>200</td>
<td>$G^{(1)}$</td>
<td>0.9825</td>
<td>0.9823</td>
<td>0.9783</td>
<td>0.9745</td>
<td>0.9734</td>
<td>0.9636</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{G}^{(1)}$</td>
<td>0.9946</td>
<td>0.9944</td>
<td>0.9926</td>
<td>0.9940</td>
<td>0.9936</td>
<td>0.9905</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.9507</td>
<td>0.9504</td>
<td>0.9450</td>
<td>0.9204</td>
<td>0.9185</td>
<td>0.9019</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(1)}$</td>
<td>0.9513</td>
<td>0.9510</td>
<td>0.9458</td>
<td>0.9214</td>
<td>0.9197</td>
<td>0.9040</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.9069</td>
<td>0.9055</td>
<td>0.8738</td>
<td>0.7924</td>
<td>0.8040</td>
<td>0.7983</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{G}^{(1)}$</td>
<td>0.9966</td>
<td>0.9880</td>
<td>0.9666</td>
<td>0.9777</td>
<td>0.9763</td>
<td>0.9569</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.8439</td>
<td>0.8407</td>
<td>0.7939</td>
<td>0.6398</td>
<td>0.6504</td>
<td>0.6548</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(1)}$</td>
<td>0.8579</td>
<td>0.8554</td>
<td>0.8136</td>
<td>0.6646</td>
<td>0.6738</td>
<td>0.6772</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>200</td>
<td>$G^{(1)}$</td>
<td>0.9082</td>
<td>0.9074</td>
<td>0.8965</td>
<td>0.7903</td>
<td>0.7987</td>
<td>0.7974</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{G}^{(1)}$</td>
<td>0.9912</td>
<td>0.9906</td>
<td>0.9858</td>
<td>0.9824</td>
<td>0.9808</td>
<td>0.9764</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.8604</td>
<td>0.8593</td>
<td>0.8382</td>
<td>0.6398</td>
<td>0.6464</td>
<td>0.6516</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(1)}$</td>
<td>0.8728</td>
<td>0.8721</td>
<td>0.8549</td>
<td>0.6642</td>
<td>0.6700</td>
<td>0.6737</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.9870</td>
<td>0.9866</td>
<td>0.9791</td>
<td>0.9840</td>
<td>0.9827</td>
<td>0.9692</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{G}^{(1)}$</td>
<td>0.9989</td>
<td>0.9985</td>
<td>0.9932</td>
<td>0.9988</td>
<td>0.9983</td>
<td>0.9926</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.9420</td>
<td>0.9409</td>
<td>0.9255</td>
<td>0.9197</td>
<td>0.9169</td>
<td>0.8864</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(1)}$</td>
<td>0.9426</td>
<td>0.9415</td>
<td>0.9266</td>
<td>0.9205</td>
<td>0.9178</td>
<td>0.8886</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>200</td>
<td>$G^{(1)}$</td>
<td>0.9871</td>
<td>0.9869</td>
<td>0.9847</td>
<td>0.9853</td>
<td>0.9845</td>
<td>0.9807</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{G}^{(1)}$</td>
<td>0.9990</td>
<td>0.9989</td>
<td>0.9975</td>
<td>0.9989</td>
<td>0.9988</td>
<td>0.9974</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.9580</td>
<td>0.9577</td>
<td>0.9529</td>
<td>0.9392</td>
<td>0.9381</td>
<td>0.9257</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(1)}$</td>
<td>0.9585</td>
<td>0.9582</td>
<td>0.9535</td>
<td>0.9398</td>
<td>0.9386</td>
<td>0.9266</td>
</tr>
</tbody>
</table>
Table 3: \( S=3, \alpha = 0.5, \phi = 0.5 \)

