Endogenous Inequality of Nations Through Asset Market Integration

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Abstract

The paper identifies an endogenous mechanism leading perfectly symmetric economies to diverge in the long run after unifying their financial asset markets. The standard OLG growth model is extended to include a financial asset used to transfer ownership of the proceeds of an exogenous random production process between generations. Consumers are risk averse, implying that consumers hold mixed portfolios of real capital and of the asset. All markets are competitive leading to a simultaneous and endogenous determination of factor prices, consumer incomes, the return on capital, and the price of the financial asset. Therefore, capital accumulation in each country depends on the two rates of return and on disposable income of consumers.

The paper demonstrates that in the absence of an international asset market, the two autarkic economies converge to a unique steady state, which is symmetric and globally stable under rational expectations dynamics. However, when the two asset markets are unified internationally additional asymmetric steady states appear and the symmetric steady state of the world economy can lose its stability, causing symmetry breaking. As a result, depending on the initial capital the world economy, perfectly symmetric economies can converge to an asymmetric steady state.

The paper derives general sufficient conditions for a saddle node bifurcation of the symmetric steady state. These reveal that the instability of the symmetric steady state occurs due to a nonlinear interaction of several elasticities in production and in asset demand. A numerical example shows that these effects occur in particular, when the production function and the function of absolute risk aversion are isoelastic.

Keywords: asset market integration, asset pricing, capital accumulation, development, divergence, symmetry-breaking.

JEL classification: E20, F36, F43, G12, O11
1 Introduction

The common convergence hypothesis in development economics suggests that countries with similar structural characteristics should exhibit similar levels of per-capita income in the long run, regardless of their initial capital stock. Several empirical studies have documented evidence against such a general convergence hypothesis. Bianchi (1997), Jones (1997) and Quah (1997) show that the shape of inter-country income distributions has transformed from a unimodal one in the early 1960s to a bimodal one in 1990s. Durlauf & Johnson (1995) confirms a positive relationship between the starting level of per capita output and subsequent growth rates, implying divergence of income levels over time. While these findings provide empirical evidence for non-convergence, it is less clear from a theoretical point of view which mechanisms cause or could explain divergence. Since the theory of pure trade seems to offer very little in this direction, the role and structure of international financial markets is often mentioned, implying that the forces in such markets cause funds to flow in wrong directions and thus, may induce intertemporal as well as interregional distortions.

Along these general lines, Matsuyama (1996) develops a model of the world economy without foreign direct investment and without uncertainty, but with an international market for deposits. He argues that the integration of inter-country deposit markets can cause a divergence of otherwise symmetric economies. He shows that imperfections in credit markets coupled with minimum physical capital investment requirement can promote deposits to flow from capital-scarce to capital-abundant countries after the deposit market are integrated. This occurs because the distortion created by the credit market inefficiency is smaller in rich countries than in poor countries. As a result, agents in rich countries are not credit constrained and can finance all profitable projects, while agents in poor countries are credit constrained and can finance only part of profitable projects. In autarky deposit rates would adjust independently in each country implying that the world economy converges to a symmetric steady state.

When capital endowments differ before deposit markets are integrated, the deposit rate can be lower in a country with scarce capital, due to the imperfection in the credit market and the indivisibility in investment. The integration implies equalization of the deposit rates. This causes funds to flow from capital scarce to capital abundant countries, setting off a mechanism so that initially rich countries increase their income gradually and lower the credit market imperfections, while initially poor countries suffer more and more from low income, low investment, and high credit market imperfections. Thus, the integration of deposit markets may cause symmetry breaking in the sense of Matsuyama (2004), i.e. the symmetric steady state of the world economy might become unstable. As a result, depending on the initial distribution of capital the world economy will converge to an asymmetric steady state.

The result in Matsuyama (2004) relies on three main assumptions, that a) investment in physical capital is non-divisible and there is a minimum investment requirement, b) consumers are credit constrained and need to borrow funds in order to become investors and c) there is no foreign direct investment. It is clear, that the result no longer holds,
if one of these assumptions is removed. In particular, dropping the assumption of the minimum capital investment requirement, one obtains that funds will not flow from capital scarce to capital abandoned countries, because of higher capital returns in capital scarce countries. Thus, credit market imperfections alone do not guaranty the result of symmetry breaking, because rich countries will not benefit from deposit market integration because deposits flow from rich to poor.

In the present paper the symmetry breaking result is obtained for quite a different model caused by a totally different mechanism. While we maintain the assumption that foreign direct investment is not allowed, we drop the assumption of a minimum requirement for physical capital investment and we eliminate the market for deposits or credit as a perfect substitute for real investment. However, we introduce uncertainty in production and a market for a financial paper asset. Therefore, each country is represented by a standard OLG growth model to which a market for a paper asset has been added. The asset is traded between generations and serves as an intertemporal device to distribute the random dividends from an exogenous production process, a so called “Lucas tree”.

The financial asset holdings of consumers are motivated by portfolio diversification in the spirit of the standard capital asset pricing model (as in Böhm & Chiarella 2005) where consumers maximize expected future consumption under rational expectations. Agents do not hold financial assets as in Bencivenga & Smith (1991) and Greenwood & Smith (1997) to insure themselves against unpredictable liquidity needs, nor by the liquidity needs of consumers à la Diamond & Dybvig (1983). Therefore, consumers face a non degenerate portfolio choice between two assets. With risk aversion this implies mixed portfolios in equilibrium, where the price of the asset is determined endogenously. Under natural conditions on the technology and on consumer’s risk aversion, in autarky, such economies converge to a unique positive steady state under rational expectations.

However, when two such economies combine their asset markets, multiple steady states can arise endogenously. This occurs because of an induced non monotonicity of the inverse demand function for the asset, as a result of an interplay of the properties of the random process, the degree of risk aversion, and the elasticity of substitution in production. Thus, asset market integration may cause symmetry breaking. As a result, depending on the initial distribution of capital the world economy will converge to an asymmetric steady state. In contrast to the findings of Matsuyama (2004), which would predict funds to flow from rich to poor countries, in this paper we obtain that funds will flow in the opposite direction financing asset purchases and crowding out (or crowding in) domestic investment in physical capital in countries with scarce (or abundant) capital.

The model uses the same type of financial asset as introduced and analyzed in Böhm, Kikuchi & Vachadze (2007b) and Kikuchi (2008). Both papers use specific functional forms for the production and the utility function to show numerical results only. In particular, Böhm, Kikuchi & Vachadze (2007b) assumes that the production function is of the CES type and that consumers have constant absolute risk aversion, while Kikuchi (2008) assumes that agents have a linear mean variance preferences and the production function is quadratic. The present paper makes no assumptions of specific functional forms for the utility and for the production functions and obtains a sufficient condition for symmetry breaking.
The paper is organized as follows. Section 2 introduces the model and discusses its main properties for general production functions and general risk preferences. Section 3 demonstrates existence, uniqueness, and the stability of an interior rational expectations equilibrium. Section 4 presents the two country model with an international financial asset market, discussing existence of rational expectations equilibria (Section 4.1) and deriving general sufficient conditions under which the symmetric equilibrium looses stability (Section 4.2). Section 5 presents global properties of a parameterized explicit example. Section 6 concludes.

2 The Model

Consider an infinite-horizon economy in discrete time $t = 0, 1, \ldots$. The economy is composed of a consumption and a production sector. The production sector consists of an infinitely lived neoclassical firm plus an exogenous production process generating a random stream of a consumption commodity in each period. Such an exogenous production process is often referred to as a so called “Lucas’ tree”, underlining the fact that the stochastic process generating the proceeds is exogenous and completely independent of the production technology described in the model. The stochastic process yields $\varepsilon_t$ units of the consumption commodity in each period payed to the owners as a dividend. We assume that $\{\varepsilon_t\}_{t=1,2,\ldots}$ is a sequence of i.i.d. random variables taking values $d > 0$ and 0 with probabilities $q \in (0, 1)$ and $1 - q$ respectively. Intertemporal ownership of the tree and the right to the proceeds is traded in the form of a financial asset (a financial contract) between successive generations of consumers. The asset is infinitely lived. It yields a random dividend and it can be sold in a competitive asset market. The total number of tradable contracts (the supply of assets) is constant over time.

