Tangible Assets, Capital Mobility, and Endogenous Fluctuations

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Abstract

This paper examines the joint dynamics of capital, factor returns, and tangible asset prices in a closed and open economies. Two period lived consumers with heterogeneous preferences hold portfolios consisting of real capital and tangible assets. It is shown that the world economy converges monotonically to a unique steady state when either (both) factor or (and) asset markets operate domestically. In contrast factor mobility coupled with the possibility of trading of the tangible asset internationally as an alternative to capital may cause the emergence of endogenous fluctuations. Thus, the so called business cycles may appear purely endogenously generated as a consequence of capital and asset market integration alone.

It is shown that the evolution of the world economy is represented by two-dimensional piecewise smooth discontinuous map, when factor and asset markets are integrated internationally. General condition for a unique steady state to collide with boundary and to stop/start existence is derived (border collision bifurcation). A numerical example shows that the integration of both factor and asset markets can cause emergence of high order cycles even when the production function is Cobb-Douglas.

Keywords: Capital accumulation; endogenous fluctuations; existence and stability of steady state; market integration; portfolio choice;

JEL classification: D91, E32, F41, F43, G12
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1 Introduction

Capital account liberalization have greatly accelerated the financial transactions worldwide. This liberalization has definitely contributed to rising living standards in many developing countries, however it has sometimes been associated with welfare loses for some developed/developing countries due to outsourcing of manufacturing jobs/outflow of capital. The costs and benefits of an open capital account liberalization appear to be one of the widely discussed topics in international economics. Critics of the liberalized capital account often point out to a series of financial crises that occurred recently in US as a mortgage crises and spread across the world and the crises that happened in Asia in the last decade.

In this paper we present a neoclassical growth model modified only to include a tangible asset in order to argue how greater integration of world’s financial and assets markets may affect the dynamics of capital accumulation of the world economy in a closed and open economy set-ups. After introducing the durable commodity as an alternative to real capital investment we show the existence of a unique and globally stable steady state. In an open economy, free capital mobility assures an instantaneous convergence of rates of return of capital. Under perfect competition this implies the instantaneous convergence in income levels among countries with homogeneous production and monotonic convergence of the world capital level. The same happens when countries open their asset markets internationally. This implies that equalization of asset prices and induces an instantaneous convergence of rates of return of capital and thus capital levels. Again convergence of the world capital level is monotonic to a unique steady state. However, the possibility of endogenous cycles can exist when countries integrate their capital and asset markets simultaneously.

The two country model analyzed in the paper is closely related to an overlapping generations models considered in Mountford (2004), Böhm & Vachadze (2008), and Kikuchi (2008). Mountford (2004) and Böhm & Vachadze (2008) analyze the role of tangible asset/land on the dynamics of capital accumulation in the closed economy and demonstrate the possibility of multiple steady states with different basins of attraction. Since the main question of this paper is whether greater integration of asset and capital markets can affect the dynamics of the capital accumulation, it is most transparent to restrict our attention to the case when the world economy in autarky has a unique steady state. In such set-up we can argue that whenever the possibility of endogenous fluctuations arises in the world economy it is due to factor and asset market integration alone. Similarly as in Kikuchi (2008), we introduce the tangible asset and ask whether the integration of asset markets among two countries can help or hinder the process of capital accumulation.
Differently from Kikuchi (2008) however, we also consider the possibility of integrating asset and capital markets simultaneously.

The paper is organized as follows. Section 2 introduces the model with a simpler, analytically tractable structure, which is capable of capturing the effect of the tangible asset on the dynamics of capital accumulation. Next we analyze the above described economy under different assumptions. In 3, we consider the benchmark case of autarky (A). In this environment, young agents allocate their wage income between domestic capital and domestic asset markets. In 4, we consider the world economy with integrated factor markets (F), where young agents allocate their portfolios between domestic asset and international capital markets. In 5, we consider a world economy with integrated asset markets (C), where young agents allocate their wage income between domestic capital and international asset markets. Finally, in section 6, we allow for integration of both factor and tangible asset markets (I). In such case young agents can invest in foreign capital market and at the same time can purchase tangible assets in foreign country. In all four cases we analyze and compare the dynamics of capital accumulation of the world economy. Section 7 concludes.

2 The Model

We consider an infinite-horizon, discrete time world economy consisting of two countries. Countries are denoted by $h$ (for home country) and by $f$ (for foreign country). There are two commodities in each country. A consumption commodity is produced by firms in each period. It is used for consumption and for physical capital investment. In contrast, a tangible asset, supply of which is constant and normalized to unity, exists in each country and lasts forever. It is not directly used in production and is traded between generations in a competitive asset market without any transaction cost. Existence of the tangible asset serves two purposes. First it is used for consumers to derive utility from holding it, and second, it is used by consumers as an investment vehicle because agents can re-sell the asset when old and increase their old age consumption level.

The consumption commodity is produced by identical and infinitely lived firms using capital and labor as inputs. The technology of each firm, within and across countries, is constant through time and is described by a constant return to scale production function $F : \mathbb{R}_+^2 \to \mathbb{R}$. The total output produced in country $j = h, f$, at time $t$, is $Y_t^j = F(K_t^j, L_t^j)$, where $K_t^j \geq 0$ and $L_t^j \geq 0$ are aggregate supplies of physical capital and labor respectively.

---

1The tangible asset can be interpreted broadly to include real estate, jewelry, paintings, and other assets not directly used in production.
Output per worker is $y_t^j = Y_t^j / L_t^j = F(K_t^j / L_t^j, 1) \equiv f(k_t^j)$, where $k_t^j = K_t^j / L_t^j$ denotes capital per worker and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is the production function in intensity form. We assume that $f(0) = 0$, $f$ is twice continuously differentiable, strictly increasing and strictly concave and satisfies, Inada conditions. Factor markets are competitive and factor rewards on physical capital and labor are given by their respective marginal products. Thus, $r_t^j = R(k_t^j) := f'(k_t^j)$ and $w_t^j = W(k_t^j) := f(k_t^j) - k_t^j f'(k_t^j)$ are country $j$'s rental rate of capital and wage income respectively. The produced commodity can be either consumed or invested in physical capital, which becomes available in the next period. Capital depreciates fully within a period.

