Optimal Sales Schemes against Interdependent Buyers*

Masaki Aoyagi†
Osaka University

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Abstract

This paper studies a monopoly pricing problem when the seller can also choose the timing of a trade with each buyer endowed with private information about the seller’s good. A buyer’s valuation of the good is the weighted sum of his and other buyers’ private signals, and is affected by the publicly observable outcomes of preceding transactions. We show that it is optimal for the seller to employ a \textit{sequential sales scheme} in which he trades with one buyer at a time. Furthermore, when the buyers differ in the weights they place on other buyers’ signals, we identify conditions under which it is optimal for the seller to trade with them in the increasing order of those weights. It is shown that the optimal scheme has a strong tendency to induce herd behavior.

Key words: monopoly pricing, staggered sales, social learning, consumption fads.
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†ISER, Osaka University, 6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan.
1 Introduction

When a seller of a good trades with multiple buyers, he often employs a dynamic sales strategy. That is, instead of serving all the buyers at once, the seller partitions them into smaller markets and supplies each market at a different timing. Examples abound in the entertainment industry where the sellers of books, video games, and movies introduce their products in one market and move on to another after generating hype in the first market. For example, on Sony’s announcement on the release of its PS3 game console, a BBC article remarks that “[n]ormally Sony staggers the release of a new console, releasing in Japan and America, with Europe coming a belated third.”\(^1\) We also observe similar sales strategies used in automobile and electronics industries.\(^2\)

There is perhaps more than one reason why a seller employs such a dynamic sales strategy. For example, the seller may engage in staggered sales simply because of a constraint on logistics. In many cases, however, we believe that it is based on more strategic motives. For example, in his classical textbook on marketing, Kotler (1988, ch. 14) states that a firm of a new product should choose a particular subset of consumers as first targets, noting that those “[e]arly adopters tend to be opinion leaders and helpful in “advertising” the new product to other potential buyers.” That is, a good sales strategy should use the adoption decisions of a small group of consumers with certain characteristics as a signal to other consumers.

In this paper, we explore the possibility of a dynamic sales strategy when there is interdependence among buyers’ valuations. More specifically, when buyers’ valuations of the seller’s good are determined in part by the publicly observable behavior of other buyers, we analyze whether the seller is better off trading with different buyers at different timings. When successful, such a trading strategy can create a chain of positive events: successful transactions with the initial set of buyers raise the valuations of the next line of buyers, success with the latter raises even further the valuation of the buyers to follow, and so on. Once in such a cycle, the seller can continually increase his offer price and raise more revenue than from static sales. Of course, the seller adopting such a scheme also faces the risk of a downward spiral where a failure in the initial markets leads to a sequence of failures in subsequent

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\(^1\)“PlayStation 3 Euro launch delayed,” BBC News, September 6, 2006. It is announced that PS3 is launched on Nov 11, 2006 in Japan, Nov 17 in the U.S. and March 30, 2007 in Europe.

\(^2\)One recent example is Toyota’s introduction of the Lexus brand to Japan after its well-publicized success in the U.S.
In our model of dynamic trading, a seller faces multiple buyers each endowed with private information about the seller’s good. Each buyer demands one unit of the good, which is produced at no cost to the seller. The private signals are independent across buyers and a buyer’s valuation of the seller’s good is a weighted sum of all buyers’ signals. As in the classical monopoly pricing problem, the seller’s trading with each buyer takes the form of price posting. The outcomes of transactions are publicly observable to subsequent buyers, and used to update the expected value of the good to them. Each buyer meets the seller once and leaves the market after accepting or rejecting the offer.

The nature of the problem can be best illustrated in a model where there are only two buyers. The seller can either trade with both at once or trade with one of them first and the other next. In the first scheme, referred to here as a *simultaneous scheme*, the seller provides the buyers no opportunity to learn about each other’s private signals. In other words, each buyer must make a purchasing decision only on his own signal. In the second scheme, referred to as a *sequential scheme*, the seller allows the second buyer to infer the private signal of his predecessor: Acceptance by the first buyer raises the second buyer’s valuation, while rejection lowers it. It should be noted that the exact amount by which the second buyer’s valuation changes depends on the level of the price offer to the first buyer: If the first buyer accepts a high price, then there will be a considerable increase in the second buyer’s valuation, while if the first buyer accepts a low price, then the increase in the valuation will be small. In this sense, the seller should choose his price offer in stage 1 so as to balance the rent to extract from buyer 1 and the information to reveal to buyer 2. If the two buyers are not ex ante identical, then the seller must also choose which buyer to serve first. With three or more buyers, the seller’s problem is similar but significantly more complex. First, besides sequential and simultaneous schemes, there are a number of intermediate schemes. Second, the choice of buyers at each stage can be contingent on the history of transactions. For example, the buyer(s) with whom the seller may wish to trade in stage 2 may be different depending on the outcome of transaction in stage 1.

The first main conclusion of the paper is that it is optimal for the seller to employ a sequential scheme. The conclusion is based on the construction of a sequential scheme that replicates any given non-sequential scheme. In the two-buyer model, for example, given any simultaneous scheme, we can construct a sequential scheme
that raises the same expected revenue. Suppose that the seller originally employs a simultaneous scheme which offers price $x_1$ to buyer 1 and $x_2$ to buyer 2. The alternative sequential scheme offers $x_1$ to buyer 1 in period 1, and makes contingent offers to buyer 2 in period 2. In particular, the seller offers player 2 a higher price when 1 accepts his offer and a lower price otherwise. The price is adjusted so that buyer 2 accepts the contingent offers with exactly the same probability as he accepts $x_2$ under the original scheme. The choice of such contingent offers is possible since buyer 2’s valuation shifts up or down by a deterministic amount as a result of the period 1 outcome under our assumptions that the valuation function is additive and the private signals are independent. The key is to show that those contingent offers yield the same expected revenue as $x_2$.

Given the optimality of a sequential scheme, the second question we address is on the optimal ordering of buyers. For this, we suppose that the buyers’ private signals have an identical distribution, and that their valuation equals the own signal plus some constant times the sum of all other buyers’ signals. The constant is the unique source of ex ante idiosyncrasy among the buyers and called the dependence weight as it measures how dependent the buyer is on others’ information. We provide a sufficient condition under which the optimal sales scheme entails an increasing order of the dependence weights: It first trades with the buyer with the smallest dependence weight, then with the buyer with the second smallest weight, and so on until it reaches the last buyer who has the largest weight. As discussed below, such a scheme is of particular interest from the point of view of marketing and monopoly theories. We call the sufficient condition the monotonicity condition as it requires certain monotonicity on the contingent price along any fixed sequence of buyers that arises at the tail of the sales scheme. For this reason, verification of the monotonicity condition involves solving for the optimal pricing strategy along a fixed sequence of buyers. We call such a problem the sequential pricing problem and provide its formulation. When the buyers’ private signals have a uniform distribution, we can identify the range of parameters for which the monotonicity condition holds by solving analytically the associated sequential pricing problem.

When the optimal sales scheme entails the increasing order of the dependence weights, the buyers who are more heavily influenced by public information are placed towards the end of the sequence and given a chance to observe more information. This has the following implications: First, the optimal scheme is likely to induce herd behavior more strongly than any other scheme with an alternative buyer ordering.
fact, it will be shown under the uniform distribution assumption that this scheme maximizes the probability that every buyer makes the same decision in the class of sequential schemes that trade with buyers in a fixed order. Second, from the perspective of classical monopoly theory, such a scheme is interpreted as suggesting an efficient way of reducing the buyers’ informational rents. That is, the seller wants a buyer with a larger dependence weight to observe more public information since reduction in his informational rents per unit of public information is larger than that of a buyer with a smaller weight.

The present model is closely related to those of social learning and monopoly pricing in sequential sales problems. These models assume sequential decision making in a fixed order by infinitely many buyers who are ex ante identical but have correlated private signals about the underlying common value of the seller’s good. It is said that cascading takes place if for some \( n \), all but the first \( n \) buyers make the same decision by ignoring their own private signals. For example, cascading takes place with probability one in the pioneering work on social learning by Bikhchandani et al. (1992), where the seller is implicit and his offer price is assumed the same against every buyer.\(^3\) On the other hand, the models of monopoly pricing in sequential sales problem as studied by Ottaviani (1999), Chamley (2004, Chap. 4), and Bose et al. (2005, 2006) suppose that the seller maximizes his profits by controlling the offer price against each buyer. These papers study how optimal pricing by the seller affects the buyers’ learning about the true value of the good. Bose et al. (2005) for example identifies the range of prior beliefs that allow for complete learning by the buyers. Sgroi (2002) studies a dual problem of a monopolist under social learning where the seller, who knows the quality of his product and offers a fixed price, partially controls public information by choosing the number of buyers to serve in period 1.\(^4\) The framework of the present paper is different from those of the above in that private signals are independent, valuations are interdependent rather than common, and the buyers can be ex ante idiosyncratic. The independence of private signals simplifies the analysis significantly. For example, it makes irrelevant the seller’s learning problem which is central to the analysis of many of the earlier models. The possible idiosyncrasy of the buyers leads to the new question of buyer ordering, and our finding provides one further support for the frequent occurrence of consumption fads in some markets not explained by the existing models.