In addition to the exogenous production process there is a neoclassical firm producing the consumption commodity using capital and labor as inputs. The technology of the firm is described by a standard production function $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ with constant returns to scale. At any time $t$, let $Y_t = F(K_t, L_t)$ denote total output produced, where $K_t \geq 0$ and $L_t \geq 0$ are aggregate supplies of physical capital and labor respectively. Output per worker is defined as $y_t = Y_t/L_t = F(K_t/L_t, 1) \equiv f(k_t)$, where $k_t = K_t/L_t$ denotes capital per worker and $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the production function in intensive form. We assume that $f(0) = 0$, i.e. capital is essential in production. In addition, $f$ is assumed to be twice continuously differentiable, strictly increasing, strictly concave, and it satisfies the Inada conditions. Both factor markets in the economy are assumed to be competitive. Therefore, under full employment, factor rewards for capital and labor are determined by their respective marginal products. Let $r_t = r(k_t) := f'(k_t)$ denote the rental rate of capital and $w_t = w(k_t) := f(k_t) - k_t f'(k_t)$ denote the wage rate in any given period. The produced commodity can be either consumed or invested in physical capital, which becomes available in the next period. Old capital depreciates fully within a period.

The consumption sector consists of overlapping generations of consumers who live for two successive periods. Thus, in any period, there are two generations alive referred to as young and old. Each generation, which consists of a continuum of homogeneous agents...
with unit mass, is identified by its date of birth. For simplicity we assume no population growth. A typical young consumer of any generation \( t = 0, 1, \ldots \) supplies one unit of labor endowment inelastically in the first period of his life for which he receives labor income \( w_t \). He does not consume in the first period, but invests all income in a portfolio consisting of physical capital and the tradable asset. Thus, the budget constraint of a young consumer is given by \( i_t + x_t p_t = w_t \), where \( i_t \geq 0 \) denotes the amount of investment in physical capital and \( x_t \geq 0 \) is the number of assets purchased at the price \( p_t \) (measured in units of consumption).

At period \( t = 0 \), each consumer of generation \(-1\) (the initial old generation which only lives for one period) is endowed with \( x > 0 \) units of the financial asset and \( k_0 \) units of capital. They consume the total receipts to both of them, which consist of the return on their capital, the proceeds from the “tree”, and the value at which they sell the asset in the market. Old consumers of succeeding generations acquire their endowment of the asset and of physical capital from saving their wage income when young.

Old consumers do not leave bequests to future generations and consume their entire wealth. Therefore, their random second period consumption is

\[
c_{t+1} = i_t r_{t+1} + x_t (p_{t+1} + \varepsilon_{t+1}),
\]

where \( i_t r_{t+1}, x_t \varepsilon_{t+1} \) and \( x_t p_{t+1} \) are the returns (in units of consumption good) received from capital investment, from asset holding as dividends, and from selling the financial asset. Depending on the realization of \( \varepsilon_{t+1} \), the old age consumption can take values \( \overline{c}_{t+1} = i_t r_{t+1} + x_t (p_{t+1} + d) \) or \( \underline{c}_{t+1} = i_t r_{t+1} + x_t p_{t+1} \) with probabilities \( q \) and \( 1 - q \) respectively. In the sequel of the paper \( \overline{c}_{t+1} \) and \( \underline{c}_{t+1} \) will be referred to as realizations of consumption in good and bad states. Since \( i_t = w_t - x_t p_t \), old age consumption can be rewritten as

\[
\overline{c}_{t+1} = w_t r_{t+1} + x_t d - x_t (p_t r_{t+1} - p_{t+1}) \quad \text{and} \quad \underline{c}_{t+1} = w_t r_{t+1} - x_t (p_t r_{t+1} - p_{t+1}).
\]

Preferences over old age consumption is described by a utility function \( u : \mathbb{R}_+ \rightarrow \mathbb{R} \). We assume that \( c \mapsto u(c) \) is twice continuously differentiable, strictly increasing, and strictly concave. For given values of wage income \( w_t \), next period’s rate of return on capital \( r_{t+1} \), and next period’s asset price \( p_{t+1} \), the consumer’s demand for the asset is defined as

\[
\varphi(w_t, r_{t+1}, p_{t+1}, p_t) \equiv \max_{x \in B(w_t, p_t)} \left\{ q u(\overline{c}_{t+1}) + (1 - q) u(\underline{c}_{t+1}) \right\},
\]

where \( \overline{c}_{t+1} \) and \( \underline{c}_{t+1} \) are consumptions in good and bad states, and \( B(w_t, p_t) = \{ x \mid x \geq 0, x p_t \leq w_t \} \) is the budget set. Given the assumptions made, asset demand of consumers takes a particularly simple form.

**Proposition 1** For any given non-negative vector, \( (w_t, r_{t+1}, p_{t+1}) \geq 0 \), asset demand is given by

\[
\varphi(w_t, r_{t+1}, p_{t+1}, p_t) = \begin{cases} 
0 & \text{if } p_t \geq p_3^* \\
\varphi_m(p_t r_{t+1} - p_{t+1}, w_t r_{t+1}) & \text{if } p_2^* < p_t < p_3^* \\
w_t/p_t & \text{if } p_t \leq p_2^*.
\end{cases}
\]
The function $\varphi_m : \mathbb{R}_+^2 \to \mathbb{R}_+$ is decreasing in its first and increasing in its second argument. The critical levels $p^*_2$, and $p^*_3$ are defined by the unique solutions $p^*_2 \varphi_m(p^*_2 r_{t+1} - p_{t+1}, w_t r_{t+1}) = w_t$ and $p^*_3 r_{t+1} = p_{t+1} + dq$.

The typical features of asset demand are visualized in Figure 1, showing that the graph of the demand function consists of three sections. For a sufficiently high price, asset demand is zero, since the expected return from the financial asset is lower than the expected return from capital investment. In this case, consumers invest all of their wage income in physical capital. For a sufficiently low price, when the expected return from the financial asset investment exceeds the expected return from capital investment, the situation is the opposite. All wage income is invested in the asset market and no new investment in physical capital occurs. For all intermediate prices, the optimal choice consists of an interior solution with a mixed portfolio containing the financial asset and physical capital. Moreover, as a consequence of portfolio theory, the function $\varphi_m$ depends only on the expected risk premium $p_t r_{t+1} - p_{t+1}$ and on the discounted wage income $w_t r_{t+1}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Asset Demand Function}
\end{figure}

3 The Closed Economy

Consider first the case of autarky, when the asset market operates only domestically. The demographic structure of consumers implies that in autarky all assets sold by old consumers are purchased by young investors. Therefore, if no new assets are added in any period, the total number of assets will be constant through time. Then, for a given non-negative vector $(w_t, r_{t+1}, p_{t+1}) \geq 0$ and for a given aggregate supply of assets $x > 0$, the asset market clearing price $p_t$ solves the equation

$$\varphi(w_t, r_{t+1}, p_{t+1}, p_t) = x. \quad (5)$$

\footnote{All proofs are provided in the appendix.}
The strict monotonicity of the asset demand function implies that equation (5) has a unique solution. Let $p_t = S(w_t, r_{t+1}, p_{t+1}, x_t)$ denote the unique market clearing asset price, where the function $S$ is usually referred to as the temporary price law. Since next period’s capital stock $k_{t+1}$ is equal to new investment, one has $k_{t+1} = w_t - p_t x_t$, where $p_t x_t$ is total spending in the asset market.

### 3.1 Stationary Rational Expectations Equilibria

Consider next the situation when stationarity and perfect foresight prevails.

**Definition 1** A Stationary Rational Expectations Equilibrium (SREE) is a pair $(k, p)$ such that

- given $k \in \mathbb{R}_+$, the price $p \in \mathbb{R}_+$ clears the asset market under perfect foresight, i.e. $p$ is a fixed point of the temporary price law
  \[ p = S(w(k), r(k), p, x), \]  
  \[ (6) \]

- given $p \in \mathbb{R}_+$, the level of capital $k \in \mathbb{R}_+$ is a fixed point of the capital accumulation equation
  \[ k = A(k, p, x) := w(k) - px. \]  
  \[ (7) \]

Even in very simple cases, the perfect foresight solutions of such economies cannot be determined explicitly, due to the interaction of the non-linearities of the price law and of the law of capital accumulation. To show that there exist exactly two such solutions, a detailed implicit analysis is required involving features of the inverse asset demand function of consumers.