The consumption sector of each country is represented by two period lived consumers, called young and old. Each generation consists of a continuum of homogeneous agents of unit mass and for simplicity we assume no population growth. Young agents are endowed with one unit of labor, which they supply inelastically to the labor market. Agents do not value first period consumption. Thus, after receiving the wage income they decide how much to invest in physical capital and how much to spend on tangible assets. Investing $i_t^j$ units of goods in capital implies the same amount of physical capital in the following period, which can be rented to the firm in exchange of $i_t^j r_{t+1}^j$ units of consumption commodities. Purchasing $x_t^j$ units of tangible assets at price $p_t^j$ implies (a) $x_t^j p_{t+1}^j$ units of consumption commodities in the next period from selling them to the market plus (b) the utility the consumer derives from holding the assets during the period. This implies the following first and second period budget constraints $i_t^j + x_t^j p_t^j = w_t^j$ and $c_t^j = i_t^j r_{t+1}^j + x_t^j p_{t+1}^j$. Agents are heterogeneous across countries and their preferences over the consumption commodity and tangible asset bundle $(c_t^j, x_t^j)$ is described by the utility functions $(c, x) \mapsto c + \pi^h x$ and $(c, x) \mapsto c + \pi^f x$, where parameters, $\pi^h$ and $\pi^f$, describe the marginal utilities, home and foreign agents obtain from asset holdings. Without loss of generality we assume that $0 < \pi^f < \pi^h < \infty$.

### 3 World Economy Under Autarky

First we analyze the situation when asset and capital markets operate only domestically. For notational convenience we drop the country index $j$. The simple demographic structure of the consumption sector implies that all assets sold by old agents are purchased by young agents. Since the number of existing assets is normalized to unity and consumers do not value first period consumption, it follows that the next period capital is given by the equation

$$k_{t+1} = \Phi_1(k_t, p_t) := W(k_t) - p_t,$$ (1)
where $p_t$ is the asset price (since the number of available assets is normalized to unity, it follows that $p_t$ is also total spending on the domestic asset market). Non-arbitrage condition between asset and capital markets implies that the triple $(p_t, p_{t+1}, r_{t+1}) \in \mathbb{R}^3_+$ satisfies the following equation $p_t r_{t+1} = p_{t+1} + \pi$. This together with (1) implies that the asset price under perfect foresight is given by the equation

$$p_{t+1} = \Phi_2(k_t, p_t) := p_t R[\Phi_1(k_t, p_t)] - \pi. \quad (2)$$

(1) and (2) imply that

$$\begin{pmatrix} k_{t+1} \\ p_{t+1} \end{pmatrix} = \Phi_n(k_t, p_t) := \begin{pmatrix} \Phi_1(k_t, p_t) \\ \Phi_2(k_t, p_t) \end{pmatrix} \quad (3)$$

is defined on the set $\mathcal{D} = \{(k, p) : k \geq 0 \text{ and } W(k) \geq p\}$. For a fixed $\pi^j > 0$, and for a given pair $(k^j_0, p^f_0) \in \mathcal{D}$, the function $\Phi$ governs the evolution of capital and asset price under perfect foresight in country $j$. In the next section we analyze the existence and uniqueness of steady states of the time one map $\Phi$.

### 3.1 Steady State Analysis

**Definition 1** A steady state of the world economy is a vector $(k^h_j, k^f_j, p^h_j, p^f_j)$ such that $(k^j_j, p^j_j) = \Phi_n(k^j_j, p^j_j)$ for $j = h, f$.

It follows from (1) and (2) that the steady state pair $(k, p)$ satisfies the following system of equations

$$p = W(k) - k \quad \text{and} \quad p = \frac{\pi}{R(k) - 1}. \quad (4)$$

Since $W(0) = 0$ and $\lim_{k \downarrow 0} R(k) = \infty$, it follows from (1) that $(0, 0, 0, 0)$ is a corner steady state.

**Assumption 1** Let $f$ be such that $k \mapsto W(k)/k$ is a strictly decreasing function with

$$\lim_{k \downarrow 0} \frac{W(k)}{k} = W'(0) > 2 \quad \text{and} \quad \lim_{k \uparrow \infty} \frac{W(k)}{k} = w < 1. \quad (5)$$

Assumption (1) restricts the production function to satisfy the minimum elasticity of substitution requirement. This condition is trivially satisfied for Cobb-Douglas, and for many other well known parametric families of production functions. In order to investigate the existence of an interior steady state, first we observe that from (1) that the interior steady state capital satisfies the equation $\Lambda(k) = \pi$, where

$$\Lambda(k) := [W(k) - k] [R(k) - 1]. \quad (6)$$
Since the function \( k \mapsto W(k) - k \) can be interpreted as the steady state spending on the asset market, while the function \( k \mapsto R(k) - 1 \) can be interpreted as a steady state discount factor, it follows from \((6)\) that the function \( k \mapsto \Lambda(k) \) can be interpreted as discounted spending on the asset.

Let us define constants \( \hat{k}_1 = R^{-1}(1) \) and \( \hat{k}_2 \), where \( \hat{k}_2 \) solves \( W(k) = k \).

**Assumption 2** Let \( f \) be such that the \( \alpha(k) < 0.5 \) for any \( k \geq 0 \), where

\[
\alpha(k) := \frac{k f'(k)}{f(k)}
\]

is the capital share in production.