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\(^3\)See also Banerjee (1992).

\(^4\)Trades with other buyers take place sequentially after period 1. Sgroi (2002) shows that the optimal number of the initial buyers varies non-monotonically with the size of the market.
By adopting the sequential scheme, the seller is committed to publicly revealing information about all past transactions to every buyer. Our finding hence parallels the well-known linkage principle in auction theory (Milgrom and Weber (1982)), which states that the auctioneer’s expected revenue is maximized when he commits to fully revealing his private information provided that the bidders’ private signals are affiliated with one another and that of the auctioneer. The principle also shows that an English auction, which publicly releases the buyers’ private information through their actions as in the sequential scheme, generates a higher revenue than a sealed-bid second-price auction. It should be noted, however, that our conclusion has no formal connection with the linkage principle in auctions. An alternative form of the linkage principle in the monopoly setting is proven by Ottaviani and Prat (2001), who show that a monopolist should optimally commit himself to publicly revealing any information before trading as long as it is affiliated with a buyer’s type and the value of the good. Our conclusion complements their theorem, which also implies the optimality of sequential trading against two buyers in an alternative environment. Regarding the optimal information revelation policy, a related observation is also made in Aoyagi (2006), who points out in a model of a dynamic tournament with a privately informed organizer that the optimal degree of information feedback to the contestants may subtly depend on the parameters of the model.

In marketing, the sequential sales scheme is sometimes called the “waterfall strategy,” while the simultaneous sales scheme is called the “sprinkler strategy.” When a firm adopts a waterfall strategy, the lead effect refers to the effect that consumer decisions in the first market have on those in subsequent markets. For example, Kalish et al. (1995) discuss the relative advantages of the two types of strategies by directly assuming the form of intertemporal dynamics of the lead effect. The present paper, on the other hand, can be seen as an attempt to generate the lead

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5In auctions, it is observed that the linkage principle holds in a rather limited environment. For example, de-Frutos and Rosenthal (1998) show that it may fail in two-stage auctions when the public information consists of the bids of the first-stage bidders. It may also fail when bidders are asymmetric (Krishna (2002, Ch. 8)), or when they demand multiple units (Perry and Reny (1999)).

6See Section 4 for more discussion.

7Discussion of simultaneous versus dynamic schemes is also seen in the literature on network externalities where agents choose whether to subscribe to a network or not. In such a model, agents’ utility depends on their own private signals as well as the number of other subscribers rather than their signals. Although issues related to social learning is absent in such a model, some common themes appear as early movers in a dynamic scheme influences the decisions of others. See, e.g., Ochs and Park (2004).
effect through value interdependence. Empirical research in the marketing literature also looks at the dynamic sales strategies used by multinational firms. In particular, the motion picture industry attracts much attention where movie makers design a detailed plan on when and how to release their products in international markets.\footnote{For example, see Elberse and Eliashberg (2003) and the references therein.}

According to the aforementioned textbook by Kotler (1980), consumers are classified by their willingness to adopt a new product. Upon emphasizing that a firm’s first target should be the innovators, whose who are the most willing to adopt, he notes that the role of personal influence is stronger on those who are less willing than the more willing. Taken together, they can be interpreted as a statement on the desirable ordering of consumers based on the degree of influence they receive from other consumers’ behavior. Our analysis on the optimal ordering of buyers provides one formal restatement of this theory.

The paper is organized as follows: The next section formulates a model of monopoly. In Section 3, we present an example with two buyers and describe some preliminary results that are used extensively in the subsequent analysis. It also gives a discussion on our choice of price-posting mechanisms over the direct mechanisms. Section 4 proves the optimality of a sequential scheme. In Section 5, we present the monotonicity condition mentioned above as a sufficient condition for the optimal sequential scheme to entail the increasing order of the dependence weights. The sequential pricing problem against a fixed sequence of buyers is formulated in Section 6. Section 7 presents the analysis of the uniform signal distribution, and demonstrates the herd-inducing property of the optimal scheme. We conclude with a discussion in Section 8.

## 2 Model

A seller of a good faces the set $I = \{1, \ldots , I\}$ of $I$ buyers each of whom has private information about the valuation of the good.\footnote{Note that $I$ represents both the set and number of buyers.} Let $s_i$ denote buyer $i$’s private signal. We assume that $s_1, \ldots , s_n$ are independent and distributed over the set of real numbers. Let $\mu_i$ be the mean value of $s_i$. When $s = (s_1, \ldots , s_n)$ denotes a signal profile, buyer $i$’s valuation of a single unit of the seller’s good is given by

$$v_i(s) = c_{i0} + c_{ii}s_i + \sum_{j \neq i} c_{ij}(s_j - \mu_j),$$
where $c_{ij} \in \mathbb{R}$ are constants. In other words, the valuations are \textit{linearly interdependent}, and buyer $i$ places weight $c_{ij}$ on buyer $j$’s signal.\textsuperscript{10} For every $i \in I$, $c_{ii} > 0$ and $c_{ij} \geq 0$ for $j \neq 0, i$. In other words, the valuation is strictly increasing in the own signal, and the buyers’ preferences are aligned in the sense that any buyer having a high signal is good news for any other buyer.\textsuperscript{11} It should also be noted that subtraction of the mean $\mu_j$ from $s_j$ for every $j \neq i$ is introduced to simplify the representation of the expected valuation.\textsuperscript{12}

We normalize the marginal cost of producing the good to zero, and assume that every buyer demands at most one unit. As discussed in the Introduction, a buyer in this model can also be interpreted as a segment of the market which has a uniform taste about the seller’s good.

The seller \textit{trades} with each buyer by posting him a price. The buyer then accepts or rejects the price and leaves the market. The price posted to each buyer and their response to it are both publicly observable.

In every period, the seller chooses target buyers and prices to offer to them as a function of past trades. Formally, denote by $I_t \subset I$ the set of buyers who are made offers in period $t$. An \textit{outcome} $y_t$ in period $t$ is a partition $(A_t, B_t)$ of the set $I_t$: $A_t$ represents the set of buyers who have accepted their offers, and $B_t$ represents those who have rejected their offers. For any subset $J$ of buyers, let $Y(J)$ denote the set of possible outcomes from the set $J$ of buyers. In other words, $Y(J)$ consists of all the two-way partitions of the set $J$ including $(J, \emptyset)$ and $(\emptyset, J)$. A \textit{history} of length $t$ consists of the outcomes in periods $1, \ldots, t$. Let $H_t$ denote the set of possible histories of length $t$, and let $H = \bigcup_{t=0}^{\infty} H_t$ be the set of all possible histories, where $H_0$ is the singleton set of the null history. Given any history $h \in H$, we denote by $I(h)$ and $U(h) = I \setminus I(h)$ the set of buyers with whom the seller has and has not, respectively, traded along $h$.

A \textit{sales scheme} of the seller, denoted $\sigma$, consists of a pair of mappings $r : H \to 2^I$ and $x = (x_i)_{i \in I} : H \to \mathbb{R}_+^I$: $r$ is the \textit{target function} with $r(h)$ specifying the subset of buyers chosen for trading at history $h$, and $x$ is the \textit{pricing function} with $x_i(h)$ specifying the price offered to buyer $i$ at $h$. Note in particular that the target buyers

\textsuperscript{10}The additive specification of the valuation function, which is common in the auction literature, can also be interpreted as a first-order approximation to a more general function.

\textsuperscript{11}Note that this is a natural assumption to make in the study of herd behavior in Section 7. Some of our conclusions hold when $c_{ij} < 0$. See Section 8 for more discussion.

\textsuperscript{12}Without the subtraction of $\mu_j$, the conclusion in Section 4 holds as is while proper adjustments are required for the conclusions in Sections 5 and 6.
in any period can be contingent on the history. In a three-buyer model, for example, after trading with buyer 1 in period 1, the seller may choose either buyer 2 or buyer 3 in period 2 depending on whether the period 1 trade is successful or not, etc. It should also be noted that the specification of the price $x_i(h)$ is relevant only if $i$ is the target buyer at $h$ (i.e., $i \in r(h)$). In order to eliminate redundancy, we require that $r$ choose at least one buyer in every period until the list of buyers is exhausted: $r(h) \neq \emptyset$ if $U(h) \neq \emptyset$. This in particular implies that all the trading ends in or before period $I$. Let $\Sigma$ be the set of all sales schemes. Two representative classes of sales schemes are the simultaneous schemes in which the seller trades with all the buyers at once (i.e., $r(h) = I$ for $h \in H_0$), and the sequential schemes in which he trades one by one with each buyer (i.e., $r(h) = \{i\}$ for some $i \in U(h)$ for each $h \in H_{t-1}$ and $t = 1, \ldots, I$).