First, we observe that young consumers must hold a mixed portfolio for the capital stock to be positive in an SREE. Let $\mathcal{L}(x) := qu(\tau_{t+1}) + (1-q)u(\xi_{t+1})$ denote the Lagrangian of the consumer’s optimization problem. Then, from the first order conditions for an interior optimum

\[ \mathcal{L}'(x) = qu'(\tau_{t+1}) (d - (p_t r_{t+1} - p_{t+1})) - (1-q)u'(\xi_{t+1}) (p_t r_{t+1} - p_{t+1}) = 0, \]  
\[ (8) \]

one obtains the relation

\[ \frac{d}{p_t r_{t+1} - p_{t+1}} = 1 + \frac{1-q}{q} \frac{u'(\tau_{t+1})}{u'(\xi_{t+1})}. \]  
\[ (9) \]

Second, we observe that stationary equilibrium consumptions in good and bad states are given by

\[ \zeta = w(k) r(k) - (w(k) - k) (r(k) - 1) = f(k) - k \]
\[ \eta = w(k) r(k) + xd - (w(k) - k) (r(k) - 1) = f(k) - k + xd. \]  
\[ (10) \]
Therefore, equations (9) and (10) imply that for a given SREE \((k, x)\), the inverse demand function of the financial asset is given by

\[
P(k, x) := \frac{d}{r(k) - 1} h(k, x) \quad \text{with} \quad h(k, x) := \frac{qu'(\tau)}{qu'(\tau) + (1 - q)u'(\omega)} \in [0, 1].
\]

The function \(h\) can be interpreted as the risk neutral probability of the good state realization. Since \(h\) is always positive, equation (11) implies that in order to guarantee a positive asset price, the equilibrium \(k\) should belong to the interval \([0, \hat{k}]\), where \(\hat{k}\) is the unique solution of \(r(k) = 1\). Equations (7) and (11) imply that capital at an interior SREE should satisfy the following equation

\[
\phi(k) = dxh(k, x),
\]

where \(\phi(k) := (w(k) - k)(r(k) - 1)\). The following two assumptions will be used to prove existence of a unique positive SREE under autarky.

**Assumption 1** The elasticity of the production function with respect to capital (or the capital share in production) \(\alpha : \mathbb{R}_+ \to [0, 1]\) defined as

\[
\alpha(k) := \frac{k f'(k)}{f(k)},
\]

satisfies the inequality \(\alpha(k) < 0.5\) for any \(k \in [0, \hat{k}]\).

**Assumption 2** The consumer’s absolute risk tolerance \(T : \mathbb{R}_+ \to \mathbb{R}_+\) defined as

\[
T(c) := -\frac{u''(c)}{u'(c)},
\]

is a non-increasing function.

Assumption 1, used in Lemma 1, implies some important properties of the function \(\phi\), while Assumption 2 will be used in Lemma 2 to establish a monotonicity property of the function \(h\). The claims of Lemmas 1 and 2 provide the main arguments to prove the following proposition.

**Proposition 2** If Assumptions 1 and 2 are satisfied, then in autarky there exists one corner and one interior SREE.

**Proof:** Proposition 2 is a direct consequence of Lemmas 1 and 2 whose proofs are given in the appendix. Without loss of generality, let \(x = 1\). Then, on the one hand, the function \(\phi(k) := (w(k) - k)(r(k) - 1)\) is strictly decreasing on the interval \(k \in [0, \hat{k}]\), \(\phi(0) = \infty\), and \(\phi(\hat{k}) = 0\) (see Lemma 1). On the other hand, the function \(k \mapsto h(k, 1)\) is positive

\[2\text{Strict monotonicity and concavity of the production function and the Inada conditions imply the existence of a unique } \hat{k} > 0 \text{ solving the equation } r(k) = 1.\]
and strictly increasing on the same interval \( k \in [0, \hat{k}] \) (see Lemma 2). This implies that equation (12) admits a unique interior solution \( k^* \in (0, \hat{k}) \) for \( x = 1 \), with an associated interior asset price \( p^* = P(k^*, 1) > 0 \).

The expressions in (11) imply that \((k, p) = (0, 0)\) is a SREE on the boundary. If \( k = 0 \), wage income and the equilibrium asset price are both zero \((w, p) = (0, 0)\), and for a zero asset price, \( k = 0 \) is a fixed point of the capital accumulation equation. QED.

Restricting the elasticity of production to be less than one half, as is done in Assumption 1, is crucial and important to obtain existence and uniqueness of an interior SREE. Its impact is striking, since there seems to be neither an immediate justification for it to hold nor an economic intuition why it has such an important influence. When it is violated, one can verify easily that the function \( \phi \) is not necessarily monotonic. Then, equation (12) can admit either no or multiple interior solutions\(^3\). To understand the significance, however, one observes that the elasticity \( \alpha(k) \) is a (first order) measure for the curvature of the production function \( f \) which determines simultaneously wages and returns in an additive way, \( f(k) = w(k) + kr(k) \). Yet, the multiplicative form of the function \( \phi \) involves also second order properties of the curvature of \( f \) which are weak for \( \alpha(k) < 0.5 \). They become strong when \( \alpha(k) \) is larger than one half.

For the remainder of the analysis Assumption 1 will be made throughout, since the purpose of the paper is to single out causes for multiplicity induced by asset market integration. Therefore, it is desirable to restrict the analysis to situations when the closed economy with a domestic asset market has a unique interior steady state (implying the existence a unique, interior, and symmetric steady state in the world economy). Then, if multiple SREE’s and instability of the symmetric equilibrium in the world economy arise after combining the domestic asset markets, the integration can be identified as the sole cause of instability and for symmetry breaking.

### 3.2 Dynamics Under Rational Expectations

In order to analyze the dynamics of the closed economy under rational expectations we apply the concept of the minimum state variable solution (MSV). From the dynamical point of view this corresponds to the associated functional rational expectations equilibrium discussed and used in the literature, see for example, Spear (1988), McCallum (1998, 1999), Böhm & Wenzelburger (2004).

Equation (5) implies that the equilibrium asset price in any \( \text{given} \) period is affected by the expectations about next periods asset price, about the future capital return, and the moments of next periods random dividend payments. One might expect this to imply that capital accumulation would become random. However, since the realizations of the random dividend affect old consumers consumption only \( \text{and since dividends are independent and identically distributed (making the moments of the random dividend constant over time)} \), it follows that capital accumulation under perfect foresight will be

\(^3\)see Böhm & Vachadze (2008) for details
deterministic. As a consequence, consumers can chose consistent deterministic (point) forecasts for next periods capital stock and its return based on the current asset price.

The essential property of the minimum state variable solution MSV stipulates that the asset price in any given period can be determined as a function of the current existing capital stock alone. If this is the case, the capital accumulation equation implies an explicit perfect predictor for next periods asset price and for the future capital return. In other words, assume for the moment that the asset market clearing price is a function of current capital alone, \( p_t = \mathcal{P}(k_t) \). Then, the capital accumulation equation implies that next periods capital

\[
k_{t+1} = \mathcal{G}(k_t) \equiv w(k_t) - \mathcal{P}(k_t)
\]

is also a function of current capital alone. As a consequence, the price and interest rate predictors can be chosen as \( p_{t+1} = \mathcal{P}(\mathcal{G}(k_t)) \) and \( r_{t+1} = r(\mathcal{G}(k_t)) \). In order for them to induce perfect foresight they must be consistent with the price law. In other words, they must satisfy the functional equation

\[
\mathcal{P}(k_t) \equiv S(w(k_t), r(\mathcal{G}(k_t)), \mathcal{P}(\mathcal{G}(k_t)), 1),
\]

for any \( k_t \in \mathbb{R}_+ \). Thus, the pair of functions \( (\mathcal{G}, \mathcal{P}) \) satisfying the system of functional equations (15) and (16) completely describes the evolution of the economy under rational expectations, which induces the so called MSV solution. In fact, with one mild additional assumption on the technology a full characterization of the rational expectations dynamics for the model here is possible.

**Assumption 3** The production function is such that \( \lim_{k \to 0} -kf''(k) = \infty \).

Since \( w'(k) = -kf''(k) \), Assumption 3 implies that the wage function has an unbounded slope at the origin.

**Proposition 3** If Assumptions 1, 2, and 3 are satisfied, then the corner equilibrium is unstable, while the interior equilibrium is globally stable under rational expectations dynamics.

Proposition 3 implies that for any economy of the given type, there exists a unique interior SREE in autarky which is globally stable under rational expectations dynamics. Thus, economies with the same characteristics of consumers and producers converge to the same positive steady state independently of initial conditions, implying identical income, identical capital returns, and an identical asset price in the long run.