It is a direct consequence of Assumptions \((1)\) and \((2)\) and of Lemma \((1)\) that

- \( \Lambda(0) = \infty \), \( \Lambda(\hat{k}_1) = 0 \), and \( \Lambda \) is strictly decreasing and positive on \((0, \hat{k}_1)\), implying the existence of a function \( K_1 : \mathbb{R}_+ \to (0, \hat{k}_1) \) such that \( \Lambda(K_1(\pi)) \equiv \pi \) for any \( \pi > 0 \);

- \( \Lambda(\hat{k}_2) = 0 \), \( \Lambda(\infty) = \infty \), and \( \Lambda \) is strictly increasing and positive on \((\hat{k}_2, \infty)\), implying the existence of a function \( K_2 : \mathbb{R}_+ \to (\hat{k}_2, \infty) \), such that \( \Lambda(K_2(\pi)) \equiv \pi \) for any \( \pi > 0 \);

The above implies that for any finite pair \((\pi^h, \pi^f)\), the world economy has two interior steady states

\[
\left( K_1(\pi^h), K_1(\pi^f), P_1(\pi^h), P_1(\pi^f) \right) \quad \text{and} \quad \left( K_1(\pi^h), K_1(\pi^f), P_1(\pi^h), P_1(\pi^f) \right),
\]

where

\[
P_1(\pi) := \frac{\pi}{R[K_1(\pi)] - 1} \quad \text{and} \quad P_2(\pi) := \frac{\pi}{R[K_2(\pi)] - 1}.
\]

Since \( R[K_1(\pi)] > 1 \) and \( R[K_2(\pi)] < 1 \) it follows that \( P_1(\pi) > 0 \) and \( P_2(\pi) < 0 \) implying that asset prices are positive/negative at first/second interior steady state.

### 3.2 Global Dynamics

First we define two loci in \((p_t, k_t)\) space, the \( KK \) and \( PP \) loci. The \( KK \) locus is the set of all pairs \((k_t, p_t) \in D\) such that \( k_{t+1} = k_t \). It follows from \((1)\) that

\[
KK = \{(k_t, p_t) \in D : p_t = W(k_t) - k_t\}
\]

\(^2\)Strict concavity of \( f \) with Inada conditions guarantee the existence and uniqueness of \( \hat{k}_1 \). While, existence and uniqueness of \( \hat{k}_2 \) is implied by Assumption \((1)\).
The $PP$ locus is the set of all pairs $(k_t, p_t) \in \mathcal{D}$ such that $p_{t+1} = k_t$. It follows from (2) that

$$PP = \left\{ (k_t, p_t) \in \mathcal{D} : k_t = W^{-1} \left[ p_t + R^{-1} \left( 1 + \frac{\pi}{p_t} \right) \right] \right\},$$

where $W^{-1}$ and $R^{-1}$ denotes inverses of functions $W$ and $R$ respectively. Figure 1 depicts $KK$ and $PP$ loci. As shown in the previous section there are two interior steady states $(K_1(\pi), P_1(\pi))$ and $(K_2(\pi), P_2(\pi))$ at which $KK$ and $PP$ loci intersect each other.

Figure 1: Existence of Globally Stable Manifold

In order to analyze the dynamics of the economy, first we establish the local stability property of each steady state.

**Proposition 1** For any given $\pi > 0$, the corner steady state is a source, steady state $(K_1(\pi), P_1(\pi))$ is a saddle, while $(K_2(\pi), P_2(\pi))$ is a stable node.

Local stability property of $(k^*_1, p^*_1) = (K_1(\pi), P_1(\pi))$ implies the existence and uniqueness
of a corresponding locally stable manifold,
\[ \mathcal{W}^{loc}(k_1^*, p_1^*) = \{(k, p) \in U | \lim_{t \to \infty} \Phi^t_x(k, p) = (k_1^*, p_1^*) \text{ and } \Phi^t_x(k, p) \in U, \text{ for any } t \in \mathbb{N} \}, \]
where \( U = B_\varepsilon(k_1^*, p_1^*) \) is an \( \varepsilon > 0 \) neighborhood of a steady state \((k_1^*, p_1^*)\) and \( \Phi^t_x(k, p) \) is the \( t^{th} \) iterate of \((k, p)\) under \( \Phi_x \). \( \mathcal{W}^{loc}(k_1^*, p_1^*) \) is a one dimensional, continuously differentiable curve, which is tangent to linear space spanned by the eigenvector of the Jacobian matrix \( J(k_1^*, p_1^*) \) with eigenvalue less than one.

**Proposition 2** The map \( \Phi_\pi \) defined in (3) is invertible.

Global properties of the dynamical system (3) can be established via the application of the Stable Manifold Theorem (see for example, Hartman (1964) and Guckenheimer & Holmes (1983)). Existence of and uniqueness of a corresponding locally stable manifold, with global invertibility property of the map \( \Phi_\pi \) implies the existence and global uniqueness of a perfect-foresight equilibrium through backward iteration of the locally stable manifold under \( \Phi_\pi^{-t} \). In particular, we obtain one dimensional globally stable manifold, which is given by \( \mathcal{W}_\pi(k_1^*, p_1^*) \), where
\[ \mathcal{W}_\pi(k_1^*, p_1^*) = \bigcup_{t \in \mathbb{N}} \{ \Phi_\pi^{-t} [ \mathcal{W}^{loc}_\pi(k_1^*, p_1^*)] \}. \]