Given a sales scheme $\sigma$, let $P^\sigma$ denote the joint probability distribution of the signal profile $s$ and the history $h$ induced by $\sigma$. Let $E^\sigma$ be the expectation with respect to the distribution $P^\sigma$. We use $P$ without the superscript to denote the marginal distribution of $s$ that does not depend on the sales scheme, and $E$ to denote the corresponding expectation. For any history $h \in H$, let

$$V_i^\sigma(s_i \mid h) = E^\sigma[v_i(s_i, \tilde{s}_{-i}) \mid h]$$

be the expected valuation of buyer $i$ with signal $s_i$ given history $h$. By assumption, it can be explicitly written as

$$V_i^\sigma(s_i \mid h) = c_{i0} + c_{ii}s_i + \sum_{j \in I(h)} c_{ij} E^\sigma[\tilde{s}_j - \mu_j \mid h].$$

Note that the summation above is over the set $I(h)$ of buyers who have already traded along $h$ since any other term involves the unconditional expectation of the private signal and hence cancels out. Buyer $i$ with signal $s_i$ accepts the seller’s offer $x_i$ at history $h$ if and only if the expected value of the good conditional on $h$ is greater than or equal to $x_i$: $V_i(s_i \mid h) \geq x_i$. The seller’s expected revenue under the sales scheme $\sigma$, denoted by $\pi(\sigma)$, is simply the sum of expected payments from the $I$ buyers.

3 Preliminaries

In this section, we first present a simple example to illustrate the problem, and then provide some preliminary results that are key to much of the subsequent analysis.
Consider first the following example. There are two buyers whose private signals \( s_1 \) and \( s_2 \) both have the uniform distribution over the unit interval \([0, 1]\) with the means \( \mu_1 = \mu_2 = 1/2 \). Suppose also that their valuation functions are given by

\[
v_1(s_1, s_2) = s_1 + c_1 \left( s_2 - \frac{1}{2} \right) \quad \text{and} \quad v_1(s_1, s_2) = s_2 + c_2 \left( s_1 - \frac{1}{2} \right),
\]

where \( 0 < c_1 \leq c_2 < 2 \). It can be seen that this is a special case of the general formulation in the previous section.

When the seller uses the simultaneous sales scheme, he will choose the price offers \( x_1 \) and \( x_2 \) so as to maximize \( x_1 P(\tilde{s}_1 \geq x_1) \) and \( x_2 P(\tilde{s}_2 \geq x_2) \), respectively. As is readily verified, the revenue maximizing prices equal \( x_1 = x_2 = 1/2 \) and the seller’s expected payoff equals

\[
\pi^0 = \frac{1}{4} \times 2 = \frac{1}{2}.
\]

On the other hand, when the seller uses the sequential sales scheme that trades with buyers 1 and 2 in this order, he needs to solve a two-step optimization problem. Consider first the problem in period 2 given the first period offer \( x_1 \in [0, 1] \). Let \( h_1 = 1 \) denote the history corresponding to buyer 1’s acceptance, and \( h_1 = 0 \) denote the history corresponding to his rejection. Depending on \( h_1 \), buyer 2’s valuation function is either

\[
V_2(s_2 | 1) = s_2 + c_2 E\left[ \tilde{s}_1 - \frac{1}{2} \mid \tilde{s}_1 \geq x_1 \right] = s_2 + c_2 \frac{x_1}{2},
\]

or

\[
V_2(s_2 | 0) = s_2 + c_2 E\left[ \tilde{s}_1 - \frac{1}{2} \mid \tilde{s}_1 < x_1 \right] = s_2 + c_2 \frac{x_1 - 1}{2}.
\]

The seller also has two prices to consider in period 2: \( x_2(1) \) is the price offer to buyer 2 when buyer 1 accepted in period 1, and \( x_2(0) \) is the offer to buyer 2 when buyer 1 rejected. The period 2 price offers hence solve

\[
x_2(1) \in \arg \max_{x_2} x_2 P(V_2(\tilde{s}_2 \mid 1) \geq x_2),
\]

and

\[
x_2(0) \in \arg \max_{x_2} x_2 P(V_2(\tilde{s}_2 \mid 0) \geq x_2).
\]

Upon substituting for \( V_2 \), we can solve these problems to obtain

\[
x_2(1) = \frac{1}{2} \left( 1 + \frac{c_2}{4} \right), \quad \text{and} \quad x_2(0) = \frac{1}{2} \left( 1 - \frac{c_2}{4} \right).
\]
Let \( \pi_2(x_1 \mid 1) \) and \( \pi_2(x_1 \mid 0) \) denote the optimized values of the period 2 expected payoffs after \( h_1 = 1 \) and \( h_1 = 0 \), respectively. They are given by

\[
\pi_2(x_1 \mid 1) = \frac{1}{4} \left(1 + \frac{c_2}{2} x_1 \right)^2, \quad \text{and} \quad \pi_2(x_1 \mid 0) = \frac{1}{4} \left(1 + \frac{c_2}{2} (x_1 - 1) \right)^2.
\]

Using them, we can also write the seller’s period 1 problem as:

\[
\max_{x_1} P(\tilde{s}_1 \geq x_1) \left\{ x_1 + \pi_2(x_1 \mid 1) \right\} + P(\tilde{s}_1 < x_1) \pi_2(x_1 \mid 0).
\]

Solve this to get

\[
x_1 = \frac{1}{2}.
\]

These optimal prices can also be obtained from Theorem 7.1 in Section 7. The optimized value of the seller’s overall expected payoff under the sequential sales scheme equals

\[
\pi^{12} = \frac{1}{2} + \frac{c_2}{64}.
\]

Likewise, when the seller uses the sequential scheme with the order of buyers 1 and 2 reversed, his maximized expected payoff is given by

\[
\pi^{21} = \frac{1}{2} + \frac{c_1}{64}.
\]

Given our assumptions on \( c_1 \) and \( c_2 \), we hence have the following ordering:

\[
\pi^0 < \pi^{21} \leq \pi^{12}. \quad (1)
\]

The inequalities in (1) show that a sequential scheme performs better than the simultaneous scheme, and that the optimal sequential scheme entails the buyer sequence in an increasing order of the weight they place on the other buyer’s signal.

We can also compute the probability that the two buyers make the same decision in respective cases. First, in the optimal simultaneous scheme, they both buy with probability 1/4 and both reject with probability 1/4 so that they make the same decision with probability 1/2. In the optimal sequential scheme with buyer 1 first, the corresponding probability is given by

\[
\frac{1}{2} P(V_2(\tilde{s}_2 \mid 1) \geq x_2(1)) + \frac{1}{2} P(V_2(\tilde{s}_2 \mid 0) < x_2(0))
\]

\[
= \frac{1}{2} \left[ P(\tilde{s}_2 \geq \frac{1}{2} - \frac{c_2}{8}) + P(\tilde{s}_2 < \frac{1}{2} + \frac{c_2}{8}) \right]
\]

\[
= \frac{1}{2} + \frac{c_2}{8}.
\]
Likewise, it is given by $\frac{1}{2} + \frac{a}{8}$ for the optimal sequential scheme with buyer 2 first. It follows that the probability of the same decision is the highest in the optimal scheme which trades with buyer 1 first, followed by the other sequential scheme and the simultaneous scheme in this order. These observations are generalized in what follows.

Before proceeding, we derive the optimal direct revelation mechanism in the above example and provide a justification for our choice of a price posting sales scheme. As is well known, the revelation principle states that the optimal mechanism can be found within the class of direct revelation mechanisms. In the current setting, a direct revelation mechanism would first solicit private signals from all the buyers, and then choose the allocation of the good and monetary transfer to each buyer based on the reported signal profile.