<table>
<thead>
<tr>
<th>S</th>
<th>M</th>
<th>Nm</th>
<th>T</th>
<th>( \rho )</th>
<th>0</th>
<th>0.5</th>
<th>0.85</th>
<th>0</th>
<th>0.5</th>
<th>0.85</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>100</td>
<td>( G^{(1)} )</td>
<td>0.8444</td>
<td>0.8392</td>
<td>0.7690</td>
<td>0.7020</td>
<td>0.7046</td>
<td>0.7008</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{G}^{(2)} )</td>
<td>0.9592</td>
<td>0.9554</td>
<td>0.9057</td>
<td>0.8807</td>
<td>0.8768</td>
<td>0.8401</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{F}^{(1)} )</td>
<td>0.8516</td>
<td>0.8442</td>
<td>0.7954</td>
<td>0.6503</td>
<td>0.6600</td>
<td>0.6717</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{F}^{(2)} )</td>
<td>0.8672</td>
<td>0.8608</td>
<td>0.8177</td>
<td>0.6725</td>
<td>0.6814</td>
<td>0.6924</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>200</td>
<td>( G^{(1)} )</td>
<td>0.8469</td>
<td>0.8461</td>
<td>0.8125</td>
<td>0.6900</td>
<td>0.6889</td>
<td>0.6868</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{G}^{(2)} )</td>
<td>0.9624</td>
<td>0.9613</td>
<td>0.9471</td>
<td>0.9046</td>
<td>0.8987</td>
<td>0.8807</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{F}^{(1)} )</td>
<td>0.8774</td>
<td>0.8751</td>
<td>0.8539</td>
<td>0.6676</td>
<td>0.6742</td>
<td>0.6834</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{F}^{(2)} )</td>
<td>0.8904</td>
<td>0.8885</td>
<td>0.8717</td>
<td>0.6947</td>
<td>0.6999</td>
<td>0.7057</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>100</td>
<td>( G^{(1)} )</td>
<td>0.9734</td>
<td>0.9709</td>
<td>0.9423</td>
<td>0.9339</td>
<td>0.9265</td>
<td>0.8685</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{G}^{(2)} )</td>
<td>0.9915</td>
<td>0.9903</td>
<td>0.9730</td>
<td>0.9856</td>
<td>0.9820</td>
<td>0.9435</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{F}^{(1)} )</td>
<td>0.9180</td>
<td>0.9157</td>
<td>0.8950</td>
<td>0.8733</td>
<td>0.8699</td>
<td>0.8295</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{F}^{(2)} )</td>
<td>0.9193</td>
<td>0.9172</td>
<td>0.8983</td>
<td>0.8796</td>
<td>0.8767</td>
<td>0.8400</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>200</td>
<td>( G^{(1)} )</td>
<td>0.9738</td>
<td>0.9735</td>
<td>0.9638</td>
<td>0.9425</td>
<td>0.9407</td>
<td>0.9114</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{G}^{(2)} )</td>
<td>0.9919</td>
<td>0.9916</td>
<td>0.9874</td>
<td>0.9893</td>
<td>0.9883</td>
<td>0.9787</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{F}^{(1)} )</td>
<td>0.9419</td>
<td>0.9414</td>
<td>0.9352</td>
<td>0.9073</td>
<td>0.9058</td>
<td>0.9841</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{F}^{(2)} )</td>
<td>0.9429</td>
<td>0.9424</td>
<td>0.9367</td>
<td>0.9113</td>
<td>0.9099</td>
<td>0.9916</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>100</td>
<td>( G^{(1)} )</td>
<td>0.8517</td>
<td>0.8474</td>
<td>0.7783</td>
<td>0.7407</td>
<td>0.7434</td>
<td>0.7387</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{G}^{(2)} )</td>
<td>0.9831</td>
<td>0.9800</td>
<td>0.9339</td>
<td>0.9772</td>
<td>0.9737</td>
<td>0.9444</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{F}^{(1)} )</td>
<td>0.8209</td>
<td>0.8147</td>
<td>0.7572</td>
<td>0.6227</td>
<td>0.6310</td>
<td>0.6389</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{F}^{(2)} )</td>
<td>0.8443</td>
<td>0.8396</td>
<td>0.7886</td>
<td>0.6521</td>
<td>0.6595</td>
<td>0.6659</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>200</td>
<td>( G^{(1)} )</td>
<td>0.8533</td>
<td>0.8509</td>
<td>0.8225</td>
<td>0.7272</td>
<td>0.7279</td>
<td>0.7234</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{G}^{(2)} )</td>
<td>0.9864</td>
<td>0.9853</td>
<td>0.9744</td>
<td>0.9829</td>
<td>0.9813</td>
<td>0.9744</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{F}^{(1)} )</td>
<td>0.8445</td>
<td>0.8417</td>
<td>0.8125</td>
<td>0.6226</td>
<td>0.6276</td>
<td>0.6338</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{F}^{(2)} )</td>
<td>0.8647</td>
<td>0.8630</td>
<td>0.8409</td>
<td>0.6493</td>
<td>0.6541</td>
<td>0.6593</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>100</td>
<td>( G^{(1)} )</td>
<td>0.9800</td>
<td>0.9788</td>
<td>0.9591</td>
<td>0.9698</td>
<td>0.9655</td>
<td>0.9240</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{G}^{(2)} )</td>
<td>0.9983</td>
<td>0.9974</td>
<td>0.9845</td>
<td>0.9979</td>
<td>0.9967</td>
<td>0.9807</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{F}^{(1)} )</td>
<td>0.9260</td>
<td>0.9239</td>
<td>0.9066</td>
<td>0.9031</td>
<td>0.8995</td>
<td>0.8658</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{F}^{(2)} )</td>
<td>0.9272</td>
<td>0.9252</td>
<td>0.9089</td>
<td>0.9050</td>
<td>0.9018</td>
<td>0.8720</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>200</td>
<td>( G^{(1)} )</td>
<td>0.9803</td>
<td>0.9801</td>
<td>0.9746</td>
<td>0.9730</td>
<td>0.9719</td>
<td>0.9590</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{G}^{(2)} )</td>
<td>0.9985</td>
<td>0.9982</td>
<td>0.9953</td>
<td>0.9983</td>
<td>0.9980</td>
<td>0.9947</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{F}^{(1)} )</td>
<td>0.9500</td>
<td>0.9494</td>
<td>0.9442</td>
<td>0.9309</td>
<td>0.9297</td>
<td>0.9166</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{F}^{(2)} )</td>
<td>0.9508</td>
<td>0.9503</td>
<td>0.9453</td>
<td>0.9321</td>
<td>0.9309</td>
<td>0.9188</td>
</tr>
</tbody>
</table>
Table 4: $S=1$, $\alpha = 0.5$, $\phi = 0.85$

<table>
<thead>
<tr>
<th>S</th>
<th>M</th>
<th>Nm</th>
<th>T</th>
<th>$\rho$</th>
<th>0</th>
<th>0.5</th>
<th>0.85</th>
<th>0</th>
<th>0.5</th>
<th>0.85</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\beta$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td>0.5</td>
<td>0.85</td>
<td>0</td>
<td>0.5</td>
<td>0.85</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.9560</td>
<td>0.9544</td>
<td>0.9488</td>
<td>0.9614</td>
<td>0.9607</td>
<td>0.9594</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(1)}$</td>
<td>0.9871</td>
<td>0.9861</td>
<td>0.9804</td>
<td>0.9880</td>
<td>0.9873</td>
<td>0.9842</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.8865</td>
<td>0.8868</td>
<td>0.8703</td>
<td>0.7372</td>
<td>0.7458</td>
<td>0.7534</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.8901</td>
<td>0.8905</td>
<td>0.8740</td>
<td>0.7392</td>
<td>0.7477</td>
<td>0.7556</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>200</td>
<td>$G^{(1)}$</td>
<td>0.9562</td>
<td>0.9561</td>
<td>0.9541</td>
<td>0.9616</td>
<td>0.9615</td>
<td>0.9602</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td>0.9878</td>
<td>0.9874</td>
<td>0.9854</td>
<td>0.9887</td>
<td>0.9884</td>
<td>0.9869</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.8995</td>
<td>0.9002</td>
<td>0.8940</td>
<td>0.7492</td>
<td>0.7544</td>
<td>0.7589</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.9027</td>
<td>0.9033</td>
<td>0.8972</td>
<td>0.7511</td>
<td>0.7561</td>
<td>0.7607</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.9910</td>
<td>0.9905</td>
<td>0.9875</td>
<td>0.9900</td>
<td>0.9896</td>
<td>0.9870</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td>0.9971</td>
<td>0.9967</td>
<td>0.9943</td>
<td>0.9969</td>
<td>0.9964</td>
<td>0.9942</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.9419</td>
<td>0.9422</td>
<td>0.9262</td>
<td>0.8951</td>
<td>0.8963</td>
<td>0.8753</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.9422</td>
<td>0.9425</td>
<td>0.9266</td>
<td>0.8954</td>
<td>0.8966</td>
<td>0.8758</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>200</td>
<td>$G^{(1)}$</td>
<td>0.9914</td>
<td>0.9913</td>
<td>0.9903</td>
<td>0.9910</td>
<td>0.9908</td>
<td>0.9898</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td>0.9973</td>
<td>0.9971</td>
<td>0.9964</td>
<td>0.9972</td>
<td>0.9970</td>
<td>0.9963</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.9547</td>
<td>0.9550</td>
<td>0.9509</td>
<td>0.9169</td>
<td>0.9173</td>
<td>0.9066</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.9549</td>
<td>0.9552</td>
<td>0.9512</td>
<td>0.9172</td>
<td>0.9176</td>
<td>0.9069</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.9646</td>
<td>0.9641</td>
<td>0.9601</td>
<td>0.9788</td>
<td>0.9785</td>
<td>0.9775</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td>0.9956</td>
<td>0.9948</td>
<td>0.9901</td>
<td>0.9971</td>
<td>0.9967</td>
<td>0.9951</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.8579</td>
<td>0.8594</td>
<td>0.8366</td>
<td>0.7031</td>
<td>0.7141</td>
<td>0.7345</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.8625</td>
<td>0.8639</td>
<td>0.8413</td>
<td>0.7046</td>
<td>0.7157</td>
<td>0.7360</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>200</td>
<td>$G^{(1)}$</td>
<td>0.9653</td>
<td>0.9653</td>
<td>0.9641</td>
<td>0.9788</td>
<td>0.9788</td>
<td>0.9787</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td>0.9962</td>
<td>0.9959</td>
<td>0.9944</td>
<td>0.9974</td>
<td>0.9972</td>
<td>0.9966</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.8724</td>
<td>0.8736</td>
<td>0.8634</td>
<td>0.6987</td>
<td>0.7054</td>
<td>0.7186</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.8764</td>
<td>0.8775</td>
<td>0.8678</td>
<td>0.7001</td>
<td>0.7068</td>
<td>0.7202</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.9938</td>
<td>0.9936</td>
<td>0.9923</td>
<td>0.9943</td>
<td>0.9941</td>
<td>0.9930</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td>0.9994</td>
<td>0.9993</td>
<td>0.9979</td>
<td>0.9994</td>
<td>0.9993</td>
<td>0.9981</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.9495</td>
<td>0.9495</td>
<td>0.9337</td>
<td>0.9198</td>
<td>0.9187</td>
<td>0.8946</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.9497</td>
<td>0.9498</td>
<td>0.9341</td>
<td>0.9200</td>
<td>0.9190</td>
<td>0.8950</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>200</td>
<td>$G^{(1)}$</td>
<td>0.9939</td>
<td>0.9940</td>
<td>0.9937</td>
<td>0.9946</td>
<td>0.9945</td>
<td>0.9941</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td>0.9995</td>
<td>0.9994</td>
<td>0.9991</td>
<td>0.9995</td>
<td>0.9994</td>
<td>0.9991</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.9619</td>
<td>0.9622</td>
<td>0.9581</td>
<td>0.9389</td>
<td>0.9382</td>
<td>0.9276</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.9621</td>
<td>0.9624</td>
<td>0.9584</td>
<td>0.9391</td>
<td>0.9384</td>
<td>0.9278</td>
</tr>
</tbody>
</table>
Table 5: S=2, $\alpha = 0.5$, $\phi = 0.85$