Globally stable manifold divides the region \( \mathcal{D} \) into two sub regions. Any point above (below) globally stable manifold diverges (converges) to infinity (the steady state \( K_2(\pi), P_2(\pi) \)). Existence of a global stable manifold also implies that for any \( k_0 \) there exists a unique \( p_0 \) such that \((k_0, p_0)\) is on the stable manifold and \((k_t, p_t) = \Phi^t_x(k_0, p_0) \in \mathcal{W}_\pi(k_1^*, p_1^*) \) for any \( t = 1, 2, ... \) and the economy converges to steady state \((K_1(\pi), P_1(\pi))\) monotonically along the stable manifold. The economy steps either on diverging path (which is inconsistent with perfect foresight equilibrium) or steady state \((K_2(\pi), P_2(\pi))\) converging path as soon as the initial pair \((k_0, p_0)\) doesn’t belong to \( \mathcal{W}_\pi(k_1^*, p_1^*) \). In other words, there exists a function \( G_\pi : \mathbb{R}_+ \to \mathbb{R}_+ \) (parameterized by \( \pi \)), which describes the law of capital accumulation in each country. \( k_{t+1} = G_\pi(k_t) \), for any initial capital stock \( k_0 \in \mathbb{R}_{++} \) and for any \( \pi > 0 \). Asset price dynamics is given by the equation \( p_{t+1} = p_t R[G_\pi(k_t)] - \pi \) for any \( p_0 \) such that \((k_0, p_0) \in \mathcal{W}_\pi(k_1^*, p_1^*) \).

## 4 World Economy with Integrated Capital Market

Now let us consider the case when capital is freely mobile across countries and young agents of a given country are allowed to purchase only the domestic asset and can hold a
portfolio containing home and foreign country capital stocks. This implies equalization of rates of returns of capital, wages, and capital stocks, across countries, i.e., \( k^h_t = k^f_t = k_t \). Since the asset market operates only domestically and the number of available assets are normalized to unity in each country, it follows that young agents of country \( h \) spends \( p_t^h \) on the asset market. This implies the following resource constraint.

\[
k_{t+1} = W(k_t) - \frac{p_t^h + p_t^f}{2}. \tag{14}
\]

Non-arbitrage between international capital and domestic asset markets implies that the evolutions of asset prices in each country is given by the following equation

\[
p_{t+1}^h = p_t^h R (k_{t+1}) - \pi^h \quad \text{and} \quad p_{t+1}^f = p_t^f R (k_{t+1}) - \pi^f. \tag{15}
\]

Let

\[
p_t = \frac{p_t^h + p_t^f}{2} \quad \text{and} \quad \pi = \frac{\pi^h + \pi^f}{2} \tag{16}
\]
de note the average world price of the asset and the average marginal utility of consumers. Then (14), (15), and (16) imply that

\[
k_{t+1} = W(k_t) - p_t \quad \text{and} \quad p_{t+1} = p_t R [W(k_t) - p_t] - \pi. \tag{17}
\]

Direct comparison of equations (1) and (2) with (17) and the analysis conducted in Section 3.1 imply the existence of a unique and interior steady state of the world economy with positive asset prices, \((k^h, k^f, p^h, p^f)\), where

\[
k^h = k^f = K_1(\pi), \quad p^h = \frac{\pi^h}{R [K_1(\pi)] - 1} \quad \text{and} \quad p^f = \frac{\pi^f}{R [K_1(\pi)] - 1}. \tag{18}
\]

Based on similar analysis, as conducted in Section 3.2 we can conclude that the convergence to the unique and interior steady state (with positive asset price) is monotonic and capital accumulation in the world economy is described by the time one map \( k_{t+1} = G_{\pi}(k_t) \), where the constant \( \pi \) is defined in (16) and the function \( G \) is introduced in Section 3.2.

5 World Economy with Integrated Asset Market

Let us now consider the case when the asset market is integrated across countries and young agents of both countries can purchase the asset at the unified asset price \( p_t^h = p_t^f = p_t \) and invest only in domestic capital stock. Let \((x_t, 2 - x_t)\), where \( x_t \in [0, 2] \), denotes the number of assets young agents of countries \( h \) and \( f \) purchase internationally.
Capital stocks in home and foreign countries and the asset price, \((k^h_t, k^f_t, p_t)\), describe the current state of the economy. We divide the state space \(\mathbb{R}^3_+\) into three disjoint sets, \(S_1\), \(S_2\), and \(S_3\), where

\[
S_1 = \left\{ (k^h, k^f, p) \in \mathbb{R}^3_+ | R \left[ W(k^h) \right] \geq R \left[ W(k^f) - 2p \right] + \frac{\pi^h - \pi^f}{p_t} \right\},
\]

\[
S_2 = \left\{ (k^h, k^f, p) \in \mathbb{R}^3_+ | R \left[ W(k^h) - 2p \right] \leq R \left[ w(k^f) \right] + \frac{\pi^h - \pi^f}{p} \right\},
\]

\(S_3 = \mathbb{R}^3_+ \setminus (S_1 \cup S_2)\). When \((k^h_t, k^f_t, p_t) \in S_1\) then young agents of country \(h\) decide not to purchase asset at all and invest all their wage income in physical capital, i.e., \(x_t = 0\). This implies that when \((k^h_t, k^f_t, p_t) \in S_1\) then the evolution of the world economy is described by the following equation

\[
\begin{pmatrix}
    k^h_{t+1} \\
    k^f_{t+1} \\
    p_{t+1}
\end{pmatrix}
= \Psi^1 \begin{pmatrix}
    k^h_t \\
    k^f_t \\
    p_t
\end{pmatrix} = \begin{pmatrix}
    \Psi^1_1 (k^h_t, k^f_t, p_t) \\
    \Psi^1_2 (k^h_t, k^f_t, p_t) \\
    \Psi^1_3 (k^h_t, k^f_t, p_t)
\end{pmatrix} := \begin{pmatrix}
    W(k^h_t) \\
    W(k^f_t) - 2p_t \\
    p_t R \left[ W(k^f_t) - 2p_t \right] - \pi^f
\end{pmatrix}.
\]