### 4 A Two Country Model

Consider now a world economy composed of two identical economies of the above type. The two countries are denoted by \( h \) (for home country) and by \( f \) (for foreign country). Consumers and firms in each country have identical characteristics. Factors of production,
capital and labor, are immobile across countries. However, consumers in both countries can now trade the consumption commodity against the financial asset in a unified international market, i.e. agents of each country can invest in domestic capital and in a financial asset from an integrated international market, but not in the foreign capital market. Notice, that this implies that there is no financial intermediation in the usual sense, with an unrestricted real credit/deposit market as a perfect substitute to real capital in each country. Therefore, the immediate mechanism equalizing deposit rates between countries and, therefore, real rates of return on capital under perfect foresight with no uncertainty is missing. Hence, there is room and potential for a diverse development of capital accumulation in the two economies. The objective is to examine whether the unification of the two asset markets alone still leads to general convergence of the two countries or under which circumstances the unification of the asset markets may induce symmetry breaking. The analysis will again be carried out when consumers in both countries have perfect foresight.

Before integration of the asset markets, young and old consumers of each country carry out their transactions only domestically. After joining the two asset markets into one, consumers of both countries will trade on the international asset market at a uniform price. The demographic structure of the model implies that all financial assets sold by old consumers of both countries are bought by young consumers. Since each country is endowed with one unit of the asset it follows that the total number of available assets in the international financial asset market is now two. This implies that for a given non-negative vector \((w_h^t, w_f^t, r_h^{t+1}, r_f^{t+1}, p_t^{t+1}) \geq 0\) of domestic and foreign wage incomes, next period’s rates of returns on capital, and next period’s asset price \(p_t^{t+1}\) (measured in units of the consumption good), an asset price \(p_t\) clearing the international asset market must solve the equation

\[
\varphi(w_h^t, r_h^{t+1}, p_t^{t+1}, p_t) + \varphi(w_f^t, r_f^{t+1}, p_t^{t+1}, p_t) = 2.
\]

Given our assumptions, equation (17) has a unique solution since the aggregate asset demand function is strictly decreasing in \(p_t\). Let

\[
p_t = S(w_h^t, w_f^t, r_h^{t+1}, r_f^{t+1}, p_t^{t+1})
\]

denote the unique asset market clearing price.

### 4.1 Stationary Rational Expectations Equilibria

**Definition 2** A Stationary Rational Expectations Equilibrium (SREE) in the world economy is a triple \((k^h, k^f, p)\) such that

- given \((k^h, k^f) \in \mathbb{R}_+^2\), the price \(p \in \mathbb{R}_+\) clears the asset market under perfect foresight, i.e. \(p\) is a fixed point of the temporary price law

\[
p = S(w(k^h), w(k^f), r(k^h), r(k^f), p);
\]
4.1 Stationary Rational Expectations Equilibria

• given \( p \in \mathbb{R}_+ \), the pair \((k^h, k^f)\) is a fixed point of each country’s capital accumulation equation

\[
\begin{align*}
  k^h &= A(k^h, p) := w(k^h) - p \varphi(w(k^h), r(k^h), p, p) \\
  k^f &= A(k^f, p) := w(k^f) - p \varphi(w(k^f), r(k^f), p, p).
\end{align*}
\]  

(20)

Since there are two steady states in each closed economy, \(0\) and \(k^*\) (where \(k^*\) solves the equation (12) with \(x = 1\)), it follows that two symmetric steady states from autarky survive after integrating the asset market, i.e. the points \((0, 0)\) and \((k^*, k^*)\) are also equilibria in the two country world economy. In addition, there are two asymmetric steady states in which one country absorbs all assets with positive capital while the other deteriorates to zero levels of capital and income. Thus, \((0, \tilde{k})\) and \((\tilde{k}, 0)\) are two asymmetric steady states in the two country economy, where \(\tilde{k}\) solves equation (12) with \(x = 2\). The interesting issue to examine is whether after integrating the asset markets internationally, there are additional equilibria where both countries hold positive quantities of the asset at positive but different levels of capital. In order to study the existence of such interior asymmetric steady states, we introduce the following concepts and notation.

For any given interior level of asset holdings \(x \in (0, 2)\), let \(k = \pi(x)\) denote the unique interior solution of equation (12) \(^4\). Then, for any distribution of asset holdings \((x, 2-x)\) among the two countries, there exist associated SREE levels of capital in each country \(k^h = \pi(x)\) and \(k^f = \pi(2-x)\). Given these capital levels and the asset holdings \((x, 2-x)\), there are corresponding supporting asset market clearing prices \(p^h = \Pi(x)\) and \(p^f = \Pi(2-x)\) in each country. The function \(\Pi\) is defined as

\[
\Pi(x) := P(\pi(x), x),
\]  

(21)

where \(P(k, x)\) is the inverse demand function as defined in equation (11). Thus, \(\Pi\) has to be interpreted as the stationary inverse demand function under perfect foresight for given asset holdings \(x\).

Finally, the asset price \(p\) at a stationary rational expectations equilibrium with asset distribution \((x, 2-x)\) after asset market integration must be the same as the two supporting asset prices in the two countries, i.e. \(\Pi(x) = p^h = p^f = \Pi(2-x)\). Therefore, at an international SREE, asset holdings \(x\) by consumers in the home country must be such that

\[
\Psi(x) := \Pi(x) - \Pi(2-x) = 0.
\]  

(22)

In other words, the asset holdings \(x\) (in the home country) at any stationary rational expectations equilibrium in the world economy must be a zero of \(\Psi(x)\).

In order to study the existence of asymmetric steady states, we first establish some properties of the continuous function \(\Psi\) which is differentiable on \((0, 2)\). By construction, one has \(\Psi(1) = 0\). Equation (12) implies that \(\lim_{x \to 0} \pi(x) = \hat{k}\), which, combined with equation (21), implies

\[
\lim_{x \to 0} \Pi(x) = \lim_{k \to \hat{k}} P(k, 0) = \lim_{k \to \hat{k}} dq_{r(k)} = \infty.
\]

\(^4\)Assumptions 1 and 2 together with Proposition 2 guarantee the existence and uniqueness of \(k\) solving the equation \(\phi(k) = dxh(k, x)\) for any \(x \in (0, 2)\).
Moreover, since \( \lim_{x \to 2} \Pi(x) \) is finite, one obtains as the boundary behavior for \( \Psi \)
\[
\lim_{x \to 0} \Psi(x) = \infty \quad \text{and} \quad \lim_{x \to 2} \Psi(x) = -\infty.
\] (23)

In order to find a sufficient condition for the existence of at least two asymmetric steady states, we consider the stationary asset demand and its elasticity. Define the stationary asset demand function
\[
X(k, p) := \begin{cases} 
0 & \text{if} \quad p \leq \underline{p} \\
x & \text{if} \quad p \in (\underline{p}, \overline{p}) \\
\infty & \text{if} \quad p \geq \overline{p}
\end{cases},
\] (24)

where \( x \) is the solution of the equation \( P(k, x) = p \) for a given pair \((k, p)\) and constants \( \underline{p} \) and \( \overline{p} \) are defined as \( \underline{p} := P(k, 0) \) and \( \overline{p} := P(k, \infty) \). The monotonicity of the function \( P \) (see Lemma 2) implies that stationary asset demand is monotonically increasing with respect to its first argument \( k \) and monotonically decreasing with respect to its second argument \( p \). Define
\[
\epsilon(k, p) := \frac{kX_k(k, p)}{X(k, p)} \quad \text{and} \quad \sigma(k) := -\frac{f'(k)(f(k) - kf''(k))}{kf''(k)f(k)},
\] (25)
as the elasticity of asset demand with respect to capital and the elasticity of factor substitution in production respectively. Now, consider the symmetric steady state \( x^* = 1 \), and let \( \epsilon^*, \alpha^*, \sigma^* \) and \( s^* \) denote the respective values of the elasticity of asset demand with respect to capital, the capital share in production, the elasticity of factor substitution, and the share of wage income spent on the asset market all evaluated at the symmetric steady state
\[
\epsilon^* = \epsilon(k^*, p^*), \quad \alpha^* = \alpha(k^*), \quad \sigma^* = \sigma(k^*) \quad \text{and} \quad s^* = p^*/w(k^*).
\] (26)
Then, one obtains the following result for the existence of interior asymmetric steady states.