Formally, a direct revelation mechanism is a mapping $(p, y) = (p_1, p_2, y_1, y_2) : S \rightarrow [0, 1]^2 \times \mathbb{R}^2$: $p_i(\hat{s})$ is the probability that buyer $i$ is allocated the good, and $y_i(\hat{s})$ is the expected monetary transfer from buyer $i$ to the seller, both when the report profile is $\hat{s}$ ($i = 1, 2$). Let $F_i$ be the distribution function of buyer $i$’s private signal $s_i$, and $f_i$ be the corresponding density. When we denote by $U_i(s_i, \hat{s}_i)$ the interim expected utility of buyer $i$ with signal $s_i$ when he participates in this mechanism by reporting $\hat{s}_i$, it can be expressed in terms of $p_i$ and $y_i$ as

$$U_i(s_i, \hat{s}_i) = \int_{S_j} \{ p_i(\hat{s}_i, s_j) v_i(s_i, s_j) - y_i(\hat{s}_i, s_j) \} f_j(s_j) \, ds_j$$

$$= s_i \bar{p}_i(\hat{s}_i) + \int_{S_j} \{ c_i(s_j - \mu) p_i(\hat{s}_i, s_j) - y_i(\hat{s}_i, s_j) \} f_j(s_j) \, ds_j,$$

where $\bar{p}_i(s_i) = \int_{S_j} p_i(s_i, s_j) f_j(s_j) \, ds_j$. The standard requirements are the incentive compatibility (IC) constraints: $U_i^*(s_i) \equiv U_i(s_i, s_i) \geq U_i(s_i, \hat{s}_i)$ for any $s_i, \hat{s}_i \in S_i$, and the interim individual rationality (IR) constraints: $U_i^*(s_i) \geq 0$ for any $s_i \in S_i$. Myerson (1981) shows that the problem of finding an optimal mechanism satisfying these requirements reduces to maximizing the following objective function

$$\sum_{i=1}^{2} \int_{S} y_i(s) f_1(s_1) f_2(s_2) \, ds_1 \, ds_2$$

$$= \sum_{i=1}^{2} \int_{S} p_i(s) \left\{ s_i + c_i(s_j - \mu) - \frac{1 - F_i(s_i)}{f_i(s_i)} \right\} f_1(s_1) f_2(s_2) \, ds_1 \, ds_2$$

with respect to $p_i : S \rightarrow [0, 1]$ ($i = 1, 2$) such that $\bar{p}_i : S_i \rightarrow [0, 1]$ is increasing.
When $F_i$ is the uniform distribution over $S_i = [0, 1]$, let $p_i$ be given by

$$p_i(s) = \begin{cases} 1 & \text{if } 2s_i + c_i(s_j - \frac{1}{2}) > 1, \\ 0 & \text{if } 2s_i + c_i(s_j - \frac{1}{2}) < 1 \end{cases}$$

for $i = 1, 2$ and $j \neq i$. It can readily be verified that this $p_i$ maximizes the objective function above, and that the associated $\bar{p}_i$ is increasing.\(^{13}\) Therefore, this $p_i$ is the allocation function used in the optimal direct revelation mechanism. The seller’s expected revenue from the optimal mechanism can be computed as

$$\pi^* = \frac{1}{2} + \frac{c_1 + c_2}{4} + \frac{c_1^2 + c_2^2}{48},$$

which is greater than any of the three profit levels obtained earlier when $c_1, c_2 > 0$.\(^{14}\) The question is if this is the “right” mechanism in the current setting. We find the answer to be negative for the following reasons. First, in a common sales situation, a seller has little power to commit buyers to an allocation contingent on their expressed preferences. For example, while market research is a common practice, its outcome is not used to force a particular allocation or price on consumers. Second, on a more technical level, the optimal direct mechanism implies one of the following two: If we want buyer $i$’s payment contingent only on his own report, then it must be positive even when he does not obtain the good. Otherwise, if we want $i$’s payment to be positive only when he obtains the good, then it must be contingent also on buyer $j$’s report. In the first case, it is difficult to justify positive payments for nothing when there is no competition for the good unlike participation fees in auctions. In the second case, on the other hand, making a buyer’s payment contingent on other buyers’ reports creates a serious credibility problem because of the seller’s incentive to misrepresent others’ reports. Third, if a buyer can leave the market after seeing the price as in the current setting, interim individual rationality assumed by the above optimal mechanism is clearly insufficient. That is, a buyer will use the allocation of the good and money suggested by the seller to update his belief about

\(^{13}\) $\bar{p}_i$ is given by

$$\bar{p}_i(s_i) = \begin{cases} 0 & \text{if } s_i \in [0, \frac{2-c_i}{4}] , \\ \frac{1}{2} + \frac{2s_i - 1}{c_i} & \text{if } s_i \in \left(\frac{2-c_i}{4}, \frac{2+c_i}{4}\right) , \\ 1 & \text{if } s_i \in \left[\frac{2+c_i}{4}, 1\right] . \end{cases}$$

\(^{14}\) When $c_1 = c_2 = 0$, the optimal direct revelation mechanism is equivalent to a price posting scheme.
the value of the good, and make a purchase only when the updated value exceeds
the suggested price. This would imply that individual rationality should be required
at the exchange stage instead of at the interim stage. Price posting is one common
and important class of sales mechanisms which are free from any of the problems
discussed above.

While the revenue from each alternative scheme in the above example can be
computed explicitly, such comparison is simply infeasible in a general problem be-
cause of technical complexity. For this reason, we take a different approach to the
problem and examine how a local change in the given scheme affects the revenue.
Given below is some preliminary analysis of the general model in this direction.

Consider a pair of sales schemes $\sigma$ and $\sigma'$, and suppose that a pair of histories
$h$ and $h'$ are induced by $\sigma$ and $\sigma'$, respectively. Suppose that along these histories,
the seller has traded with the same set of buyers with exactly the same outcomes.
That is, the set of buyers who have accepted the seller’s offers along $h$ is the same
as that along $h'$ ($A(h) = A(h')$), and also the set of the buyers who have rejected
the offers along $h$ is the same as that along $h'$ ($B(h) = B(h')$). The following lemma
states that if, for every one of those buyers, the probability that he accepts the offer
is the same under both schemes, then so are the valuation functions of subsequent
buyers conditional on $h$ and $h'$. It is based on the following simple logic: No matter
what the history up to buyer $j$ is, if the probability that $j$ accepts his offer under
one scheme is the same as that under another scheme, then the inference drawn
about $j$’s private signal when he accepts (resp. rejects) the offer in the first scheme
is the same as that when he accepts (resp. rejects) the offer in the second scheme.

Formally, given any sales scheme $\sigma = (r, x) \in \Sigma$ and any history $h \in H$, let

$$z^\sigma_i(h) = P(V^\sigma_i(s_i | h) \geq x_i(h))$$

be the probability that buyer $i$ accepts the seller’s offer $x_i(h)$ given his valuation
conditional on history $h \in H$.

**Lemma 3.1.** Let $\sigma = (r, x)$ and $\sigma' = (r', x')$ be any sales schemes and $h$ and $h'$ be
any histories induced by $\sigma$ and $\sigma'$, respectively, with the same set of buyers along
them and the same outcomes (i.e., $A(h) = A(h')$ and $B(h) = B(h')$). For any buyer
$j \in J \equiv I(h) = I(h')$, let $h_j$ and $h'_j$ denote the truncations of $h$ and $h'$, respectively,
with which the seller trades with $j$: $j \in r(h_j) \cap r'(h'_j)$. If $z^\sigma_j(h_j) = z^{\sigma'}_j(h'_j)$ for every
$j \in J$, then for any remaining buyer $i \notin J$,

$$V^\sigma_i(s_i | h) = V^{\sigma'}_i(s_i | h') \text{ for every } s_i.$$
Our next observation concerns the expected change in a buyer’s valuation as a function of the decision of his predecessor. For any sales scheme \( \sigma \), history \( h \), and buyer \( i \) that \( \sigma \) trades with at \( h \), let

\[
\kappa_i(\sigma)(h) = E[\tilde{s}_i - \mu_i \mid V_i(\tilde{s}_i \mid h) \geq x_i(h)],
\]

\[
\lambda_i(\sigma)(h) = E[\tilde{s}_i - \mu_i \mid V_i(\tilde{s}_i \mid h) < x_i(h)].
\]

(2)

That is, \( \kappa_i(\sigma)(h) \) denotes the expected value of bidder \( i \)’s private signal (minus its unconditional mean \( \mu_i \)) when he accepted the seller’s offer \( x_i(h) \) at history \( h \). Likewise, \( \lambda_i(h) \) is the expected value of his private signal (minus \( \mu_i \)) when he rejected the offer. It should be noted that for any buyer \( j \) that comes after \( i \), \( c_{ji} \kappa_i(\sigma)(h) \) equals the change in his valuation when \( i \) accepts the offer, and \( c_{ji} \lambda_i(\sigma)(h) \) equals the change in his valuation when \( i \) rejects the offer. Since

\[
z_i(h)\kappa_i(\sigma)(h) + (1 - z_i(h)) \lambda_i(\sigma)(h)
= E \left[ (\tilde{s}_i - \mu_i) 1\{V_i(\tilde{s}_i \mid h) \geq x_i(h)\} + (\tilde{s}_i - \mu_i) 1\{V_i(\tilde{s}_i \mid h) < x_i(h)\} \right]
= 0,
\]

the expected change in \( j \)’s valuation is zero conditional on \( h \). We summarize this simple martingale property in the next lemma.

**Lemma 3.2.** For any \( \sigma = (r, x) \), \( h \in H \), and \( i \in I \) such that \( i \in r(h) \), \( z_i(h)\kappa_i(\sigma)(h) + (1 - z_i(h)) \lambda_i(\sigma)(h) = 0 \).

Lemma 3.2 also implies that \( \kappa_i(\sigma)(h) \geq 0 \) and \( \lambda_i(\sigma)(h) \leq 0 \). That is, acceptance by any buyer has a positive impact on the valuation of his successors, while every rejection has a negative impact. As seen in the next section, given any scheme that trades with buyers \( i \) and \( j \) simultaneously, Lemma 3.2 allows us to construct an alternative scheme that trades with them in sequence, but yields exactly the same revenue.