When \((k^h_t, k^f_t, p_t) \in S_2\), then young agents of country \(f\) decide not to buy asset at all, i.e., \(x_t = 2\). This implies that when \((k^h_t, k^f_t, p_t) \in S_2\) then the evolution of the world economy is described by the equation

\[
\begin{pmatrix}
    k^h_{t+1} \\
    k^f_{t+1} \\
    p_{t+1}
\end{pmatrix}
= \Psi^2 \begin{pmatrix}
    k^h_t \\
    k^f_t \\
    p_t
\end{pmatrix} = \begin{pmatrix}
    \Psi^2_1 (k^h_t, k^f_t, p_t) \\
    \Psi^2_2 (k^h_t, k^f_t, p_t) \\
    \Psi^2_3 (k^h_t, k^f_t, p_t)
\end{pmatrix} := \begin{pmatrix}
    W(k^h_t) - 2p_t \\
    W(k^f_t) \\
    p_t R \left[ W(k^h_t) - 2p_t \right] - \pi^h
\end{pmatrix}.
\]

In contrast, when \((k^h_t, k^f_t, p_t) \in S_3\) then young agents of both countries decide to hold a mixed portfolio and thus the evolution of the economy is described by the following equation

\[
\begin{pmatrix}
    k^h_{t+1} \\
    k^f_{t+1} \\
    p_{t+1}
\end{pmatrix}
= \Psi^3 \begin{pmatrix}
    k^h_t \\
    k^f_t \\
    p_t
\end{pmatrix} = \begin{pmatrix}
    \Psi^3_1 (k^h_t, k^f_t, p_t) \\
    \Psi^3_2 (k^h_t, k^f_t, p_t) \\
    \Psi^3_3 (k^h_t, k^f_t, p_t)
\end{pmatrix} := \begin{pmatrix}
    W(k^h_t) - x_t p_t \\
    W(k^f_t) - (2 - x_t) p_t \\
    p_t R \left[ W(k^h_t) - x_t p_t \right] - \pi^h
\end{pmatrix},
\]

where \(x_t = \phi(k^h_t, k^f_t, p_t) \in (0, 2)\) solves the equation

\[
p_t R \left[ W(k^h_t) - x_t p_t \right] - \pi^h = p_t R \left[ W(k^f_t) - (2 - x_t) p_t \right] - \pi^f.
\]

\(\dagger\) Since \(R \left[ W(k^h) \right] < R \left[ W(k^h) - 2p \right]\) and \(R \left[ W(k^f) \right] < R \left[ w(k^f) - 2p \right]\), it follows that \(S_1\) and \(S_2\) are disjoint sets.
where 5 equations observe that the capital stocks at any interior steady state should satisfy the following
deteriorates to zero capital level. These steady states are:
in which one of the countries absorbs all assets with positive capital stock while the other
Young agents get no wage income and asset holdings is passed to the future generations
state (0 0), in which both countries hold zero capital stock and the asset price is zero.

The steady state analysis conducted in section 3.1 implies the existence of a corner steady
state asset holding where the function \( \Psi \) is described by the following time-one map

\[
\begin{pmatrix}
    k^h_{t+1} \\
    k^f_{t+1} \\
    p_{t+1}
\end{pmatrix} = \Psi(k^h_t, k^f_t, p_t) := \begin{cases}
    \Psi^1(k^h_t, k^f_t, p_t) & \text{if } (k^h_t, k^f_t, p_t) \in S_1 \\
    \Psi^2(k^h_t, k^f_t, p_t) & \text{if } (k^h_t, k^f_t, p_t) \in S_2 \\
    \Psi^3(k^h_t, k^f_t, p_t) & \text{if } (k^h_t, k^f_t, p_t) \in S_3.
\end{cases}
\]

5.1 Steady State Analysis

**Definition 2** A steady state of the world economy is a stationary triple \((k^h, k^f, p)\), such that \((k^h, k^f, p) = \Psi(k^h, k^f, p)\).

In order to demonstrate the existence and uniqueness of an interior steady state first we observe that the capital stocks at any interior steady state should satisfy the following equations

\[
k^h = K_1[\pi^h x], \quad \text{and} \quad k^f = K_1[\pi^f(2 - x)],
\]

with \(x \in (0, 2)\). For a given steady state pair \((k^h, k^f)\), and for a given asset holding \(x \in (0, 2)\), there exists a corresponding supporting asset market clearing prices

\[
p^h = \frac{\pi^h}{K_1(\pi^h x)} - 1 \quad \text{and} \quad p^f = \frac{\pi^f}{K_1(\pi^f(2 - x))} - 1,
\]

where the function \(K_1\) is defined in (3) and (9). Asset market integration requires a steady state asset holding \(x\) to solve the equation \(\Delta(x) = 0\), where

\[
\Delta(x) := \frac{\pi^h}{K_1(\pi^h x)} - 1 - \frac{\pi^f}{K_1(\pi^f(2 - x))} - 1.
\]

---

4It is straightforward to check that (a) \((0, \hat{k}^f, \hat{p}^f) \in S_1\) and \((0, \hat{k}^f, \hat{p}^f) = \Psi_1(0, \hat{k}^f, \hat{p}^f)\) and (b) \((\hat{k}^h, 0, \hat{p}^h) \in S_2\) and \((\hat{k}^h, 0, \hat{p}^h) = \Psi_2(\hat{k}^h, 0, \hat{p}^h)\).

5Functions \(K_1\) and \(P_1\) are defined in (3) and (9).
Proposition 3 If Assumptions 2 is satisfied then there exists a unique \( x^* \in (0, 2) \) solving \( \Delta(x) = 0 \).

Proof: Since \( \Lambda \) is monotonically decreasing on \( k \in (0, \hat{k}_1) \), it follows that the function \( K_1 \), defined in (8), is also monotonically decreasing. This with monotonicity of \( R \) implies that the function \( \Delta \) is strictly decreasing on \( x \in (0, 2) \). In addition, \( \Delta(2) < 0 < \Delta(0) \). This with monotonicity of \( \Delta \) implies the existence and uniqueness of \( x^* \) solving \( \Delta(x) = 0 \).