**Proposition 4** If Assumptions 1 and 2 are satisfied, then the inequality

\[
\delta^* := \epsilon^* - \frac{1}{s^*} \left( \frac{\alpha^*}{\sigma^*} - 1 \right) - 1 < 0
\]  

(27)

implies the existence of at least two interior asymmetric steady states in the world economy.

The proposition states a sufficient condition for the existence of interior asymmetric steady states. This must hold locally at the symmetric steady state. It identifies general requirements for the production function and the utility function implying a positive slope of the function \( \Psi \) at the symmetric steady state. Multiplicity then follows from continuity and from the boundary behavior of \( \Psi \). Observe that the condition (27) implies that interior asymmetric steady states are more likely to co-exist with the symmetric one whenever the elasticity of asset demand \( \epsilon^* \) and the elasticity of factor substitution \( \sigma^* \) are both small at the same time. Figure 2(a) portrays the situation when \( \Psi'(1) > 0 \). Then, there exist at least two additional interior steady states in the world economy. Clearly, condition (27) is only a sufficient condition for the existence of interior asymmetric steady states. As Figure 2(b) indicates there may well exist asymmetric steady states even when \( \Psi'(1) \) is negative and (27) fails to hold.

Inequality (27) reveals that asset demand needs to be sufficiently inelastic at the symmetric steady state to guarantee the existence of interior asymmetric steady states. In order to study the role of the assumptions needed for the above result, notice that the definition of asset demand given in equation (24) implies the identity

\[
P(k, X(k, p)) \equiv p.
\]  

(28)

Applying the implicit function theorem to equation (28) reveals that the elasticity of asset demand at the symmetric steady state can be represented as the ratio of elasticities of the inverse demand function with respect to capital and with respect to asset holdings

\[
\epsilon^* = \frac{k^* P_k(k^*, x^*)}{x^* P_x(k^*, x^*)},
\]  

(29)

both evaluated at \((k, x) = (k^*, x^*)\). Thus, asset demand is inelastic when the inverse demand function \( P \) is very sensitive with respect to asset holdings and insensitive with respect to capital. One direct consequence of equation (29) is that the assumption of the randomness of dividends and of the concavity of the utility function are absolutely necessary for (27) to hold.

Suppose there were no uncertainty in dividend payments. Then, \( q = 1 \) and the inverse demand function given in (11) becomes

\[
P(k, x) = \frac{d}{r(k) - 1}.
\]  

(30)
4.2 Dynamics Under Rational Expectations

Equation (30) immediately implies infinitely elastic asset demand, because $P_x(k, x) = 0$ for any $(k, x) > 0$. When $\epsilon^* = \infty$, inequality (27) fails to be satisfied for any $(k, x) > 0$. In other words, asset price equalization from equation (30) implies equalization of capital stocks as well and immediate convergence to the symmetric steady state. Thus, symmetry breaking cannot occur.

An analogous result holds if agents are risk neutral. With linear utility, equation (11) implies that the risk adjusted probability of a good state realization is constant and independent of the pair $(k, x)$ with $h(k, x) = q$. This again implies immediate convergence of the world economy to the symmetric equilibrium after asset market integration, because the asset demand is infinitely elastic and thus inequality (27) is never satisfied.

Summarizing this discussion, when there is no uncertainty or when agents are risk neutral, the equilibrium asset price does not depend on the level of asset holdings in the two countries. The price must be equal to the discounted value of the expected dividend (equation (30)). This means that asset price equalization implies the equalization of capital returns and the equalization of capital stocks. When $q \neq 1$ and agents are risk averse, the inverse demand function $P(k, x)$ depends positively on $k$ and negatively on $x$. Therefore, after asset market integration, returns on capital are equalized with risk adjusted returns on financial assets within each country. However, risk adjusted returns can differ in stationary equilibria, which implies the possibility of asymmetric steady states and of symmetry breaking.

4.2 Dynamics Under Rational Expectations

In order to analyze the dynamics of the economy we proceed as in the case of autarky and use the minimum state variable solution (MSV). Suppose there exists a function $\mathcal{P} : R_+^2 \to R_+$ such that the uniform asset price can be determined by the capital stocks in each country, $p_t = \mathcal{P}(k^h_t, k^f_t)$. Then, the capital accumulation equations imply that $(k^h_{t+1}, k^f_{t+1})$ should satisfy

$$k^h_{t+1} = w(k^h_t) - s(k^h_t, k^h_{t+1}, k^f_{t+1})$$
$$k^f_{t+1} = w(k^f_t) - s(k^f_t, k^h_{t+1}, k^f_{t+1})$$

(31)

where $s(k^h_t, k^h_{t+1}, k^f_{t+1})$ and $s(k^f_t, k^h_{t+1}, k^f_{t+1})$ are total spending on the international financial market by young agents of countries $h$ and $f$. These functions are defined as

$$s(k^h_t, k^h_{t+1}, k^f_{t+1}) := \varphi \left( w(k^h_t), r(k^h_{t+1}), \mathcal{P}(k^h_{t+1}, k^f_{t+1}), \mathcal{P}(k^h_t, k^f_t) \right) \mathcal{P}(k_h, k_f)$$
$$s(k^f_t, k^h_{t+1}, k^f_{t+1}) := \varphi \left( w(k^f_t), r(k^f_{t+1}), \mathcal{P}(k^h_{t+1}, k^f_{t+1}), \mathcal{P}(k^h_t, k^f_t) \right) \mathcal{P}(k_h, k_f).$$

(32)

In order for the price predictor to be perfect it should be consistent with the price law, i.e. it should satisfy the following functional equation

$$\mathcal{P}(k^h_t, k^f_t) \equiv S(w(k^h_t), w(k^f_t), r(k^h_{t+1}), r(k^f_{t+1}), \mathcal{P}(k^h_{t+1}, k^f_{t+1})).$$

(33)
Proposition 5 \textit{If Assumptions 1, 2, and 3 are satisfied, then the symmetric steady state $(0, 0)$ is a source while the asymmetric steady states $(\bar{k}, 0)$ and $(0, \bar{k})$ are unstable saddles.}

The instability of the corner steady states implies that no matter what is the initial capital distribution in the economy, the steady states $(0, 0)$, $(\bar{k}, 0)$ and $(0, \bar{k})$ cannot be reached from interior capital allocations. In order to analyze the stability of the symmetric interior steady state, we evaluate the Jacobian matrix at the symmetric steady state. One finds that the trace $T$ and the determinant $D$ of the Jacobian matrix at the symmetric steady state are

$$T = 2G_1 \quad \text{and} \quad D = G_1^2 - G_2^2,$$

where $G_1 \equiv G_1(k^*, k^*)$ and $G_2 \equiv G_2(k^*, k^*)$ are the derivatives of the function $G$ with respect to its first and second argument respectively, evaluated at the symmetric steady state. Since $T^2 - 4D = 4G_2^2 > 0$ it follows that both roots of the characteristic polynomial

$$\lambda^2 - T\lambda + D = 0,$$

are real with $\lambda_1 = G_1 + G_2$ and $\lambda_2 = G_1 - G_2$. Equations (31) and (32) imply that the functions $G_1$ and $G_2$ satisfy the system of equations

$$\begin{cases}
G_1 & = w' - (\varphi_1 w' + \varphi_2 r' G_1 + \varphi_3 (P_1 G_1 + P_2 G_2) + \varphi_4 P_1) P - \varphi P_1 \\
G_2 & = - (\varphi_2 r' G_2 + \varphi_3 (P_1 G_2 + P_2 G_1) + \varphi_4 P_2) P - \varphi P_2.
\end{cases}$$

(38)

where

$$\begin{cases}
P_1 & = S_1 w' + S_3 r' G_1 + S_4 r' G_2 + S_5 (P_1 G_1 + P_2 G_2) \\
P_2 & = S_2 w' + S_3 r' G_2 + S_4 r' G_1 + S_5 (P_1 G_2 + P_2 G_1).
\end{cases}$$

(39)

Clearly, at the symmetric steady state, $S_1 = S_2$, $S_3 = S_4$, and therefore, $P_1 = P_2$ holds. This property together with equation (39) implies that

$$P_1 = S_1 w' + S_3 r' (G_1 + G_2) + S_5 P_1 (G_1 + G_2).$$

(40)

Applying similar arguments as those used in the proof of Proposition 3, one finds that $\lambda_1$ satisfies $\lambda_1 = G_1 + G_2 \in (0, 1)$. From the system of equations (38) one obtains that the second root of the characteristic equation satisfies

$$\lambda_2 = G_1 - G_2 = \frac{w' (1 - \varphi_1 \varphi_1 P)}{1 + \varphi_2 r' P} > 0,$$

(41)
since \( \varphi_1 \mathcal{P} < 1, \ varphi_2 < 0, \) and \( r^\prime < 0. \) Therefore, \( \lambda_1 \in (0, 1) \) and \( \lambda_2 > 0 \) implies that the symmetric steady state can lose its stability only by undergoing a fold bifurcation.