### 4 Sequential Sales Scheme

In this section, we show that the seller’s expected payoff is maximized when he employs a sequential scheme.
Theorem 4.1. The seller’s expected revenue is maximized when he employs a sequential scheme: There exists a sequential sales scheme $\sigma^*$ such that $\pi(\sigma^*) = \max_{\sigma \in \Sigma} \pi(\sigma)$. 

Proof. See the Appendix. 

The proof of the theorem shows that given any non-sequential scheme $\sigma$, there exists a sequential scheme that performs at least as well as $\sigma$. Suppose for simplicity that $\sigma$ induces some history $\underline{h} \in H_{n-1} \ (n \geq 1)$ at which it trades with two buyers $i$ and $j$. Let $x_j \equiv x_j(\underline{h})$ denote the price offer to buyer $j$ under the original scheme. Consider the following alternative scheme $\sigma^* = (r^*, x^*)$: In period $n$ at history $\underline{h}$, $\sigma^*$ trades only with buyer $i$ by offering the same price as under $\sigma$. In period $n+1$, $\sigma^*$ trades with buyer $j$ with the price offer adjusted according to the outcome of trade with buyer $i$. Specifically, the offer to $j$ under $\sigma^*$ equals $x_j + c_{ji} \kappa_i^\sigma(\underline{h})$ when buyer $i$ accepted the offer, and it equals $x_j + c_{ji} \lambda_i^\sigma(\underline{h})$ when buyer $i$ rejected the offer. Since buyer $j$’s valuation changes by either $c_{ji} \kappa_i^\sigma(\underline{h})$ or $c_{ji} \lambda_i^\sigma(\underline{h})$ as a result of the period $n$ outcome, the probability that he accepts the modified offer after each contingency is equal to the probability that he accepts the offer $x_j$ at $\underline{h}$ under the original scheme. Specifically, if we denote by $(\underline{h}, 1) \in H_n$ the history under $\sigma^*$ which takes place when $i$ accepts the offer at $\underline{h}$, then

$$z_j^*(\underline{h}, 1) = P\left(V_j^{\sigma^*}(\bar{s}_j \mid \underline{h}, 1) \geq x_j + c_{ji} \kappa_i^\sigma(\underline{h})\right)$$

$$= P\left(V_j^{\sigma^*}(\bar{s}_j \mid \underline{h}) \geq x_j\right)$$

$$= z_j(\underline{h}).$$

Likewise, if $(\underline{h}, 0) \in H_n$ denotes the $n$-length history under $\sigma^*$ which takes place when $i$ rejects the offer at $\underline{h}$, then $z_j^*(\underline{h}, 0) = z_j(\underline{h})$. It then follows that the seller’s expected revenue from buyer $j$ under $\sigma^*$ conditional on $\underline{h}$ is computed as

$$z_i(\underline{h}) z_j(\underline{h}) \{x_j + c_{ji} \kappa_i^\sigma(\underline{h})\} + (1 - z_i(\underline{h})) z_j(\underline{h}) \{x_j + c_{ji} \lambda_i^\sigma(\underline{h})\}$$

$$= z_j(\underline{h}) \left[x_j + c_{ji} \left(z_i(\underline{h}) \kappa_i^\sigma(\underline{h}) + (1 - z_i(\underline{h})) \lambda_i^\sigma(\underline{h})\right)\right]$$

$$= z_j(\underline{h}) x_j,$$

where the second equality follows from Lemma 3.2. Note that $z_j(\underline{h}) x_j = z_j(\underline{h}) x_j(\underline{h})$ is just the expected revenue from buyer $j$ under the original scheme. It also follows from Lemma 3.1 that the valuation functions of all the buyers that come after $(\underline{h}, 1)$ or $(\underline{h}, 0)$ are the same as those under $\sigma$ since regardless of $i$’s decision, the probability
that $j$ accepts the offer under $\sigma^*$ is the same under the original scheme. Therefore, if $\sigma^*$ posts the same price to each of those buyers as $\sigma$, then the seller’s expected revenue from them is just the same. As seen in the formal proof in the Appendix, these arguments generalize to the case where $\sigma$ chooses more than two buyers at $h$. Hence, if $\sigma^*$ trades with more than one buyer in any period, we can repeatedly apply the above argument to conclude that there is a sequential scheme that yields the same expected revenue as $\sigma$.

As mentioned in the Introduction, Ottaviani and Prat (2001) prove the linkage principle for the monopoly problem when the monopolist can publicly reveal information which is affiliated with the value of the good as well as the buyer type. Their theorem specifically implies that a sequential scheme dominates a simultaneous scheme since the former provides the buyers with more information. However, it cannot be applied to our framework for the following reasons. First, the analysis of Ottaviani and Prat (2001) builds on the incentive compatibility conditions associated with finite buyer types. It is hence not clear how we can generalize their argument to the continuous type distribution. Second, and more important, their result cannot be used to rank various intermediate schemes that may arise when there are three or more buyers. To see this point, suppose that we want to compare the performance of the following two schemes against three buyers: In the first scheme, the seller trades with buyers 1 and 2 in stage 1 and then with buyer 3 in stage 2. In the second sequential scheme, the seller trades with buyer 1 in stage 1, buyer 2 in stage 2, and buyer 3 in stage 3. The theorem of Ottaviani and Prat (2001) shows that the seller’s revenue from buyer 2 is higher in the second scheme than in the first scheme since buyer 2 is provided with more information in the second scheme. However, it is not clear if the seller’s revenue from buyer 3 is likewise increased since he observes the decisions of the other two buyers in both schemes. On the other hand, we use Lemma 3.1 to show that a local expansion of a non-sequential scheme can be done in such a manner that the expected revenue from buyer 3 remains the same.

5 Optimal Buyer Sequence

Given the optimality of sequential sales schemes demonstrated in the previous section, we now turn to the question of optimal buyer sequencing in such schemes. In particular, when the buyers differ only in the weights they place on others’ signals,
how should the seller order them in terms of those weights? In what follows, we focus on this question by supposing that each buyer’s private signal has a common distribution, and that his valuation \( v_i(s) \) given the signal profile \( s = (s_1, \ldots, s_I) \) equals

\[
v_i(s) = c_0 + s_i + c_i \sum_{j \neq i} (s_j - \mu),
\]

where \( \mu \) is the common mean of \( s_j \). That is, we set \( c_{ii} = 1, c_{ij} = c_i \) for \( j \neq i, 0, \) and \( c_{i0} = c_0 \) in the general formulation of Section 2. Note that \( c_i \) is the only source of difference across buyers and is an unambiguous measure of the degree of dependence of buyer \( i \)'s valuation on others’ information.

Given a sequential sales scheme \( \sigma = (r, x) \), we redefine \( r(h) \) to be the buyer (an element of \( I \)) that \( r \) chooses at history \( h \). We also express a history induced by \( \sigma \) as a sequence of 0’s and 1’s: At history \( h \in H_{t-1} \), outcome 1 in period \( t \) implies that buyer \( r(h) \) accepted the seller’s offer and outcome 0 implies that he rejected it. For example, \((1, 0) \in H_2 \) denotes the history induced by \( \sigma \) in which buyer \( r(h_0) \) accepts the offer \( x(h_0) \), and then buyer \( r(1) \) rejects the offer \( x(1) \). Likewise, given any history \( h \in H_{t-1} \) induced by \( \sigma \), \((h, 1) \) and \((h, 0) \) represent the histories obtained by appending to \( h \) buyer \( r(h) \)'s acceptance and rejection, respectively, in period \( t \).

Let a sales scheme \( \sigma \in \Sigma \) be given. For any history \( h \in H_{t-1} \), let

\[
\alpha^\sigma(h) = \sum_{j \in I(h)} E^\sigma[\tilde{s}_j - \mu | h]
\]

be the sum of the expected values of private signals (minus \( \mu \)) conditional on history \( h \). It can be readily verified that buyer \( i \)'s valuation function conditional on history \( h \) can be expressed as

\[
V_i^\sigma(s_i | h) = c_0 + s_i + c_i \alpha^\sigma(h).
\]

In the sense that \( \alpha^\sigma(h) \) completely determines a buyer’s valuation at \( h \), it is referred to as the state at \( h \). When \( r(h) = i \), the state transition is described as follows:

\[
\alpha^\sigma(h, 1) = \alpha^\sigma(h) + E[\tilde{s}_i - \mu | V_i^\sigma(\tilde{s}_i | h) \geq x_i(h)] = \alpha^\sigma(h) + \kappa_i^\sigma(h)
\]

if \( i \) accepts,

and

\[
\alpha^\sigma(h, 0) = \alpha^\sigma(h) + E[\tilde{s}_i - \mu | V_i^\sigma(\tilde{s}_i | h) < x_i(h)] = \alpha^\sigma(h) + \lambda_i^\sigma(h)
\]

if \( i \) rejects.
Since $\kappa_i^\sigma(h) \geq 0 \geq \lambda_j^\sigma(h)$ as seen earlier, the state variable goes up whenever an offer is accepted and goes down whenever it is rejected. Furthermore, since the initial state is $\alpha_0 = 0$, the state remains positive as long as all previous transactions have been successful, and remains negative as long as they have all failed.