Q.E.D.

Proposition 3 implies the existence and uniqueness of an interior \( (\hat{h}^h, \hat{h}^f) \in (0, 2) \) solving equation \( \Delta(x) = 0 \). Then \( (\pi^h, \pi^f) \) with equations (27) and (28) determines the unique steady state of the world economy.

5.2 Global Dynamics

As above, first we evaluate the Jacobian matrix at each steady state and conclude about the local stability property. Let \( \lambda_1, \lambda_2, \) and \( \lambda_3 \), denote characteristic roots of the Jacobian matrix evaluates at a given steady state. Then, it is straitforward to show that all three characteristic roots are unbounded at \((0, 0, 0)\). By evaluation the Jacobian matrix of \( \Psi^1 \) at \((0, \hat{k}^f, \hat{p}^f)\), we obtain that the first characteristic root is unbounded, while the second and third characteristic roots satisfy inequality \( 0 < \lambda_2 < 1 < \lambda_3 < \infty \). This implies that the corner steady state \((0, \hat{k}^f, \hat{p}^f)\) is a saddle and there exists a one dimensional stable manifold along which the economy can converge to this steady state. Since \( k^h_0 \) and \( k^f_0 \) are two independent variables, one can conclude that the steady state \((0, \hat{k}^f, \hat{p}^f)\) is an unstable one. Similar analysis shows that the second characteristic root of the Jacobian matrix of \( \Psi^2 \) at \((\hat{k}^h, 0, \hat{p}^h)\) is unbounded, while first and third characteristic roots satisfy inequality \( 0 < \lambda_1 < 1 < \lambda_3 < \infty \). This implies that the corner steady state \((\hat{k}^h, 0, \hat{p}^h)\) is an unstable one.

By evaluation the Jacobian matrix of \( \Psi^3 \) at the interior steady state \((k^h, k^f, p)\) we obtain that \( \lambda_1 \in (0, 1), \lambda_2 \in (0, 1), \) and \( \lambda_3 > 1 \). This implies the existence of two dimensional locally stable manifold along which the economy converges to the interior steady state. This with continuity and invertibility \( \Psi \) implies the existence of a globally stable manifold along which the economy converges monotonically to the unique and interior steady state for any initial values \((k^h_0, k^f_0)\).
6 World Economy with Integrated Capital and Asset Markets

In this section we analyze the situation when capital and asset markets operate internationally. This means equalization of rates of returns on capital, wages, capital stocks, and asset prices, i.e., \( k_t^h = k_t^f = k_t \) and \( p_t^h = p_t^f = p_t \) for any \( t \). This implies the following resource constraint

\[
k_{t+1} = \Theta_1(k_t, p_t) := W(k_t) - p_t. \tag{30}
\]

If the pair \((k_t, p_t)\) is such that \( W(k_t) \leq 2p_t \) then \( 2k_{t+1} \leq W(k_t) \) and thus the measure of young agents who invest in physical capital is no more than one. Since \( \pi^f < \pi^h \) it follows that only part of the young agents of country \( f \) invest in physical capital, while the rest of the young agents of country \( f \) and all young agents of country \( h \) invest in the international asset market. Since young agents of country \( f \) are indifferent between investing in capital or in asset, it follows from non-arbitrage condition that the next period asset price is

\[
p_{t+1} = p_t R[\Theta_1(k_t, p_t)] - \pi^f. \tag{31}
\]

If the pair \((k_t, p_t)\) is such that \( 2p_t > W(k_t) \) then \( 2k_{t+1} > W(k_t) \) and thus the measure of young agents who invest in physical capital exceeds one. Since young agents of country \( h \) value asset more than young agents of country \( f \) it follows that all young agents of country \( f \) and only part of country \( h \) invest in physical capital. While the rest of young agents of country \( h \) invest in international asset market. Since young agents of country \( h \) are indifferent between investing in capital and asset markets, it follows from non-arbitrage condition that the next period asset price is

\[
p_{t+1} = p_t R[\Theta_1(k_t, p_t)] - \pi^h. \tag{32}
\]

This implies that the next period asset price can be expressed as

\[
p_{t+1} = \Theta_2(k_t, p_t) := \begin{cases} 
p_t R[\Theta_1(k_t, p_t)] - \pi^f & \text{if } 2p_t \geq W(k_t) \\
p_t R[\Theta_1(k_t, p_t)] - \pi^h & \text{if } 2p_t < W(k_t). \end{cases} \tag{33}
\]

and thus the evolution of the economy under perfect foresight is described by the following time one map

\[
\begin{pmatrix} k_{t+1} \\ p_{t+1} \end{pmatrix} = \Theta(k_t, p_t) := \begin{pmatrix} \Theta_1(k_t, p_t) \\ \Theta_2(k_t, p_t) \end{pmatrix}. \tag{34}
\]

We define two disjoint regions \( \Sigma_f = \{(k, p)|2p > W(k)\} \), \( \Sigma_h = \{(k, p)|2p < W(k)\} \) and \( \Sigma := \{(k, p)|2p = W(k)\} \), where \( \Sigma \) denotes the boundary of the region where continuity of \( \Theta \) is lost.
6.1 Steady State Analysis

**Definition 3** A steady state of the world economy is a stationary pair \((k, p)\), such that \((k, p) = \Theta(k, p)\).