Let \( \Omega = \mathbb{R}_+^+ \times (0, 1) \) denote the space of parameters \((d, q)\) characterizing the exogenous production process (the “Lucas tree”). Let us define sets

\[
\begin{align*}
\Omega^\ast & := \{ (d, q) \in \Omega | \delta^* < 0 \}, \\
\Omega^c & := \{ (d, q) \in \Omega | \delta^* = 0 \}, \\
\Omega^s & := \{ (d, q) \in \Omega | \delta^* > 0 \},
\end{align*}
\]

(42)

where \( \delta^* \) is defined in equation (27). The function \( \Psi \) defined in equation (22) and displayed on Figure 2 slopes upwards (downward) at \( x = 1 \) when \((d, q) \in \Omega^u\) (when \((d, q) \in \Omega^s\)), while it is tangent to the zero line at \( x = 1 \) when \((d, q) \in \Omega^c\). It is evident from the inverse demand function and the stationarity condition (equations (11) and (12)) that the parameter values of the random process \((d, q)\) interact in an important non linear way with the production function and the utility function determining the critical slope of the excess price map \( \Pi \) in each country. Thus, as soon as the parameters do not belong to the region \( \Omega^s \), the values of \((d, q)\) also induce a non linear impact on the local stability of the mapping \( \mathcal{G} \) at the symmetric steady state, a feature which is common to many symmetric dynamical systems of the form under consideration here. As a consequence one obtains the following final result on symmetry breaking.

**Proposition 6** If Assumptions 1, 2, and 3 are satisfied, then the interior and symmetric steady state of the world economy is asymptotically stable only if \((d, q) \in \Omega^s\). As the parameter values leave the region \( \Omega^s \) the symmetric steady state looses its stability by undergoing a fold bifurcation.

The main result of this section can be summarized as follows. Given Assumptions 1 – 3, the analysis shows that symmetry breaking in the sense of Matsuyama (1996) occurs whenever the parameters of the exogenous production process leave a certain well defined set \( \Omega^s \). Propositions 4 and 6 together imply that the set \( \Omega^s \) is non empty and that non cyclical divergence occurs in the neighborhood of the symmetric steady state. Thus, the capital stocks of any two countries in a world economy with capital endowments arbitrarily close to the symmetric steady state will not converge to the symmetric steady state but rather diverge to an asymmetric stationary allocation of capital. As a consequence, output per capita, wages, and rates of return on capital will differ in the two countries.

Finally, one may also ask in which way asset market integration may affect the welfare in each country. Suppose \( \bar{c}^i \) and \( \underline{c}^i \) denote the steady state consumption levels in good and bad states in countries \( i = h, f \) respectively. Then, equation (10) implies that

\[
\underline{c}^i = f(k^i) - k^i \quad \text{and} \quad \bar{c}^i = f(k^i) - k^i + x^id. \tag{43}
\]

Clearly, when the world economy converges to a symmetric steady state then \((k^h, x^h) = (k^f, x^f) = (k^*, 1)\) and thus, the steady state welfare levels would be identical implying neither gain nor loss of welfare due to the asset market integration.

In contrast, the steady state welfare level will worsen in the initially poor country while it will improve in the initially rich country as a result of asset market integration. This
5 A Numerical Example

may be seen from the following argument. Suppose the symmetry breaking takes place and the world economy converges to an asymmetric steady state \((k^h, k^f)\). Clearly \(k^h > k^f\) \((k^h < k^f)\) if and only if \(k^h > k^f\) \((k^h < k^f)\) at the time of asset market integration. The equilibrium asset demand function, defined in equation (24), is monotonic with respect to steady state capital. Thus, asset price equalization implies that equilibrium asset holdings satisfy \(x^h > x^f\) when \(k^h > k^f\) and \(x^h < x^f\) when \(k^h < k^f\). Since the steady state capital stock satisfies the inequality \(r(k_i) > 1\) if follows from equation (43) that \((c^h, \bar{c}^h) > (c^f, \bar{c}^f)\) only if \(k^h > k^f\) at the time of asset market integration. Thus, the initially poor country will never improve, while the initially rich country will never worsen the steady state welfare levels as a result of asset market integration.

5 A Numerical Example

This section presents a parameterized version of the economy described above, providing a large and robust class of examples of economies with stable asymmetric steady states for an admissible configuration of parameters. The example also reveals further insight into the qualitative features of the non-linearities appearing, giving evidence of the occurrence of symmetry breaking for such economies.

Let the production function be isoelastic and of the form \(f(k) := Ak^\alpha\), and assume that the utility function \(u\) is such that its first derivative is

\[
 u'(c) := \begin{cases}
 \exp\left(-a \frac{c^{1-b}}{1-b}\right) & \text{if } b \neq 1 \\
 c^{-a} & \text{if } b = 1.
\end{cases}
\] (44)

The derivative form (44) implies an absolute risk aversion function given by

\[
 T(c) = -\frac{u''(c)}{u'(c)} = ac^{-b}.
\] (45)

Thus, the absolute risk aversion function has a constant elasticity equal to \(-b\). The parameter \(a\) measures the scale and \(b\) measures the curvature of absolute risk aversion. When \(b = 0\), the absolute risk aversion is constant, \(T(c) = a\), while when \(b = 1\) absolute risk aversion is \(T(c) = ac^{-1}\). Therefore, this specification includes both the CARA and CRRA utility functions as special cases.

It is straightforward to verify that the production function satisfies Assumptions 1 and 3 when \(0 < \alpha < 0.5\), and that the utility function satisfies Assumption 2 when \(a > 0\) and \(b \geq 0\). In this case, Proposition 2 implies that for any \(x \in (0, 2)\) there exists a unique \(k\) satisfying equation (12). In order to obtain and analyze the functions \(\pi(x)\) and \(\Pi(x)\), we use numerical procedures for which we choose the parameter values given in Table 1.

Figure 3 displays the graphs of the two functions, which have been calculated numerically for the standard parameter set. The diagram provides the basic intuition of the role of the non-monotonicity of the function \(\Pi\) for the existence of an asymmetric SREE.
Given the values of the parameters, one finds that the level of stationary capital described by the function $\pi(x)$ is first decreasing in a non linear way for low values of $x$ and steeply increasing for large $x$, see panel (a) of Figure 3. The primary effect of an increase in asset holdings is a decrease in the equilibrium asset price. As asset holdings increase, young consumers have to bear more risk. However, in spite of the decrease of the asset price, their willingness to pay for the asset declines more quickly than the increase in their asset demand. As a result, their asset demand becomes relatively inelastic causing asset market spending to decline. This in turn causes the stationary capital level $k = \pi(x)$ to rise again.

In other words, as asset holding increases the induced change of the stationary level of capital reverses and increases again, caused by a decrease in spending on assets. This reversal effect on the stationary capital is reinforced when transmitted into the inverse demand function $P(k, x)$, as can be seen from the definition of $\Pi(x) := P(\pi(x), x)$, equation (21). If this effect is sufficiently strong it generates first a decline of the supporting price at the symmetric steady state $x = 1$ which reverses to a steep increase for large asset holdings (see Figure 3.(b)). This effect combined with the boundary behavior of the function $\Pi$ then causes the occurrence of asymmetric steady states.

In addition to the graphical representation, it may be informative to compute the numerical values at the steady states for the parameters chosen. At the symmetric steady state, investment in physical capital and spending on the asset market account for 8.3% and 91.7% of the wage income respectively. The annual rate of return on capital is 5.2% and the elasticity of asset demand with respect to capital is 0.208. This together with inequality (27) implies a critical value $\delta^* = -0.06$, which is a sufficient condition for symmetry breaking. At the asymmetric steady state, the initially rich and the initially poor countries hold $x^h = 1.42$ and $x^f = 0.58$ units of the asset respectively. Investment in physical capital is 26.2% and 2.4% of their wage income respectively. While, annual rates of return on capital in the rich and in the poor country are 1.82% and 9.05% respectively.
Steady state levels of the capital stock, of wage income, and of asset holdings are higher in the initially rich country, implying that the country with a high stationary level of capital attains a high level of welfare as well.