In what follows, we will focus on a class of sequential schemes in which the pricing function $x$ satisfies certain monotonicity conditions. As will be seen later, these conditions hold for the optimal pricing function under the uniform signal distribution. Formally, given any history $h \in H_{n-1}$ ($n \leq I - 1$), the sales scheme $\sigma = (r, x)$ is non-contingent after $h$ if the buyers $r$ chooses in periods $n+1, \ldots, I$ are independent of the outcomes in periods $n, \ldots, I - 1$, i.e., there exist $\rho_{n+1}, \ldots, \rho_I \in I \setminus I(h)$ such that for any sequence of outcomes $y_n, \ldots, y_t$ in periods $n, \ldots, t$,

$$r(h, y_n, \ldots, y_{t-1}) = \rho_t. \quad (4)$$

Intuitively, $\sigma$ is non-contingent after $h$ if the target buyers in all future periods are known at $h$. The target function $\sigma$ is non-contingent if it is non-contingent after the null history. In other words, if $\sigma$ is non-contingent, then the buyer sequence is fixed a priori. It should be noted that for any history $h$ of length $I - 2$, every sequential scheme $\sigma$ is non-contingent after $h$ since then no matter what happens with buyer $r(h)$ in period $I - 1$, the only remaining buyer must be chosen in period $I$.

Given any sales scheme $\sigma \in \Sigma$ and history $h \in H$, recall that $z_i(h) \equiv z_i^\sigma(h)$ denotes the probability that the seller’s price offer $x_i(h)$ is accepted by buyer $i$ at history $h \in H$. Suppose that $r$ is non-contingent after some history $h$, and satisfies the following inequality:

$$z_j(h)\alpha^\sigma(h) \leq z_j(h) z_i(h, 1) \{\alpha^\sigma(h) + \kappa_j^\sigma(h)\} + (1 - z_j(h)) z_i(h, 0) \{\alpha^\sigma(h) + \lambda_j^\sigma(h)\}. \quad (5)$$

where $j$ is the target buyer at $h$ and $i$ is the target buyer in the following period (i.e., $j = r(h)$ and $i = r(h, 0) = r(h, 1)$). As noted earlier, $\alpha^\sigma(h) + \kappa_j^\sigma(h)$ is the state at history $(h, 1)$ when $j$ accepts at $h$, and $\alpha^\sigma(h) + \lambda_j^\sigma(h)$ is the state at history $(h, 0)$ when $j$ rejects at $h$. It follows that (5) captures the movement of the (expected) value of the product [probability of acceptance] $\times$ [state]: The left-hand side gives the value of this product at history $h$, while the right-hand side is the expected value of the product in the following period (conditional on $h$). (5) requires that

$^{15}$Note that the target buyer $r(h)$ in period $n$ is known at $h$ whether $r$ is non-contingent or not.
this value be increasing, and hence is referred to as the *monotonicity condition* at \( h \).

Now define \( \Sigma^0(h) \) to be the class of sequential sales schemes such that

\[
\Sigma^0 = \left\{ \sigma = (r, x) : \text{For any history } h \text{ induced by } \sigma, \right. \\
\text{if } \sigma \text{ is non-contingent after } h, \text{ then (5) holds at } h \right\}
\]

(6)

Take any pair of buyers \( i \) and \( j \) such that buyer \( j \) is more dependent on others’ signals than buyer \( i \): \( c_j \geq c_i \). The following lemma states that for any \( \sigma \in \Sigma^0 \), if \( r \) is non-contingent after some history \( h \) and trades with buyers \( j \) and \( i \) in this order at \( h \), then there exists an alternative scheme \( \sigma^* \) that reverses the order of \( i \) and \( j \) and yields a higher revenue than \( \sigma \).

**Lemma 5.1.** Let \( \sigma = (r, x) \in \Sigma^0 \). Suppose that \( \sigma \) induces history \( h \in H_{n-1} \) (1 \( \leq \) \( n \leq I - 1 \)) such that \( r \) is non-contingent after \( h \), and

\[
r(h) = j \quad \text{and} \quad r(h, 0) = r(h, 1) = i
\]

for some \( i \neq j \) such that \( c_j \geq c_i \). Then there exists \( \sigma^* = (r^*, x^*) \in \Sigma^0 \) such that \( r^* \) is non-contingent after \( h \), \( r^*(h) = i \), \( r^*(h, 0) = r^*(h, 1) = j \), and \( \pi(\sigma^*) \geq \pi(\sigma) \).

**Proof.** See the Appendix. \( \blacksquare \)

The proof of this lemma consists of showing that when the alternative scheme \( \sigma^* \) is chosen so that the probability of acceptance by \( i \) under \( \sigma^* \) equals that by \( j \) under \( \sigma \), and that the probability of acceptance by \( j \) under \( \sigma^* \) equals that by \( i \) under \( \sigma \), then the difference in expected revenue between the two schemes is equivalent to (5).

Take any \( \sigma = (r, x) \in \Sigma^0 \) and suppose \( c_j \geq c_i \). Since \( r \) is non-contingent at any \( h \in H_{I-2} \) where there are only two buyers left as noted above, it follows from Lemma 5.1 that revenue improvement is possible if \( \sigma \) trades with \( j \) and \( i \) in this order. In other words, whenever there are two buyers left, it is always optimal to trade first with the one with the smaller dependence weight. This suggests that at any history \( h \in H_{I-3} \) where there are three buyers left, the optimal scheme \( \sigma \) is non-contingent after \( h \): No matter what the outcome with buyer \( r(h) \) in period \( I-2 \), choose the buyer with the smaller weight in period \( I - 1 \). If \( \sigma \) is non-contingent at \( h \in H_{I-3} \), however, Lemma 5.1 suggests that revenue improvement is possible if \( \sigma \) trades with a buyer with the smallest weight among the three first at \( h \). Repeating
this argument, we can show that it is optimal to trade with the buyer with the smallest weight in period 1, one with the second smallest weight in period 2, and so on. The following theorem formalizes this argument and shows that among the sales schemes in $\Sigma^0$, the seller’s expected revenue is maximized when he trades with the buyers in the increasing order of their weights $c_i$.

**Theorem 5.2.** Suppose that $c_1 \leq \cdots \leq c_I$. Among the sales schemes in $\Sigma^0$, the seller’s expected revenue is maximized when he employs a non-contingent scheme that trades with buyer $t$ in period $t$ for every $t \in I$.

**Proof.** See the Appendix. ■

This theorem is useful in that it reduces the problem of finding an optimal sales scheme to that of solving the optimal pricing problem along the fixed sequence of buyers as described in the next section.

### 6 Sequential Pricing Problem

As seen above, the monotonicity condition concerns the acceptance probabilities along a fixed sequence of buyers at the tail. In this section, we consider the constrained optimization problem in which the seller trades sequentially with a subset of buyers taking the buyer order as given. The interpretation is that these buyers are at the tail of the sequence in the original maximization problem. We will argue that if the optimal pricing function in this constrained problem satisfies the monotonicity condition (5), then the optimal sales scheme in the unconstrained problem belongs to the set $\Sigma^0$ defined in (6). This in turn implies through Theorem 5.2 that the optimal scheme is indeed the non-contingent scheme that entails the increasing sequence of dependence weights. Assume in what follows that the common cumulative distribution function of the private signal $s_i$, denoted $F$, is strictly increasing.

In order to define a sequential pricing problem, we begin by rewriting a buyer’s valuation function in terms of the state $\alpha$. Specifically, since the valuation at any history $h$ is completely determined by $\alpha$ at $h$ as in (3), write $V_i(s_i | \alpha)$ for buyer $i$’s valuation in state $\alpha$:

$$V_i(s_i | \alpha) = c_0 + s_i + c_i \alpha.$$  

In what follows, we will treat the state $\alpha$ as a continuous variable and for any $t = 0, \ldots , I - 1$, let $C_t = [t(\bar{s} - \mu), t(\bar{s} - \mu)]$ for any $t = 0, 1, \ldots , I - 1$. It can be seen
that if $\alpha$ is any state at the completion of the trades with the first $t$ buyers, then $\alpha \in C_t$. For any subset $J \subseteq I$ with $J \neq \emptyset$, suppose that the seller has traded with the buyers in $I \setminus J$ and will face the buyers in $J$. Let $n = |J|$ and $\rho = (\rho_1, \ldots, \rho_n)$ be any permutation over $J$. Let also the state $\alpha_0 \in C_{I-n}$ be given. The sequential pricing problem given $q \equiv (J, \rho, \alpha_0)$ is the problem of finding an optimal contingent price when the seller trades with buyers $\rho_1, \ldots, \rho_n \in J$ in this order given the initial state $\alpha_0 \in C_{I-n}$. Since the buyers’ valuations are determined by $\alpha$, we also redefine the pricing function as a function of $\alpha$. Specifically, let $x_t(\alpha)$ be any permutation over $J$.