<table>
<thead>
<tr>
<th></th>
<th>M</th>
<th>Nm</th>
<th>T</th>
<th>$\rho$</th>
<th>0</th>
<th>0.5</th>
<th>0.85</th>
<th>0</th>
<th>0.5</th>
<th>0.85</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\beta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.8989</td>
<td>0.8962</td>
<td>0.8594</td>
<td>0.7348</td>
<td>0.7364</td>
<td>0.7472</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(1)}$</td>
<td>0.9724</td>
<td>0.9696</td>
<td>0.9415</td>
<td>0.9033</td>
<td>0.8888</td>
<td>0.8658</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.8540</td>
<td>0.8499</td>
<td>0.8125</td>
<td>0.6316</td>
<td>0.6420</td>
<td>0.6617</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.8644</td>
<td>0.8605</td>
<td>0.8245</td>
<td>0.6577</td>
<td>0.6650</td>
<td>0.6774</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>200</td>
<td>$G^{(1)}$</td>
<td>0.9004</td>
<td>0.8980</td>
<td>0.8860</td>
<td>0.7229</td>
<td>0.7197</td>
<td>0.7215</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td>0.9747</td>
<td>0.9736</td>
<td>0.9661</td>
<td>0.9181</td>
<td>0.9124</td>
<td>0.8954</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.8814</td>
<td>0.8806</td>
<td>0.8674</td>
<td>0.6686</td>
<td>0.6747</td>
<td>0.6845</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.8902</td>
<td>0.8895</td>
<td>0.8778</td>
<td>0.6973</td>
<td>0.7010</td>
<td>0.7074</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.9819</td>
<td>0.9807</td>
<td>0.9692</td>
<td>0.9702</td>
<td>0.9676</td>
<td>0.9467</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td>0.9942</td>
<td>0.9933</td>
<td>0.9850</td>
<td>0.9924</td>
<td>0.9905</td>
<td>0.9753</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.9181</td>
<td>0.9169</td>
<td>0.8942</td>
<td>0.8696</td>
<td>0.8686</td>
<td>0.8396</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.9190</td>
<td>0.9178</td>
<td>0.8956</td>
<td>0.8713</td>
<td>0.8705</td>
<td>0.8421</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>200</td>
<td>$G^{(1)}$</td>
<td>0.9823</td>
<td>0.9822</td>
<td>0.9778</td>
<td>0.9736</td>
<td>0.9735</td>
<td>0.9631</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td>0.9946</td>
<td>0.9943</td>
<td>0.9918</td>
<td>0.9938</td>
<td>0.9932</td>
<td>0.9881</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.9424</td>
<td>0.9423</td>
<td>0.9360</td>
<td>0.9046</td>
<td>0.9043</td>
<td>0.8895</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.9430</td>
<td>0.9430</td>
<td>0.9369</td>
<td>0.9059</td>
<td>0.9056</td>
<td>0.8917</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.9049</td>
<td>0.9026</td>
<td>0.8639</td>
<td>0.8045</td>
<td>0.8088</td>
<td>0.8087</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td>0.9875</td>
<td>0.9846</td>
<td>0.9530</td>
<td>0.9634</td>
<td>0.9571</td>
<td>0.9228</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.8258</td>
<td>0.8244</td>
<td>0.7814</td>
<td>0.6371</td>
<td>0.6482</td>
<td>0.6651</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.8417</td>
<td>0.8400</td>
<td>0.7984</td>
<td>0.6634</td>
<td>0.6732</td>
<td>0.6839</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>200</td>
<td>$G^{(1)}$</td>
<td>0.9075</td>
<td>0.9070</td>
<td>0.8934</td>
<td>0.7960</td>
<td>0.7981</td>
<td>0.8063</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td>0.9905</td>
<td>0.9896</td>
<td>0.9822</td>
<td>0.9776</td>
<td>0.9742</td>
<td>0.9655</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.8513</td>
<td>0.8514</td>
<td>0.8335</td>
<td>0.6428</td>
<td>0.6501</td>
<td>0.6657</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.8648</td>
<td>0.8648</td>
<td>0.8497</td>
<td>0.6691</td>
<td>0.6760</td>
<td>0.6885</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.9869</td>
<td>0.9862</td>
<td>0.9784</td>
<td>0.9834</td>
<td>0.9823</td>
<td>0.9705</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td>0.9988</td>
<td>0.9983</td>
<td>0.9925</td>
<td>0.9986</td>
<td>0.9980</td>
<td>0.9909</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.9265</td>
<td>0.9249</td>
<td>0.9033</td>
<td>0.8965</td>
<td>0.8943</td>
<td>0.8640</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.9273</td>
<td>0.9257</td>
<td>0.9045</td>
<td>0.8975</td>
<td>0.8953</td>
<td>0.8659</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>200</td>
<td>$G^{(1)}$</td>
<td>0.9870</td>
<td>0.9868</td>
<td>0.9844</td>
<td>0.9851</td>
<td>0.9844</td>
<td>0.9803</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td>0.9990</td>
<td>0.9988</td>
<td>0.9973</td>
<td>0.9989</td>
<td>0.9987</td>
<td>0.9971</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.9496</td>
<td>0.9495</td>
<td>0.9437</td>
<td>0.9266</td>
<td>0.9256</td>
<td>0.9131</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.9502</td>
<td>0.9500</td>
<td>0.9443</td>
<td>0.9272</td>
<td>0.9263</td>
<td>0.9141</td>
</tr>
</tbody>
</table>
Table 6: S=3, $\alpha = 0.5$, $\phi = 0.85$