6 Summary and Conclusions

The paper analyzes possible implications of unifying a market of a financial paper asset internationally in a two country model of economic growth. The standard neoclassical growth model of two identical countries with capital accumulation and OLG consumers is extended to include a market of a financial asset, enabling the transfer of ownership among generations of an exogenous random production process (a so-called “Lucas” tree). Under complete separation and autarky, with perfect competition and rational expectations, both economies converge under general assumptions to the same globally attracting steady state with identical capital levels, incomes, consumptions, and asset returns. Thus, when the asset markets are separate, capital accumulation, and asset price development adjust independently in each country leading to inter-country income convergence. However, when the two asset markets are integrated (neutralizing all size and randomness effects between the two identical economies), there exist general conditions of consumer preferences and of production technologies, such that stable asymmetric steady states appear endogenously, while the symmetric steady state becomes unstable. In this case, initial capital endowments become crucial in determining the long run development of otherwise identical economies. Thus, while the clearing of the international asset market still guarantees a uniform asset price and return in that market, capital accumulation and incomes processes in the two countries diverge making long run incomes, consumption levels, and rates of return on capital unequal.

The result is shown to be robust and to occur for a general setting. The paper identifies the integration of the market for the financial asset as a cause uniquely, most transparently, and without disguise. By choosing an environment with uncertainty and risk aversion of consumers in an otherwise standard convex environment with perfect competition and rational expectations, the model contains two distinct investment opportunities which are not perfect substitutes. Therefore, consumers hold mixed portfolios in general with endogenous substitution effects between the financial asset and real capital. The paper identifies two sided spill over effects between real markets and asset markets induced by portfolio behavior of rational consumers as the major villain of an un-equalizing force of growth between otherwise identical countries. While under autarky these forces are stabilizing within each country, they can create diverging effects after allowing trade and the integration of markets for financial assets.

While the literature has discussed other structural causes like indivisibilities, liquidity constraints, etc. (as for example in Matsuyama 2004) leading to asymmetries in the development of nations, the results here do not require the interaction with such features. However, in both models there is no foreign direct investment. In such situations differences in rates of returns on capital across countries can persist, if markets for real deposits or credit either do not exist, or if they are restricted, as in Matsuyama (2004). If these
restrictions are removed symmetry breaking no longer occurs in either model. It seems to be a challenging question to investigate a more extended model with uncertainty, a proper market of a financial asset and with real credit market restrictions. A fortiori, the same endogenous mechanism as found here would exist in any more general setting as well. Thus, whether symmetry breaking occurs in such cases depends on the strength and possible countervailing forces between the mechanisms.

7 Appendix

Lemma 1 Let \( \hat{k} \) denote the unique solution of \( r(k) = 1 \) and define the function \( \phi : [0, \hat{k}] \to \mathbb{R}_+ \) as

\[
\phi(k) := (w(k) - k)(r(k) - 1),
\]

where \( w \) and \( r \) are the functions of the wage and the interest rate respectively. If Assumption 1 is satisfied then \( \phi(0) = \infty, \phi(\hat{k}) = 0, \) and \( \phi \) is non-negative and strictly decreasing.

Proof: Since \( r(\hat{k}) = 1 \) it follows that \( \phi(\hat{k}) = 0 \). Assumption 1 implies that \( kr(k) < w(k) \) on the interval \( [0, \hat{k}] \). This with inequality \( r(k) \geq 1 \) implies that \( w(k) > k \) and thus \( \phi(k) \geq 0 \) for all \( k \in [0, \hat{k}] \).

In order to show that \( \lim_{k \to 0} \phi(k) = \infty \) we use the following argument. On the one hand \( \lim_{k \to 0} \phi(k) = \lim_{k \to 0} w(k)r(k) \). On the other hand, Assumption 1 implies that for sufficiently small \( k \), \( f(k) = Ck^\alpha(0) \) where \( C > 0 \) is some constant and \( \alpha(0) \) is the elasticity of production at \( k = 0 \). This implies that for sufficiently small \( k \), \( r(k) = C\alpha(0) k^{\alpha(0) - 1} \), and \( w(k) = C(1 - \alpha(0)) k^{\alpha(0)} \). Combined with the inequality \( \alpha(k) < 0.5 \) for any \( k \in [0, \hat{k}] \) implies

\[
\lim_{k \to 0} w(k)r(k) = C^2 \alpha(0)(1 - \alpha(0)) \lim_{k \to 0} k^{2\alpha - 1} = \infty.
\]

In order to show that \( \phi \) is a strictly decreasing function we rewrite equation (46) as follows.

\[
\phi(k) = w(k)r(k) - kr(k) - w(k) + k = w(k)r(k) - (f(k) - k).
\]

On the one hand \( r'(k) < 0 \) and \( w(k) > kr(k) \) implying

\[
[w(k)r(k)]' = w'(k)r(k) + w(k)r'(k) = -kr'(k)r(k) + w(k)r'(k) = r'(k)[w(k) - kr(k)] < 0.
\]

On the other hand \( r(k) \geq 1 \) for \( k \in [0, \hat{k}] \) implying

\[
-[f(k) - k]' = -r(k) + 1 < 0.
\]

It follows from inequalities (49) and (50) and from equation (48) that \( \phi' < 0 \) for \( k \in [0, \hat{k}] \).
Lemma 2 Define the function $h : [0, \hat{k}] \times \mathbb{R}_+ \to [0, 1]$ as

$$h(k, x) := \frac{qu'(g(k) + xd)}{qu'(g(k) + xd) + (1-q)u'(g(k))},$$  \hspace{1cm} (51)

where $u$ is the utility function, $g(k) := f(k) - k$, and $f$ is the production function. If Assumption 2 is satisfied, then the function $h$ is non-decreasing with respect to its first and non-increasing with respect to its second argument.

**Proof:** Continuous differentiability of the $h$ function follows since $u$ and $g$ are twice continuously differentiable. Differentiating (51) with respect to $k$ implies

$$h_k(k, x) = q(1-q)\frac{u''(g(k) + xd)u'(g(k)) - u'(g(k) + xd)u''(g(k))}{[qu'(g(k) + xd) + (1-q)u'(g(k))]^2}g'(k).$$  \hspace{1cm} (52)

The nominator of the can be further simplified.

$$u''(g(k) + xd)u'(g(k)) - u'(g(k) + xd)u''(g(k)) =$$

$$= \frac{u'(g(k))}{u'(g(k) + xd)}(T(g(k)) - T(g(k) + xd)).$$  \hspace{1cm} (53)

Since $T(c)$ is non-increasing and $g'(k) > 0$ on the interval $[0, \hat{k}]$, equations (52) and (53) imply that $h_k(k, x) \geq 0$.

In order to show that $h_x(k, x) \leq 0$ for a given $k \in [0, \hat{k}]$, we take the natural log on both sides of equation (51) and then differentiate it. We obtain

$$\frac{h_x(k, x)}{h(k, x)} = -T(g(k) + xd)(1 - h(k, x))g'(k).$$  \hspace{1cm} (54)

Expression (54) implies that

$$h_x(k, x) = -T(g(k) + xd)(1 - h(k, x))h(k, x)g'(k).$$  \hspace{1cm} (55)

Since $h(k, x) \in [0, 1]$ and $g'(k) \geq 0$, equation (55) with Assumption 2 implies that $h_x(k, x) \leq 0$.

**Proof of Proposition 1:** Rewrite the consumer’s optimization problem as

$$\max_{x \in B(w,p)} qu(\overline{c}) + (1-q)u(c),$$  \hspace{1cm} (56)

where $\overline{c}_1 = wr_1 + xd - x(pr_1 - p_1)$ and $\overline{c}_1 = wr_1 - x(pr_1 - p_1)$. Let $L(x) \equiv qu(\overline{c}_1) + (1-q)u(\overline{c}_1)$, then $L'(x) = qu'(\overline{c}_1)(d - (pr_1 - p_1)) - (1-q)u'(\overline{c}_1)(pr_1 - p_1)$. From concavity of the utility function it follows that $L''(x) < 0$ for any $x \geq 0$.\]
1) Suppose \( pr_1 \geq p_1 + qd \) then the optimal asset demand is zero. Since
\[
\mathcal{L}'(0) = u'(wr_1) [qd - (pr_1 - p_1)] \leq 0
\]
and \( \mathcal{L}'' < 0 \) it follows from the Kuhn-Tucker conditions that \( x = 0 \) is an optimal solution.