Let also the state $\alpha_0 \in C_{I-n}$ be given. The sequential pricing problem given $q \equiv (J, \rho, \alpha_0)$ is the problem of finding an optimal contingent price when the seller trades with buyers $\rho_1, \ldots, \rho_n \in J$ in this order given the initial state $\alpha_0 \in C_{I-n}$. Since the buyers’ valuations are determined by $\alpha$, we also redefine the pricing function as a function of $\alpha$. Specifically, let $x_t(\alpha)$ be any permutation over $J$.

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Let also the state $\alpha_0 \in C_{I-n}$ be given. The sequential pricing problem given $q \equiv (J, \rho, \alpha_0)$ is the problem of finding an optimal contingent price when the seller trades with buyers $\rho_1, \ldots, \rho_n \in J$ in this order given the initial state $\alpha_0 \in C_{I-n}$. Since the buyers’ valuations are determined by $\alpha$, we also redefine the pricing function as a function of $\alpha$. Specifically, let $x_t(\alpha)$ be any permutation over $J$.

Let also the state $\alpha_0 \in C_{I-n}$ be given. The sequential pricing problem given $q \equiv (J, \rho, \alpha_0)$ is the problem of finding an optimal contingent price when the seller trades with buyers $\rho_1, \ldots, \rho_n \in J$ in this order given the initial state $\alpha_0 \in C_{I-n}$. Since the buyers’ valuations are determined by $\alpha$, we also redefine the pricing function as a function of $\alpha$. Specifically, let $x_t(\alpha)$ be any permutation over $J$.

Let also the state $\alpha_0 \in C_{I-n}$ be given. The sequential pricing problem given $q \equiv (J, \rho, \alpha_0)$ is the problem of finding an optimal contingent price when the seller trades with buyers $\rho_1, \ldots, \rho_n \in J$ in this order given the initial state $\alpha_0 \in C_{I-n}$. Since the buyers’ valuations are determined by $\alpha$, we also redefine the pricing function as a function of $\alpha$. Specifically, let $x_t(\alpha)$ be any permutation over $J$.

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It follows that the monotonicity condition (5) in period $t$ at state $\alpha$ is expressed in terms of $\kappa$ and $\lambda$ as

$$
    z_t(\alpha) \alpha \leq z_t(\alpha) z_{t+1}(\alpha + \kappa(z_t(\alpha))) \{ \alpha + \kappa(z_t(\alpha)) \}
    + (1 - z_t(\alpha)) z_{t+1}(\alpha + \lambda(z_t(\alpha))) \{ \alpha + \lambda(z_t(\alpha)) \}.
$$

(8)

The sequential pricing problem given $q \equiv (J, \rho, \alpha_0)$ is described as follows. Let $\pi_n(z_n, \alpha_{n-1})$ denote the seller’s expected revenue from the last buyer $\rho_n$ in state $\alpha_{n-1}$ when he makes an offer that is accepted with probability $z_n$. In view of (7), it can be written as

$$
    \pi_n(z_n, \alpha_{n-1}) = g(z_n) + z_n c_\rho_n \alpha_{n-1} + c_0 z_n,
$$

where $g : [0, 1] \to \mathbb{R}$ is defined by $g(z) = z F^{-1}(1 - z)$. Let $\pi^*_n(\alpha_{n-1})$ denote the maximized value of $\pi_n(z_n, \alpha_{n-1})$:

$$
    \pi^*_n(\alpha_{n-1}) = \max_{z_n \in [0, 1]} \pi_n(z_n, \alpha_{n-1}).
$$

For $t = 1, \ldots, n - 1$, the seller’s expected revenue over periods $t, \ldots, n$ is recursively defined by

$$
    \pi_t(z_t, \alpha_{t-1}) = g(z_t) + z_t c_\rho \alpha_{t-1} + c_0 z_t + f_{t+1}(z_t, \alpha_{t-1}),
$$

where

$$
    f_{t+1}(z_t, \alpha_{t-1}) = z_t \pi^*_{t+1}(\alpha_{t-1} + \kappa(z_t)) + (1 - z_t) \pi^*_{t+1}(\alpha_{t-1} + \lambda(z_t))
$$

is the seller’s expected revenue over periods $t + 1, \ldots, n$ when he chooses $z_t$ in period $t$, and then follows the optimal course of action in subsequent periods. The optimized value of $\pi_t(z_t, \alpha_{t-1})$ is denoted by

$$
    \pi^*_t(\alpha_{t-1}) = \max_{z_t \in [0, 1]} \pi_t(z_t, \alpha_{t-1}).
$$

(9)

Let $z^{q^*}$ be the solution to (9) when the parameters are $q = (J, \rho, \alpha_0)$. The following corollary states that if $z^{q^*}$ satisfies (8) for $t = 1, \ldots, n$ and for every possible parameter combination $q = (J, \rho, \alpha_0)$, then the optimal sales scheme $\sigma^*$ is indeed non-contingent and entails an increasing sequence of the $c_i$’s.

**Corollary 6.1.** Suppose that $c_1 \leq \cdots \leq c_I$ and that for every possible combination of parameters $q = (J, \rho, \alpha_0)$, $z^{q^*}$ satisfies (8) for $t = 1, \ldots, |J| - 1$. Then among all the sales schemes, the seller’s expected revenue is maximized by the non-contingent scheme that trades with buyer $t$ in period $t$ for every $t \in I$. 

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Proof. Let \( \sigma = (r, x) \) be an optimal scheme and take any history \( \underline{h} \) induced by \( \sigma \) after which \( \sigma \) is non-contingent. Let \( J = I \setminus I(\underline{h}) \) be the set of remaining buyers at \( \underline{h} \) and \( n = |J| \). Furthermore, let \( \rho = (\rho_1, \ldots, \rho_n) \) be the non-contingent sequence of buyers that \( \sigma \) trades with after \( \underline{h} \). Then the continuation pricing function \( x \) after \( \underline{h} \) solves the sequential pricing problem with \( q = (J, \rho, \alpha_0 = \alpha^\sigma(\underline{h})) \). By assumption, \( z \) associated with this \( x \) satisfies (5) for \((i, j) = (\rho_{t+1}, \rho_t)\) for \( t = 1, \ldots, n - 1 \), and hence \( \sigma \in \Sigma^0 \). It then follows from Theorem 5.2 that the seller’s expected revenue is maximized by the non-contingent scheme that trades with buyer \( t \) in period \( t \). □

7 Uniform Distribution

In this section, we suppose that a buyer’s private signal is drawn from a uniform distribution and present an analytical solution to the sequential pricing problem formulated in the previous section. Sufficient conditions for the solution to satisfy the monotonicity condition are given. We will then show that the optimal scheme induces herd behavior among buyers more often than any other scheme that entails an alternative buyer sequence.

Formally, suppose that all buyers are identical except for the weights \( c_i \), and that their signals are drawn from the uniform distribution over the interval \([s, \bar{s}]\), i.e.,

\[
F(s_i) = \frac{s_i - \bar{s}}{\Delta},
\]

where \( \Delta = \bar{s} - \underline{s} \). In this case, \( \kappa(z) \) and \( \lambda(z) \) can be written explicitly as:

\[
\kappa(z) = \frac{\Delta}{2} (1 - z) \quad \text{and} \quad \lambda(z) = -\frac{\Delta}{2} z.
\]

For any \( n = 2, \ldots, I \), take any set \( J \) of \( n \) buyers and any ordering \( \rho = (\rho_1, \ldots, \rho_n) \) of them. Let \( a_n = \Delta + \underline{s} + c_0 \), \( b_n = c_{\rho_n} \), and

\[
\begin{align*}
a_t &= \Delta + \frac{s + c_0}{1 + \frac{\Delta}{16} \sum_{k=t+1}^{n} b_k c_{\rho_k}}, \quad \text{and} \quad b_t = \frac{c_{\rho_t}}{1 + \frac{\Delta}{16} \sum_{k=t+1}^{n} b_k c_{\rho_k}}
\end{align*}
\]

for \( t = 1, \ldots, n - 1 \). The following theorem explicitly describes the solution to the sequential pricing problem when it has an interior solution for every \( t = 1, \ldots, n \).

Theorem 7.1. Suppose that every \( s_i \) has the uniform distribution over \([\underline{s}, \bar{s}]\). Take any subset \( J \) of \( n \) buyers, any ordering \( \rho = (\rho_1, \ldots, \rho_n) \) of them, and any initial
state $\alpha_0 \in C_{I-n}$. If
\[
b_t < \frac{2}{\Delta(I-n+t-1)} \min \{\Delta, 2\Delta - a_t\} \quad \text{for every } t = 1, \ldots, n, \tag{10}\]
the solution to the sequential pricing problem with $q = (J, \rho, \alpha_0)$ is given by
\[
z_t(\alpha) = \frac{1}{2\Delta}(a_t + b_t \alpha) \tag{11}\]
for any $\alpha \in C_{I-n+t-1}$ and $t = 1, \ldots, n$.