<table>
<thead>
<tr>
<th></th>
<th>M</th>
<th>Nm</th>
<th>T</th>
<th>$\rho$</th>
<th>0</th>
<th>0.5</th>
<th>0.85</th>
<th>0</th>
<th>0.5</th>
<th>0.85</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>1</td>
<td>5</td>
<td>20</td>
<td>100</td>
<td>0.8419</td>
<td>0.8362</td>
<td>0.7551</td>
<td>0.7036</td>
<td>0.7050</td>
<td>0.6990</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>200</td>
<td></td>
<td>0.8470</td>
<td>0.8438</td>
<td>0.8118</td>
<td>0.6880</td>
<td>0.6851</td>
<td>0.6804</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>100</td>
<td></td>
<td>0.9725</td>
<td>0.9705</td>
<td>0.9417</td>
<td>0.9285</td>
<td>0.9230</td>
<td>0.8758</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>200</td>
<td></td>
<td>0.9735</td>
<td>0.9727</td>
<td>0.9635</td>
<td>0.9431</td>
<td>0.9365</td>
<td>0.9123</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>100</td>
<td></td>
<td>0.8482</td>
<td>0.8398</td>
<td>0.7619</td>
<td>0.7445</td>
<td>0.7511</td>
<td>0.7419</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>200</td>
<td></td>
<td>0.8539</td>
<td>0.8494</td>
<td>0.8149</td>
<td>0.7285</td>
<td>0.7257</td>
<td>0.7223</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>100</td>
<td></td>
<td>0.9799</td>
<td>0.9784</td>
<td>0.9581</td>
<td>0.9682</td>
<td>0.9644</td>
<td>0.9273</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>200</td>
<td></td>
<td>0.9803</td>
<td>0.9799</td>
<td>0.9742</td>
<td>0.9728</td>
<td>0.9714</td>
<td>0.9592</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table 7: S=1, $\alpha = 0.85$, $\phi = 0.5$

<table>
<thead>
<tr>
<th>S</th>
<th>M</th>
<th>Nm</th>
<th>T</th>
<th>$\rho$</th>
<th>$\beta$</th>
<th>$G^{(1)}$</th>
<th>$G^{(2)}$</th>
<th>$F^{(1)}$</th>
<th>$F^{(2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>100</td>
<td>0.9520</td>
<td>0.0</td>
<td>0.9525</td>
<td>0.9497</td>
<td>0.9477</td>
<td>0.9498</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9861</td>
<td>0.0</td>
<td>0.9859</td>
<td>0.9814</td>
<td>0.9851</td>
<td>0.9847</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8902</td>
<td>0.0</td>
<td>0.8897</td>
<td>0.8698</td>
<td>0.7457</td>
<td>0.7507</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8936</td>
<td>0.0</td>
<td>0.8931</td>
<td>0.8739</td>
<td>0.7481</td>
<td>0.7528</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>200</td>
<td>0.9545</td>
<td>0.0</td>
<td>0.9548</td>
<td>0.9538</td>
<td>0.9555</td>
<td>0.9562</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9873</td>
<td>0.0</td>
<td>0.9873</td>
<td>0.9860</td>
<td>0.9877</td>
<td>0.9874</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9016</td>
<td>0.0</td>
<td>0.9017</td>
<td>0.8934</td>
<td>0.7545</td>
<td>0.7575</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9047</td>
<td>0.0</td>
<td>0.9048</td>
<td>0.8971</td>
<td>0.7566</td>
<td>0.7593</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>100</td>
<td>0.9902</td>
<td>0.0</td>
<td>0.9898</td>
<td>0.9868</td>
<td>0.9888</td>
<td>0.9883</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9968</td>
<td>0.0</td>
<td>0.9966</td>
<td>0.9938</td>
<td>0.9965</td>
<td>0.9961</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9438</td>
<td>0.0</td>
<td>0.9434</td>
<td>0.9294</td>
<td>0.9071</td>
<td>0.9063</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9441</td>
<td>0.0</td>
<td>0.9437</td>
<td>0.9299</td>
<td>0.9074</td>
<td>0.9067</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>200</td>
<td>0.9909</td>
<td>0.0</td>
<td>0.9910</td>
<td>0.9899</td>
<td>0.9904</td>
<td>0.9902</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9972</td>
<td>0.0</td>
<td>0.9971</td>
<td>0.9965</td>
<td>0.9971</td>
<td>0.9969</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9552</td>
<td>0.0</td>
<td>0.9552</td>
<td>0.9507</td>
<td>0.9238</td>
<td>0.9230</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9555</td>
<td>0.0</td>
<td>0.9555</td>
<td>0.9510</td>
<td>0.9240</td>
<td>0.9232</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>100</td>
<td>0.9622</td>
<td>0.0</td>
<td>0.9628</td>
<td>0.9607</td>
<td>0.9765</td>
<td>0.9768</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9554</td>
<td>0.0</td>
<td>0.9551</td>
<td>0.9912</td>
<td>0.9969</td>
<td>0.9967</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8609</td>
<td>0.0</td>
<td>0.8601</td>
<td>0.8279</td>
<td>0.6842</td>
<td>0.6925</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8652</td>
<td>0.0</td>
<td>0.8645</td>
<td>0.8332</td>
<td>0.6855</td>
<td>0.6940</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>200</td>
<td>0.9638</td>
<td>0.0</td>
<td>0.9637</td>
<td>0.9637</td>
<td>0.9777</td>
<td>0.9776</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9960</td>
<td>0.0</td>
<td>0.9959</td>
<td>0.9948</td>
<td>0.9972</td>
<td>0.9971</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8737</td>
<td>0.0</td>
<td>0.8734</td>
<td>0.8590</td>
<td>0.6859</td>
<td>0.6906</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8776</td>
<td>0.0</td>
<td>0.8774</td>
<td>0.8639</td>
<td>0.6872</td>
<td>0.6920</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>100</td>
<td>0.9934</td>
<td>0.0</td>
<td>0.9935</td>
<td>0.9916</td>
<td>0.9938</td>
<td>0.9935</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9994</td>
<td>0.0</td>
<td>0.9993</td>
<td>0.9971</td>
<td>0.9994</td>
<td>0.9993</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9507</td>
<td>0.0</td>
<td>0.9504</td>
<td>0.9373</td>
<td>0.9284</td>
<td>0.9266</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9510</td>
<td>0.0</td>
<td>0.9506</td>
<td>0.9377</td>
<td>0.9286</td>
<td>0.9268</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>200</td>
<td>0.9937</td>
<td>0.0</td>
<td>0.9938</td>
<td>0.9935</td>
<td>0.9943</td>
<td>0.9942</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9995</td>
<td>0.0</td>
<td>0.9994</td>
<td>0.9989</td>
<td>0.9995</td>
<td>0.9994</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9623</td>
<td>0.0</td>
<td>0.9623</td>
<td>0.9582</td>
<td>0.9435</td>
<td>0.9425</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9625</td>
<td>0.0</td>
<td>0.9625</td>
<td>0.9584</td>
<td>0.9437</td>
<td>0.9427</td>
</tr>
</tbody>
</table>
Table 8: S=2, $\alpha = 0.85$, $\phi = 0.5$