On the one hand, it follows from \((30)\) that \(p = W(k) - k\). On the other hand, it follows from \((33)\), that steady state asset price satisfies the following equation

\[
p = \begin{cases} 
\frac{\pi^f}{R(k) - 1} & \text{if } \frac{2\pi^f}{R(k) - 1} \geq W(k) \\
\frac{\pi^h}{R(k) - 1} & \text{if } \frac{2\pi^f}{R(k) - 1} < W(k).
\end{cases} \tag{35}
\]

The above equations implies that \((0, 0)\) is always a corner steady state of the world economy. Since \(R [K_2(\pi^f)] < 1\), it follows from \((35)\) that \((K_2(\pi^h), P_2(\pi^h))\) is never an interior steady state. In contrast, \((K_2(\pi^h), P_1(\pi^h))\) is always an interior steady state, because \(R [K_2(\pi^h)] < 1\).

\((K_1(\pi^f), P_1(\pi^f))\) is an interior steady state when \(2P_1(\pi^f) \geq W [K_1(\pi^f)]\)

\((K_1(\pi^h), P_1(\pi^h))\) is an interior steady state when

\[2P_1(\pi^f) < W [K_1(\pi^f)] \quad \text{and} \quad 2P_1(\pi^h) \frac{\pi^f}{\pi^h} < W [K_1(\pi^h)] \tag{36}\]

Steady state does not exist when \(W [K_1(\pi^h)] \leq \frac{2\pi^f}{R[K_1(\pi^f)]} - 1 < W [K_1(\pi^f)]\)

6.2 Dynamics

On the one hand, stability property of the pseudo steady state \(K(\pi^h)\) implies that parameters of the local approximation satisfy the following restrictions \(a_{20} > 0, a_{21} \in (0, 1)\). On the other hand, stability property of the pseudo steady state \(K(\pi^f)\) implies that parameters of the local approximation satisfy the following restrictions \(a_{10} > 0\), while \(a_{11} \in (a_{21}, 1)\) or \(a_{11} \geq 1\).

Clearly local dynamics of the capital stock under linear approximation (given in \((??)\)), strongly depends on two slopes \(a_{11}\) and \(a_{21}\). The case when \(0 < a_{21} < a_{11} < 1\) is investigated in Ferri, Greenberg & Day (2001), where the authors show that stable cycles of any period may exist. Even more so, cycles may exist not only with any period, but also with any number of points in the left and right regions. In order words, for any
integers $m$ and $n$ there exists values of $a_{11}$ and $a_{21}$ such that $0 < a_{21} < a_{11} < 1$ and the economy converges to a cycle of order $m + n$, where $m$ points are in the left while $n$ points in the right region. Each cycle of period $m + n$ is always globally asymptotically stable. The case when $0 < a_{21} < 1 < a_{11}$ is investigated in Tramontana, Gardini & Ferri (2009), where the authors show that in addition to asymptotically stable cycle of any period $m + n$ ($m$ points are in the left, while $n$ points are in the right region) there might exist pure chaos with a unique and absolutely continuous invariant ergodic measure.

### 6.3 Numerical Example

In this section we consider a numerical example through which the model can be simulated globally and the existence of cycles of high order can be demonstrated numerically. Suppose the production function is Cobb-Douglas, $f(k) = Ak^\alpha$, where $A > 0$ is the Hick’s neutral productivity level and $\alpha \in (0, 0.5)$ is the share of capital income.

![One-dimensional orbit diagram of $k_t$ as a function of $\pi^f$](image1)

![One-dimensional orbit diagram of $k_t$ as a function of $\pi^h$](image2)

**Figure 2: One-dimensional Bifurcation**

Figure 2 displays a one dimensional bifurcation of the capital stock with respect to parameters $\pi^f$ and $\pi^h$, while keeping other parameter values constant, as given in Table 1. As Table 1 indicates the bifurcation parameter value $\pi^f_*$ at which the unique steady state $K(\pi^f)$ hits the boundary is

<table>
<thead>
<tr>
<th>$A$</th>
<th>$\alpha$</th>
<th>$\pi^f_*$</th>
<th>$\pi^h$</th>
<th>$k_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.00</td>
<td>0.40</td>
<td>0.10</td>
<td>2.00</td>
<td>2.00</td>
</tr>
</tbody>
</table>

Table 1: Standard parameter set
Common folklore suggests that international capital mobility allows countries with limited savings to attract financing for productive domestic investment, strengthening the domestic income generating process and thus increase the long run welfare. In contrast, we show that capital market integration can sometimes cause the reduction of welfare in a rich country without affecting the long run welfare in a poor country. Take off of a poor country from an underdevelopment trap is also possible after the capital market integration when the capital stock is sufficiently high in a rich country at the moment of market integration. Intermediate situation is also possible when the rich country looses
while the poor country benefits from an capital market integration. The possibility of asset market integration reduces the chance that the capital market integration will play a positive role in improvement of the long run welfare in both poor and rich countries. This is so because after capital and asset market integration the foreign capital flow finances the foreign asset purchases rather than capital investment.

In an overlapping generations setup we demonstrate that the tradeoff between investment in capital and durable commodity can create a potential for a poverty trap and move rational investment strategies away from a social point of view optimal capital investment strategy. Thus, perfect competition, perfect foresight and openness of capital and asset markets are no guarantee for Pareto improving trades for economies with asset markets. As a result, policymakers should often give a second thought before suggesting integration of capital and asset markets.