2) Suppose \( pr_1 \leq p_1 \). Then consumptions in a good and bad states, \( \bar{\tau} \) and \( \underline{\tau} \), are both increasing functions of \( x \). Since the utility function is strictly increasing, it follows that the agent will invest all his wage income in the asset market and make no investment in physical capital and thus the optimal demand is \( x = w/p \).

3) Suppose \( p_1 < pr_1 < p_1 + qd \) and let \( \tau \equiv wr_1/(pr_1 - p_1) > 0 \). Then depending on whether \( \mathcal{L}'(\tau) \) is positive or negative we can have either a corner or an interior solution. A unique corner solution \( x = w/p \) exists when
\[
\mathcal{L}'(\tau) = qu' (\tau d) (d - (pr_1 - p_1)) - (1 - q) u'(0) (pr_1 - p_1) > 0.
\]
Otherwise there exists a unique and interior solution solving the equation \( \mathcal{L}'(x) = 0 \). Let \( x = \varphi(pr_1 - p_1, wr_1) \) denote the solution. Applying the Implicit Function Theorem one finds that the function \( \varphi_m \) is increasing with respect to its first and decreasing with respect to its second argument. In addition, asset demand satisfies the boundary condition. As \( p \downarrow p_1 \equiv p_1/r_1 \) then the asset demand grows unboundedly. This implies that there exists a constant \( p_2^* \in (p_1^*, p_3^*) \) such that \( p_2^* \varphi_m(p_2^*r_1 - p_1, wr_1) = w \).

\[\square\]

**Proof of Proposition 3:** Let us first show that \( \mathcal{G}' \leq 0 \) implies a contradiction. Differentiating the price law (16) we obtain that at an SREE, \( k = \mathcal{G}(k) \), the following equation should be satisfied
\[
\mathcal{P}'(k) - S_1 w'(k) = S_2 r'(k) \mathcal{G}'(k) + S_3 \mathcal{P}'(k) \mathcal{G}'(k),
\]
where \( S_1, S_2, \) and \( S_3 \) are the partial derivatives of the function \( \mathcal{S} \) with respect to its first, second, and third arguments respectively.

Since \( S_1 \in [0, 1] \) and \( \mathcal{G}' < 0 \), it follows that the left hand side of equation (59) is positive because,
\[
\mathcal{P}'(k) - S_1 w'(k) = w'(k) - \mathcal{G}'(k) - S_1 w'(k) = (1 - S_1) w'(k) - \mathcal{G}'(k) > 0.
\]
The inequalities \( S_2 < 0, S_3 > 0, r' < 0, \mathcal{P}' = w' - \mathcal{G}' > 0 \) and \( \mathcal{G}' < 0 \), imply that the right hand side of equation (59) is non-positive, because
\[
S_2 r'(k) \mathcal{G}'(k) + S_3 \mathcal{P}'(k) \mathcal{G}'(k) = (S_2 r'(k) + S_3 \mathcal{P}'(k)) \mathcal{G}'(k) \leq 0.
\]
But the inequalities (60) and (61) contradict the equation (59) and thus \( \mathcal{G}' > 0 \).

Now, let us show that \( 0 < \mathcal{G}'(0) = \gamma < \infty \) implies a contradiction. By dividing both sides of equation (59) by \( w'(k) \) we obtain
\[
\frac{\mathcal{P}'(k)}{w'(k)} = \frac{S_1}{S_1} \left( S_2 \frac{r'(k)}{w'(k)} + S_3 \frac{\mathcal{P}'(k)}{w'(k)} \right) \mathcal{G}'(k).
\]
Taking the limit of both sides of equation (62) as \( k \to 0 \) we obtain
\[
\lim_{k \to 0} \frac{\mathcal{P}'(k)}{w'(k)} - S_1 = 1 - S_1 \in [0, 1],
\]
(63)
and
\[
\lim_{k \to 0} \left( S_2 \frac{r'(k)}{w'(k)} + S_3 \frac{\mathcal{P}'(k)}{w'(k)} \right) \mathcal{G}'(k) = \lim_{k \to 0} \left( -S_2 \frac{1}{k} + S_3 \right) \gamma = \infty.
\]
(64)
Equations (62), (63), and (64) imply a contradiction and thus \( \mathcal{G}'(0) = \infty \).

Since the time one map of capital accumulation is a strictly increasing function with two fixed points \( k = 0 \) and \( k = \hat{k} \) and \( \mathcal{G}'(0) = \infty \) implies the instability of the corner steady state and the stability of the interior SREE.

**Proof of Proposition 4:** In order to show existence of interior asymmetric steady states we rely on the property of the function \( \Psi \). Since \( \Psi(1) = 0, \Psi(0) = \infty \) and \( \Psi(2) = -\infty \) it follows that the condition \( \Psi'(1) > 0 \) is sufficient for existence of interior asymmetric steady states. Equation (22) implies that \( \Psi'(1) = 2\Pi'(1) \) and thus \( \Pi'(1) > 0 \) is sufficient for existence of interior asymmetric steady states.

Since \( P(k, X(k, p)) \equiv p \) we obtain that \( X_k(k^*, p^*) = -P_k(k^*, 1)/P_x(k^*, 1) \) and the inequality (27) implies
\[
\epsilon^* < \frac{1}{s^*} \left( \frac{\alpha^*}{\sigma^*} - 1 \right) + 1 \quad \Leftrightarrow \quad -\frac{k P_k(k^*, 1)}{P_x(k^*, 1)} < \frac{w(k^*)}{p^*} \frac{w'(k^*)k^* - k^*}{w(k^*)}. \tag{65}
\]
Inequality (65) implies
\[
-\frac{P_k(k^*, 1)}{P_x(k^*, 1)} < \frac{w'(k^*) - 1}{p^*} \quad \Leftrightarrow \quad p^* P_k(k^*, 1) + (w'(k^*) - 1) P_x(k^*, 1) < 0. \tag{66}
\]
The following two identities \( \Pi(x) \equiv P(\pi(x), x) \) and \( w(\pi(x)) - \pi(x) \equiv \Pi(x) x \) imply that
\[
\Pi'(1) = P_k(k^*, 1)\pi'(1) + P_x(k^*, 1) \quad \text{and} \quad (w'(k^*) - 1) \pi'(1) = \Pi(1) + \Pi'(1). \tag{67}
\]
By solving the above system with respect to \( \pi'(1) \) and \( \Pi'(1) \) we obtain
\[
\Pi'(1) = \frac{p^* P_k(k^*, 1) + (w'(k^*) - 1) P_x(k^*, 1)}{w'(k^*) - 1 - P_k(k^*, 1)} \quad \text{and} \quad \pi'(1) = \frac{p^* + P_x(k^*, 1)}{w'(k^*) - 1 - P_k(k^*, 1)}. \tag{68}
\]
Stability of the unique interior steady state in the closed economy implies that \( w'(k^*) - 1 - P_k(k^*, 1) < 0 \), which with inequality (68) implies that if inequality (67) is satisfied then \( \Pi'(1) > 0 \).

**Proof of Proposition 6:** First we show that when \( \delta^* = 0 \) then \( \lambda_2 \) defined in equation (41) satisfies, \( \lambda_2 = 1 \). Equation \( \delta^* = 0 \) implies that
\[
\epsilon^* = \frac{1}{s^*} \left( \frac{\alpha^*}{\sigma^*} - 1 \right) + 1 \quad \Leftrightarrow \quad k^* X(k^*, p^*) = \frac{w^*}{p^*} \left( \frac{k^* f'(k^*) f(k^*) w'(k^*)}{f(k^*) f'(k^*) w(k^*) - 1} \right) + 1. \tag{69}
\]
Equation (69) implies

\[ k^* X(k^*, p^*) = \frac{k^* w'(k^*) - w(k^*) + p^*}{p^*} \iff p^* X(k^*, p^*) = w'(k^*) - 1. \]  

(70)

Since \( X(k, p) = \varphi(w(k), r(k), p, p) \), it follows from equation (70) that

\[ p^* (\varphi_1 w'(k^*) + \varphi_2 r'(k^*)) = w'(k^*) - 1 \]

Combined with equation (41), this implies \( \lambda_2 = 1 \). Therefore, from equations (69) and (70) one has that \((d, q) \in \Omega^a\) implies \( \lambda_2 > 1 \), and \((d, q) \in \Omega^s\) implies \( \lambda_2 < 1 \).

\[ \square \]

References


REFERENCES