<table>
<thead>
<tr>
<th>S</th>
<th>M</th>
<th>Nm</th>
<th>T</th>
<th>$\rho$</th>
<th>0</th>
<th>0.5</th>
<th>0.85</th>
<th>0</th>
<th>0.5</th>
<th>0.85</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.8972</td>
<td>0.9719</td>
<td>0.8603</td>
<td>0.8704</td>
<td>0.8997</td>
<td>0.9744</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td>0.8972</td>
<td>0.9710</td>
<td>0.8586</td>
<td>0.8689</td>
<td>0.8994</td>
<td>0.9742</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(1)}$</td>
<td>0.8734</td>
<td>0.9511</td>
<td>0.8353</td>
<td>0.8482</td>
<td>0.8897</td>
<td>0.9689</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{G}^{(2)}$</td>
<td>0.7339</td>
<td>0.9119</td>
<td>0.6652</td>
<td>0.6905</td>
<td>0.7259</td>
<td>0.9288</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.7486</td>
<td>0.9123</td>
<td>0.6827</td>
<td>0.7066</td>
<td>0.7270</td>
<td>0.9247</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.7634</td>
<td>0.8908</td>
<td>0.7021</td>
<td>0.7230</td>
<td>0.7386</td>
<td>0.9157</td>
</tr>
</tbody>
</table>
Table 9: S=3, $\alpha = 0.85$, $\phi = 0.5$

<table>
<thead>
<tr>
<th>S</th>
<th>M</th>
<th>Nm</th>
<th>T</th>
<th>$\rho$</th>
<th>0</th>
<th>0.5</th>
<th>0.85</th>
<th>0</th>
<th>0.5</th>
<th>0.85</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>100</td>
<td>$\tilde{C}^{(1)}$</td>
<td>0.8433</td>
<td>0.8433</td>
<td>0.8034</td>
<td>0.7449</td>
<td>0.7526</td>
<td>0.7644</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{C}^{(2)}$</td>
<td>0.9562</td>
<td>0.9542</td>
<td>0.9130</td>
<td>0.8764</td>
<td>0.8761</td>
<td>0.8648</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(1)}$</td>
<td>0.8314</td>
<td>0.8264</td>
<td>0.7925</td>
<td>0.6418</td>
<td>0.6598</td>
<td>0.6924</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(2)}$</td>
<td>0.8479</td>
<td>0.8441</td>
<td>0.8132</td>
<td>0.6567</td>
<td>0.6762</td>
<td>0.7125</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>200</td>
<td>$\tilde{C}^{(1)}$</td>
<td>0.8471</td>
<td>0.8495</td>
<td>0.8276</td>
<td>0.7218</td>
<td>0.7235</td>
<td>0.7282</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{C}^{(2)}$</td>
<td>0.9614</td>
<td>0.9613</td>
<td>0.9494</td>
<td>0.8969</td>
<td>0.8967</td>
<td>0.8840</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(1)}$</td>
<td>0.8671</td>
<td>0.8660</td>
<td>0.8484</td>
<td>0.6628</td>
<td>0.6734</td>
<td>0.6911</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(2)}$</td>
<td>0.8811</td>
<td>0.8799</td>
<td>0.8670</td>
<td>0.6821</td>
<td>0.6954</td>
<td>0.7122</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>100</td>
<td>$\tilde{C}^{(1)}$</td>
<td>0.9715</td>
<td>0.9709</td>
<td>0.9428</td>
<td>0.9126</td>
<td>0.9128</td>
<td>0.8786</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{C}^{(2)}$</td>
<td>0.9909</td>
<td>0.9902</td>
<td>0.9679</td>
<td>0.9780</td>
<td>0.9758</td>
<td>0.9401</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(1)}$</td>
<td>0.9308</td>
<td>0.9307</td>
<td>0.9262</td>
<td>0.8929</td>
<td>0.8927</td>
<td>0.8793</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(2)}$</td>
<td>0.9320</td>
<td>0.9319</td>
<td>0.9279</td>
<td>0.8990</td>
<td>0.8992</td>
<td>0.8881</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>200</td>
<td>$\tilde{C}^{(1)}$</td>
<td>0.8525</td>
<td>0.8537</td>
<td>0.8147</td>
<td>0.7741</td>
<td>0.7814</td>
<td>0.7903</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{C}^{(2)}$</td>
<td>0.9815</td>
<td>0.9788</td>
<td>0.9352</td>
<td>0.9777</td>
<td>0.9729</td>
<td>0.9408</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(1)}$</td>
<td>0.8043</td>
<td>0.8002</td>
<td>0.7618</td>
<td>0.6190</td>
<td>0.6316</td>
<td>0.6537</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(2)}$</td>
<td>0.8280</td>
<td>0.8253</td>
<td>0.7908</td>
<td>0.6437</td>
<td>0.6564</td>
<td>0.6788</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>100</td>
<td>$\tilde{C}^{(1)}$</td>
<td>0.8551</td>
<td>0.8553</td>
<td>0.8364</td>
<td>0.7502</td>
<td>0.7509</td>
<td>0.7549</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{C}^{(2)}$</td>
<td>0.9860</td>
<td>0.9851</td>
<td>0.9747</td>
<td>0.9839</td>
<td>0.9820</td>
<td>0.9737</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(1)}$</td>
<td>0.8359</td>
<td>0.8340</td>
<td>0.8119</td>
<td>0.6233</td>
<td>0.6298</td>
<td>0.6433</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(2)}$</td>
<td>0.8562</td>
<td>0.8554</td>
<td>0.8395</td>
<td>0.6470</td>
<td>0.6538</td>
<td>0.6676</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>200</td>
<td>$\tilde{C}^{(1)}$</td>
<td>0.9794</td>
<td>0.9791</td>
<td>0.9562</td>
<td>0.9609</td>
<td>0.9552</td>
<td>0.9168</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{C}^{(2)}$</td>
<td>0.9981</td>
<td>0.9973</td>
<td>0.9779</td>
<td>0.9972</td>
<td>0.9958</td>
<td>0.9709</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(1)}$</td>
<td>0.9051</td>
<td>0.9038</td>
<td>0.8905</td>
<td>0.8815</td>
<td>0.8791</td>
<td>0.8569</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(2)}$</td>
<td>0.9067</td>
<td>0.9055</td>
<td>0.8932</td>
<td>0.8850</td>
<td>0.8836</td>
<td>0.8651</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>100</td>
<td>$\tilde{C}^{(1)}$</td>
<td>0.9800</td>
<td>0.9803</td>
<td>0.9750</td>
<td>0.9703</td>
<td>0.9680</td>
<td>0.9510</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{C}^{(2)}$</td>
<td>0.9984</td>
<td>0.9983</td>
<td>0.9940</td>
<td>0.9982</td>
<td>0.9978</td>
<td>0.9929</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(1)}$</td>
<td>0.9387</td>
<td>0.9385</td>
<td>0.9347</td>
<td>0.9193</td>
<td>0.9187</td>
<td>0.9096</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\tilde{F}^{(2)}$</td>
<td>0.9397</td>
<td>0.9396</td>
<td>0.9359</td>
<td>0.9210</td>
<td>0.9206</td>
<td>0.9129</td>
</tr>
</tbody>
</table>
Table 10: S=1, $\alpha = 0.85$, $\phi = 0.85$