8 Appendix

Lemma 1 If Assumptions 1 and 2 are satisfied then

- \( \Lambda(0) = \infty \), \( \Lambda(\hat{k}_1) = 0 \), and \( \Lambda \) is strictly decreasing and positive on \((0, \hat{k}_1)\);
- \( \Lambda(\hat{k}_2) = 0 \), \( \Lambda(\infty) = \infty \), and \( \Lambda \) is strictly increasing and positive on \((\hat{k}_2, \infty)\);

Proof: It is straightforward to see that \( \Lambda(k) > 0 \) for \( k \in (0, \hat{k}_1) \cup (\hat{k}_2, \infty) \) and \( \Lambda(k) < 0 \) for \( k \in (\hat{k}_1, \hat{k}_2) \). If \( k \in (0, \hat{k}_1) \) then \( R(k) \geq 1 \). As a result, it follows from Assumption 2 that \( kR(k) < W(k) \) and thus

\[
\Lambda'(k) = -R'(k) [kR(k) - W(k)] - [R(k) - 1] < 0, \tag{37}
\]

for \( k \in (0, \hat{k}_1) \). If \( k \in (\hat{k}_2, \infty) \) then \( R(k) \leq 1 \), \( W(k) < k \) and \( W'(k) < 1 \). As a result,

\[
\Lambda'(k) = -R'(k) [kR(k) - W(k)] - [R(k) - 1] > [1 - R(k)][1 - W'(k)] > 0, \tag{38}
\]

for \( k \in (\hat{k}_2, \infty) \). In addition \( \Lambda \) satisfies the following boundary behavior

\[
\lim_{k \downarrow 0} \Lambda(k) = \lim_{k \downarrow 0} W(k)R(k) = \lim_{k \downarrow 0} k^{2\alpha(0)-1} = \infty, \tag{39}
\]

and

\[
\lim_{k \uparrow \infty} \Lambda(k) = \lim_{k \uparrow \infty} k \left[\frac{W(k)}{k} - 1\right] [R(k) - 1] = \infty. \tag{40}
\]

Q.E.D.

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Proof of Proposition 1

In order to analyze the dynamics of the economy, first we establish the local stability property of each steady state. For this reason, we evaluate the Jacobian matrix at the corner \((0,0)\) and at two interior steady states \((K_1(\pi), P_1(\pi))\) and \((K_2(\pi), P_2(\pi))\). (1) and (2) imply

\[
\Phi_{11}(k,p) = W'(k) \quad \text{and} \quad \Phi_{12}(k,p) = -1,
\]

and

\[
\Phi_{21}(k,p) = pR'(k)W'(k) \quad \text{and} \quad \Phi_{22}(k,p) = R(k) - pR'(k).
\]

It follows from (10) and (11) that the trace and determinant of the Jacobian Matrix

\[
J(k,p) = \begin{pmatrix}
\Phi_{11}(k,p) & \Phi_{12}(k,p) \\
\Phi_{21}(k,p) & \Phi_{22}(k,p)
\end{pmatrix}
= \begin{pmatrix}
W'(k) & -1 \\
pR'(k)W'(k) & R(k) - pR'(k)
\end{pmatrix}
\]

evaluated at a given steady state, are

\[
T(k,p) = W'(k) + R(k) - pR'(k) = R(k) - W(k)R'(k) \quad \text{and} \quad D(k,p) = W'(k)R(k).
\]

(44) implies that the corner steady state is a source.

In order to establish the local stability property of \((k,p) = (K_1(\pi), P_1(\pi))\), we observe that (a) \(T(k,p) > R(k) > 1\), (b) \(D(k,p) > 0\), and (c)

\[
1 - T(k,p) + D(k,p) = 1 - W'(k) - R(k) + pR'(k) + W'(k)r(k) =
\]

\[
= [W'(k) - 1] [R(k) - 1] + [W(k) - k] R'(k) = \Lambda'(k) < 0.
\]

(a), (b), and (c) implies that \((k,p) = (K_1(\pi), P_1(\pi))\) is a saddle.

In order to establish the local stability property of \((k,p) = (K_2(\pi), P_2(\pi))\), we observe that (a) \(T(k,p) > 0\), (b) \(0 < D(k,p) = W'(k)R(k) < 1\) (because \(W'(k) < 1\) and \(R(k) < 1\)), and (c)

\[
1 - T(k,p) + D(k,p) = 1 - W'(k) - R(k) + pR'(k) + W'(k)r(k) =
\]

\[
= [W'(k) - 1] [R(k) - 1] + [W(k) - k] R'(k) = \Lambda'(k) > 0.
\]

(a), (b), and (c) implies that \((k,p) = (K_1(\pi), P_1(\pi))\) is a stable node.

Q.E.D.
Proof of Proposition 2 In order to show the invertibility of the map \( \Phi_{\pi} \), we show that the system of equations
\[
\begin{align*}
 u &= W(k) - p \\
 v &= pR(W(k) - p) - \pi
\end{align*}
\]
has a unique solution with respect to \((k, p)\) for any pair \((u, v)\) \(\in \mathbb{R}_{+}^2\). Equation (47) implies
\[
p = \frac{v + \pi}{R(u)} \quad \text{and} \quad k = W^{-1}\left[u + \frac{v + \pi}{R(u)}\right].
\]
Since the function \(W\) is monotonically increasing with \(W(0) = 0\) and \(W(\infty) = \infty\), claim of the proposition follows from (48).

Q.E.D.

Proof of Proposition If \(\pi^f \geq \Lambda(k^c) \Rightarrow K(\pi^f) \leq k^c\) and thus \(2K(\pi^f) \leq W[K(\pi^f)]\). This with (36) implies that \(K(\pi^f)\) is a unique steady state.

If \(\pi^f < \Lambda(k^c) \Rightarrow K(\pi^f) > k^c\) and thus \(2K(\pi^f) > W[K(\pi^f)]\). This with (36) implies that a unique candidate for steady state is \(K(\pi^h)\). If \(\pi^f < \pi^h < \Lambda(k^c) \Rightarrow K(\pi^h) > k^c\) and thus \(2K(\pi^h) > W[K(\pi^f)]\). This with (36) implies that \(K(\pi^h)\) is a unique steady state.

When \(\pi^f < \Lambda(k^c) \leq \pi^h \Rightarrow K(\pi^h) \leq k^c < K(\pi^f)\) and thus \(2K(\pi^h) \leq W[K(\pi^f)]\) and \(2K(\pi^h) > W[K(\pi^f)]\). This with (36) implies that neither \(K(\pi^f)\) nor \(K(\pi^h)\) is a steady state of the economy.

Q.E.D.
References