<table>
<thead>
<tr>
<th>S</th>
<th>M</th>
<th>Nm</th>
<th>T</th>
<th>$\rho$</th>
<th>$\beta$</th>
<th>$^G(1)$</th>
<th>$^G(2)$</th>
<th>$^F(1)$</th>
<th>$^F(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>100</td>
<td></td>
<td></td>
<td>0.9517</td>
<td>0.9855</td>
<td>0.8571</td>
<td>0.8613</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9523</td>
<td>0.9848</td>
<td>0.8580</td>
<td>0.8620</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9477</td>
<td>0.9786</td>
<td>0.8428</td>
<td>0.8467</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9509</td>
<td>0.9849</td>
<td>0.7057</td>
<td>0.7082</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9525</td>
<td>0.9844</td>
<td>0.7165</td>
<td>0.7190</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9491</td>
<td>0.9784</td>
<td>0.7296</td>
<td>0.7319</td>
</tr>
</tbody>
</table>

| 1 | 5 | 20 | 200|        |        | 0.9547 | 0.9871 | 0.8831 | 0.8866 |
|   |   |    |    |        |        | 0.9547 | 0.9868 | 0.8841 | 0.8874 |
|   |   |    |    |        |        | 0.9529 | 0.9849 | 0.8785 | 0.8821 |
|   |   |    |    |        |        | 0.9549 | 0.9872 | 0.7320 | 0.7339 |
|   |   |    |    |        |        | 0.9554 | 0.9869 | 0.7375 | 0.7398 |
|   |   |    |    |        |        | 0.9555 | 0.9849 | 0.7467 | 0.7491 |

| 1 | 5 | r.c. | 100|        |        | 0.9900 | 0.9967 | 0.9108 | 0.9113 |
|   |   |      |    |        |        | 0.9894 | 0.9963 | 0.9113 | 0.9117 |
|   |   |      |    |        |        | 0.9856 | 0.9927 | 0.8951 | 0.8958 |
|   |   |      |    |        |        | 0.9883 | 0.9963 | 0.8625 | 0.8628 |
|   |   |      |    |        |        | 0.9876 | 0.9957 | 0.8644 | 0.8648 |
|   |   |      |    |        |        | 0.9834 | 0.9920 | 0.8466 | 0.8470 |

| 1 | 5 | r.c. | 200|        |        | 0.9909 | 0.9971 | 0.9377 | 0.9380 |
|   |   |      |    |        |        | 0.9911 | 0.9970 | 0.9381 | 0.9385 |
|   |   |      |    |        |        | 0.9898 | 0.9961 | 0.9342 | 0.9345 |
|   |   |      |    |        |        | 0.9903 | 0.9970 | 0.8992 | 0.8995 |
|   |   |      |    |        |        | 0.9900 | 0.9968 | 0.9000 | 0.9003 |
|   |   |      |    |        |        | 0.9886 | 0.9957 | 0.8905 | 0.8910 |

| 1 | 20 | 20 | 100|        |        | 0.9618 | 0.9947 | 0.8276 | 0.8327 |
|   |   |    |    |        |        | 0.9619 | 0.9939 | 0.8294 | 0.8343 |
|   |   |    |    |        |        | 0.9584 | 0.9882 | 0.8090 | 0.8137 |
|   |   |    |    |        |        | 0.9765 | 0.9966 | 0.6741 | 0.6757 |
|   |   |    |    |        |        | 0.9766 | 0.9961 | 0.6863 | 0.6879 |
|   |   |    |    |        |        | 0.9758 | 0.9939 | 0.7103 | 0.7121 |

| 1 | 20 | 20 | 200|        |        | 0.9637 | 0.9958 | 0.8559 | 0.8602 |
|   |   |    |    |        |        | 0.9640 | 0.9955 | 0.8573 | 0.8531 |
|   |   |    |    |        |        | 0.9635 | 0.9939 | 0.8484 | 0.8473 |
|   |   |    |    |        |        | 0.9777 | 0.9971 | 0.6833 | 0.6847 |
|   |   |    |    |        |        | 0.9774 | 0.9969 | 0.6906 | 0.6920 |
|   |   |    |    |        |        | 0.9774 | 0.9962 | 0.7065 | 0.7081 |

| 1 | 20 | r.c. | 100|        |        | 0.9933 | 0.9994 | 0.9183 | 0.9187 |
|   |   |      |    |        |        | 0.9933 | 0.9992 | 0.9185 | 0.9188 |
|   |   |      |    |        |        | 0.9910 | 0.9967 | 0.9023 | 0.9028 |
|   |   |      |    |        |        | 0.9937 | 0.9993 | 0.8875 | 0.8878 |
|   |   |      |    |        |        | 0.9933 | 0.9991 | 0.8871 | 0.8874 |
|   |   |      |    |        |        | 0.9912 | 0.9969 | 0.8649 | 0.8653 |

| 1 | 20 | r.c. | 200|        |        | 0.9937 | 0.9995 | 0.9449 | 0.9452 |
|   |   |      |    |        |        | 0.9937 | 0.9994 | 0.9453 | 0.9456 |
|   |   |      |    |        |        | 0.9934 | 0.9988 | 0.9413 | 0.9416 |
|   |   |      |    |        |        | 0.9942 | 0.9995 | 0.9214 | 0.9216 |
|   |   |      |    |        |        | 0.9941 | 0.9994 | 0.9210 | 0.9212 |
|   |   |      |    |        |        | 0.9935 | 0.9988 | 0.9113 | 0.9116 |
Table 11: S=2, $\alpha = 0.85$, $\phi = 0.85$

<table>
<thead>
<tr>
<th>S</th>
<th>M</th>
<th>Nm</th>
<th>T</th>
<th>$\rho$</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td>0.5</td>
<td>0.85</td>
<td>0</td>
<td>0.5</td>
<td>0.85</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.8969</td>
<td>0.8964</td>
<td>0.8680</td>
<td>0.7558</td>
<td>0.7635</td>
<td>0.7790</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td>0.9687</td>
<td>0.9669</td>
<td>0.9379</td>
<td>0.8905</td>
<td>0.8843</td>
<td>0.8673</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.7975</td>
<td>0.7953</td>
<td>0.7621</td>
<td>0.5875</td>
<td>0.6010</td>
<td>0.6307</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.8098</td>
<td>0.8074</td>
<td>0.7746</td>
<td>0.6041</td>
<td>0.6175</td>
<td>0.6443</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>200</td>
<td>$G^{(1)}$</td>
<td>0.8997</td>
<td>0.8995</td>
<td>0.8890</td>
<td>0.7355</td>
<td>0.7352</td>
<td>0.7469</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td>0.9731</td>
<td>0.9724</td>
<td>0.9648</td>
<td>0.9093</td>
<td>0.9081</td>
<td>0.8933</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.8496</td>
<td>0.8491</td>
<td>0.8373</td>
<td>0.6415</td>
<td>0.6471</td>
<td>0.6670</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.8597</td>
<td>0.8591</td>
<td>0.8487</td>
<td>0.6658</td>
<td>0.6727</td>
<td>0.6891</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.9809</td>
<td>0.9804</td>
<td>0.9665</td>
<td>0.9580</td>
<td>0.9562</td>
<td>0.9319</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td>0.9937</td>
<td>0.9929</td>
<td>0.9814</td>
<td>0.9873</td>
<td>0.9850</td>
<td>0.9630</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.8576</td>
<td>0.8565</td>
<td>0.8320</td>
<td>0.8013</td>
<td>0.8022</td>
<td>0.7767</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.8590</td>
<td>0.8578</td>
<td>0.8339</td>
<td>0.8051</td>
<td>0.8059</td>
<td>0.7810</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>200</td>
<td>$G^{(1)}$</td>
<td>0.9818</td>
<td>0.9820</td>
<td>0.9774</td>
<td>0.9694</td>
<td>0.9692</td>
<td>0.9555</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td>0.9943</td>
<td>0.9941</td>
<td>0.9911</td>
<td>0.9930</td>
<td>0.9921</td>
<td>0.9847</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.9090</td>
<td>0.9091</td>
<td>0.9027</td>
<td>0.8686</td>
<td>0.8693</td>
<td>0.8566</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.9100</td>
<td>0.9101</td>
<td>0.9038</td>
<td>0.8708</td>
<td>0.8714</td>
<td>0.8599</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.9040</td>
<td>0.9018</td>
<td>0.8730</td>
<td>0.8087</td>
<td>0.8160</td>
<td>0.8259</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td>0.9842</td>
<td>0.9809</td>
<td>0.9479</td>
<td>0.9572</td>
<td>0.9509</td>
<td>0.9171</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.7727</td>
<td>0.7716</td>
<td>0.7355</td>
<td>0.5962</td>
<td>0.6104</td>
<td>0.6328</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.7895</td>
<td>0.7882</td>
<td>0.7513</td>
<td>0.6213</td>
<td>0.6334</td>
<td>0.6494</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>200</td>
<td>$G^{(1)}$</td>
<td>0.9068</td>
<td>0.9073</td>
<td>0.8982</td>
<td>0.7972</td>
<td>0.8042</td>
<td>0.8155</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td>0.9891</td>
<td>0.9883</td>
<td>0.9806</td>
<td>0.9755</td>
<td>0.9727</td>
<td>0.9624</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.8213</td>
<td>0.8219</td>
<td>0.8069</td>
<td>0.6203</td>
<td>0.6301</td>
<td>0.6493</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.8360</td>
<td>0.8366</td>
<td>0.8234</td>
<td>0.6471</td>
<td>0.6556</td>
<td>0.6721</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>100</td>
<td>$G^{(1)}$</td>
<td>0.9863</td>
<td>0.9861</td>
<td>0.9755</td>
<td>0.9801</td>
<td>0.9778</td>
<td>0.9604</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td>0.9986</td>
<td>0.9980</td>
<td>0.9890</td>
<td>0.9981</td>
<td>0.9971</td>
<td>0.9851</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.8668</td>
<td>0.8653</td>
<td>0.8414</td>
<td>0.8343</td>
<td>0.8330</td>
<td>0.8046</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.8681</td>
<td>0.8665</td>
<td>0.8431</td>
<td>0.8361</td>
<td>0.8350</td>
<td>0.8077</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>200</td>
<td>$G^{(1)}$</td>
<td>0.9868</td>
<td>0.9868</td>
<td>0.9841</td>
<td>0.9837</td>
<td>0.9828</td>
<td>0.9776</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$G^{(2)}$</td>
<td>0.9989</td>
<td>0.9987</td>
<td>0.9965</td>
<td>0.9988</td>
<td>0.9985</td>
<td>0.9960</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(1)}$</td>
<td>0.9162</td>
<td>0.9162</td>
<td>0.9100</td>
<td>0.8921</td>
<td>0.8918</td>
<td>0.8805</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{F}^{(2)}$</td>
<td>0.9171</td>
<td>0.9171</td>
<td>0.9109</td>
<td>0.8931</td>
<td>0.8929</td>
<td>0.8820</td>
<td></td>
</tr>
</tbody>
</table>
Table 12: $S=3, \alpha = 0.85, \phi = 0.85$

<table>
<thead>
<tr>
<th>$S$</th>
<th>$M$</th>
<th>$Nm$</th>
<th>$T$</th>
<th>$\rho$</th>
<th>$\beta$</th>
<th>$G(1)$</th>
<th>$G(2)$</th>
<th>$F(1)$</th>
<th>$F(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>100</td>
<td>0.8421</td>
<td>0.8412</td>
<td>0.7957</td>
<td>0.7544</td>
<td>0.7612</td>
<td>0.7654</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9481</td>
<td>0.9437</td>
<td>0.8863</td>
<td>0.8599</td>
<td>0.8599</td>
<td>0.8429</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.7427</td>
<td>0.7342</td>
<td>0.6760</td>
<td>0.5306</td>
<td>0.5516</td>
<td>0.5791</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.7626</td>
<td>0.7536</td>
<td>0.6935</td>
<td>0.5405</td>
<td>0.5638</td>
<td>0.5948</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>20</td>
<td>200</td>
<td>0.8492</td>
<td>0.8478</td>
<td>0.8231</td>
<td>0.7260</td>
<td>0.7270</td>
<td>0.7301</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9585</td>
<td>0.9574</td>
<td>0.9380</td>
<td>0.8770</td>
<td>0.8753</td>
<td>0.8575</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8197</td>
<td>0.8158</td>
<td>0.7893</td>
<td>0.5948</td>
<td>0.6105</td>
<td>0.6294</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8365</td>
<td>0.8330</td>
<td>0.8094</td>
<td>0.6122</td>
<td>0.6283</td>
<td>0.6482</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>100</td>
<td>0.9714</td>
<td>0.9706</td>
<td>0.9372</td>
<td>0.9045</td>
<td>0.9079</td>
<td>0.8758</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9902</td>
<td>0.9887</td>
<td>0.9601</td>
<td>0.9621</td>
<td>0.9602</td>
<td>0.9206</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8069</td>
<td>0.8028</td>
<td>0.7601</td>
<td>0.7257</td>
<td>0.7297</td>
<td>0.6986</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8095</td>
<td>0.8053</td>
<td>0.7639</td>
<td>0.7359</td>
<td>0.7383</td>
<td>0.7063</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r.c.</td>
<td>200</td>
<td>0.9731</td>
<td>0.9732</td>
<td>0.9646</td>
<td>0.9310</td>
<td>0.9293</td>
<td>0.9065</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9915</td>
<td>0.9911</td>
<td>0.9851</td>
<td>0.9825</td>
<td>0.9813</td>
<td>0.9645</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8811</td>
<td>0.8805</td>
<td>0.8699</td>
<td>0.8308</td>
<td>0.8319</td>
<td>0.8132</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8805</td>
<td>0.8822</td>
<td>0.8720</td>
<td>0.8384</td>
<td>0.8391</td>
<td>0.8219</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>100</td>
<td>0.8489</td>
<td>0.8489</td>
<td>0.8029</td>
<td>0.7871</td>
<td>0.7915</td>
<td>0.7967</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9703</td>
<td>0.9638</td>
<td>0.8978</td>
<td>0.9496</td>
<td>0.9368</td>
<td>0.8943</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.7199</td>
<td>0.7124</td>
<td>0.6599</td>
<td>0.5543</td>
<td>0.5688</td>
<td>0.5865</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.7481</td>
<td>0.7400</td>
<td>0.6822</td>
<td>0.5812</td>
<td>0.5931</td>
<td>0.6052</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>200</td>
<td>0.8544</td>
<td>0.8529</td>
<td>0.8279</td>
<td>0.7546</td>
<td>0.7560</td>
<td>0.7609</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9820</td>
<td>0.9802</td>
<td>0.9607</td>
<td>0.9745</td>
<td>0.9699</td>
<td>0.9501</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.7902</td>
<td>0.7878</td>
<td>0.7602</td>
<td>0.5910</td>
<td>0.5991</td>
<td>0.6173</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8146</td>
<td>0.8130</td>
<td>0.7898</td>
<td>0.6220</td>
<td>0.6302</td>
<td>0.6467</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>100</td>
<td>0.9791</td>
<td>0.9788</td>
<td>0.9520</td>
<td>0.9560</td>
<td>0.9515</td>
<td>0.9144</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9976</td>
<td>0.9965</td>
<td>0.9742</td>
<td>0.9943</td>
<td>0.9910</td>
<td>0.9577</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8184</td>
<td>0.8142</td>
<td>0.7750</td>
<td>0.7820</td>
<td>0.7772</td>
<td>0.7356</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8209</td>
<td>0.8166</td>
<td>0.7785</td>
<td>0.7877</td>
<td>0.7832</td>
<td>0.7430</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>r.c.</td>
<td>200</td>
<td>0.9800</td>
<td>0.9801</td>
<td>0.9746</td>
<td>0.9691</td>
<td>0.9668</td>
<td>0.9518</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9982</td>
<td>0.9980</td>
<td>0.9933</td>
<td>0.9978</td>
<td>0.9972</td>
<td>0.9908</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8894</td>
<td>0.8887</td>
<td>0.8788</td>
<td>0.8645</td>
<td>0.8633</td>
<td>0.8478</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.8911</td>
<td>0.8904</td>
<td>0.8806</td>
<td>0.8672</td>
<td>0.8663</td>
<td>0.8523</td>
</tr>
</tbody>
</table>
Table 13: Simulation for Factor Number Selection, RMSE

<table>
<thead>
<tr>
<th>$(\rho, \beta)$</th>
<th>$(\alpha, \phi)$</th>
<th>(0.5, 0.5)</th>
<th>(0.5, 0.85)</th>
<th>(0.85, 0.5)</th>
<th>(0.85, 0.85)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ICp2</td>
<td>G</td>
<td>R</td>
<td>G</td>
<td>R</td>
</tr>
<tr>
<td>(0, 0)</td>
<td>0.000</td>
<td>0.119</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.452</td>
<td>0.000</td>
<td>0.127</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.292</td>
<td>0.000</td>
<td>0.043</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.028</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.015</td>
<td>0.000</td>
<td>0.020</td>
<td>0.000</td>
</tr>
<tr>
<td>(0.5, 0)</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.356</td>
<td>0.000</td>
<td>0.010</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.217</td>
<td>0.000</td>
<td>0.020</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.094</td>
<td>0.000</td>
<td>0.101</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.217</td>
<td>0.000</td>
<td>0.020</td>
<td>0.000</td>
</tr>
<tr>
<td>(0, 0.2)</td>
<td>0.000</td>
<td>1.000</td>
<td>0.000</td>
<td>1.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.393</td>
<td>0.000</td>
<td>0.116</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.239</td>
<td>0.000</td>
<td>0.072</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.121</td>
<td>0.000</td>
<td>0.116</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.068</td>
<td>0.000</td>
<td>0.072</td>
<td>0.000</td>
</tr>
<tr>
<td>(0.5, 0.2)</td>
<td>0.722</td>
<td>1.000</td>
<td>0.892</td>
<td>1.000</td>
<td>0.744</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.322</td>
<td>0.000</td>
<td>0.350</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.913</td>
<td>1.000</td>
<td>0.913</td>
<td>1.000</td>
<td>0.913</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.118</td>
<td>0.000</td>
<td>0.307</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.068</td>
<td>0.000</td>
<td>0.072</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.322</td>
<td>0.000</td>
<td>0.350</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.118</td>
<td>0.000</td>
<td>0.307</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.068</td>
<td>0.000</td>
<td>0.072</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.322</td>
<td>0.000</td>
<td>0.350</td>
<td>0.000</td>
</tr>
</tbody>
</table>