**Proof.** See the Appendix. ■

It can be readily verified that the condition (10) guarantees that for any $t$, the optimal probability is an interior solution: $z_t(\alpha) \in (0, 1)$ for any $\alpha \in C_{I-n+t-1}$. Since $b_t \leq c_\rho$, this condition holds when $s + c_0 < \Delta$ and all the weights $c_i$ are small. It should be noted however that the total weight $(I-1)c_i$ placed by $i$ on others’ signals need not be small: For example, when $s + c_0 = 0$, (10) holds if $(I-1)c_i < 2$ for every $i \in I$.

Since $z_t$ is an affine function by Theorem 7.1, we have for any $\alpha$,
\[
z_t(\alpha) z_{t+1}(\alpha + \kappa(z_t(\alpha))) + (1 - z_t(\alpha)) z_{t+1}(\alpha + \lambda(z_t(\alpha))) = z_{t+1}(\alpha + \kappa(z_t(\alpha))) + (1 - z_t(\alpha)) \lambda(z_t(\alpha)) \tag{12}\]
where the second equality follows from Lemma 3.2. Using (11) and (12), we see that the monotonicity condition (8) holds at any state in any period if
\[
\{16(b_{t+1} - b_t) - b_t^2 b_{t+1}\} \alpha^2 + 2\{8(a_{t+1} - a_t) + b_t b_{t+1}(\Delta - a_t)\} \alpha + a_t b_{t+1}(2\Delta - a_t) \geq 0 \tag{13}\]
for any $\alpha \in C_{I-n+t-1}$ and $t = 1, \ldots, n$.

The next lemma presents sufficient conditions for (13).

**Lemma 7.2.** In the sequential pricing problem $(J, \rho, \alpha_0)$, the monotonicity condition (13) holds if
\[
\frac{b_t}{b_{t+1}} < 1 - \frac{b_t^2}{16} + \frac{1}{4(t-1)^2} \quad \text{for } t = 1, \ldots, I-1, \tag{14}\]
and either $s + c_0 = 0$, or $\Delta$ is sufficiently large compared with $s$, $c_0$ and $\max_i c_i$.  

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Proof. See the Appendix. ■

Since \( \frac{b_t}{b_{t+1}} < \frac{c_{pt}}{c_{pt+1}} \) and \( b_t \leq c_{pt} \), (14) holds if

\[
\frac{c_i}{c_j} \leq 1 - \frac{c_i^2}{16} + \frac{1}{4(I-1)^2} \quad \text{for any } i \neq j.
\]

(15)

In turn, these inequalities hold when \( \max_{i \in I} c_i < \frac{1}{2}I-1 \) and \( c_i \) and \( c_j \) are sufficiently close to each other. The following theorem identifies the optimal scheme under the uniform distribution.

**Theorem 7.3.** Suppose that every \( s_i \) has the uniform distribution over \([s, \bar{s}]\), and that (10) and (13) hold for the solution to every sequential pricing problem \((J, \rho, \alpha_0)\). If \( c_1 \leq \ldots \leq c_I \), then the seller’s expected revenue is maximized when he employs a non-contingent sequential scheme and trades with buyer \( t \) in period \( t \) (i.e., \( \rho_t = t \)) by posting a price that is accepted with probability \( z_t \) given in Theorem 7.1.

There are more than one way to see that the optimal scheme identified above amplifies the buyers’ tendency to herd. First, it can be shown that the probability of acceptance in the optimal scheme moves in the direction of the dominant action in the past. In other words, a buyer accepts with a higher probability than his predecessor when most buyers before him have chosen to accept, and rejects with a higher probability when most buyers before him have chosen to reject. Second, there is a more direct effect of buyer ordering: the buyers placed late in the sequence are those who have larger dependence weights and hence are more heavily influenced by public information. It can be shown that the probability that every buyer makes the same decision in the optimal scheme is larger than that in any other scheme which trades with them in an alternative order. For technical reasons, our discussion below assumes that the dependence weights \( c_1, \ldots, c_I \) are all small. Specifically, suppose that for some \( \varepsilon > 0 \) and \( \hat{c}_1, \ldots, \hat{c}_I > 0 \), we can write

\[
c_i = \varepsilon \hat{c}_i \quad \text{for every } i \in I.
\]

(16)

We will consider the limit as \( \varepsilon \to 0 \) while keeping \( \hat{c}_1, \ldots, \hat{c}_I \) fixed. It can be seen that (15) will be satisfied for a sufficiently small \( \varepsilon > 0 \) if

\[
\frac{\max_{i \in I} \hat{c}_i}{\min_{i \in I} \hat{c}_i} < 1 + \frac{1}{4(I-1)^2}.
\]

(17)

For example, when \( s + c_0 = 0 \), (17) guarantees that the monotonicity condition (13) is satisfied for a small \( \varepsilon \) by Lemma 7.2.

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For any \( m = 0, 1, \ldots \), let \( o(\varepsilon^m) \) denote any term such that \( \lim_{\varepsilon \to 0} |o(\varepsilon^m)|/\varepsilon^m = 0 \). Consider now the sequential pricing problem \((I, \rho, \alpha_0 = 0)\) against all buyers. We have \( a_I = \Delta + s + c_0 \) and \( b_I = c_{\rho_I} \) by definition, and can also show by induction that \( a_t = \Delta + s + c_0 + o(\varepsilon) \) and \( b_t = c_{\rho_t} + o(\varepsilon^2) \), for \( t = 1, \ldots, I-1 \).\(^{17}\) First, note from (12) that the expected value of the acceptance probability \( z_{t+1} \) in period \( t+1 \) conditional on the state \( \alpha_{t-1} \) at the beginning of period \( t \) equals

\[
E^\sigma[z_{t+1}(\tilde{a}_t) | \alpha_{t-1}] = z_{t+1}(\alpha_{t-1}).
\]

It hence follows that

\[
E^\sigma[z_{t+1}(\tilde{a}_t) | \alpha_{t-1}] - z_t(\alpha_{t-1}) = \frac{1}{2\Delta} \{ a_{t+1} - a_t + (b_{t+1} - b_t) \alpha_{t-1} \}
= \frac{1}{2\Delta} (c_{\rho_{t+1}} - c_{\rho_t}) \alpha_{t-1} + o(\varepsilon).
\]

Since \( c_{\rho_{t+1}} - c_{\rho_t} > 0 \) in the optimal scheme identified in Theorem 7.3, we readily obtain the following proposition, which identifies one source of herd behavior through the optimal pricing strategy: When the state \( \alpha_{t-1} > 0 \) as a result of many buyers having accepted, the expected probability that the next buyer accepts is higher than the probability that the current buyer accepts. Conversely, when the state \( \alpha_{t-1} < 0 \) as a result of many buyers having rejected, the expected probability that the next buyer rejects is higher than the probability that the current buyer rejects. In other words, when one of the two decisions is dominant in the history, the probability that the dominant action is chosen tends to go up.

**Proposition 7.1.** Suppose that \( c_1 < \cdots < c_I \), and denote by \( \sigma \) the optimal sequential scheme that trades with buyer \( t \) in period \( t \). Then for any \( \delta > 0 \), there exists

\[
b_t = c_{\rho_t} + o(\varepsilon^2) \quad \text{for} \quad k = t + 1, \ldots, I - 1,
\]

and

\[
a_t = \Delta + s + c_0 - (s + c_0) \frac{\sum_{k=t+1}^{I} b_k c_{\rho_k}}{1 + \sum_{k=t+1}^{I} b_k c_{\rho_k}} = \Delta + s + c_0 - o(\varepsilon^2).
\]

Furthermore,

\[
b_k = c_{\rho_k} + o(\varepsilon^2) \quad \text{for} \quad k = t + 1, \ldots, I - 1,
\]

and

\[
a_t = \Delta + s + c_0 - (s + c_0) \frac{\sum_{k=t+1}^{I} b_k c_{\rho_k}}{1 + \sum_{k=t+1}^{I} b_k c_{\rho_k}} = \Delta + s + c_0 + o(\varepsilon).
\]
\( \bar{\varepsilon} > 0 \) such that if \( \varepsilon < \bar{\varepsilon} \), then \( \alpha_{t-1} > \delta \) implies \( E^\sigma[z_{t+1}(\tilde{\alpha}_t) \mid \alpha_{t-1}] > z_t(\alpha_{t-1}) \), and \( \alpha_{t-1} < -\delta \) implies \( E^\sigma[z_{t+1}(\tilde{\alpha}_t) \mid \alpha_{t-1}] < z_t(\alpha_{t-1}) \).

We next evaluate the effect of buyer ordering by computing the probability that every buyer makes the same decision for each buyer sequence. Formally, let the buyer ordering \( \rho = (\rho_1, \ldots, \rho_I) \) be given. Let \( \bar{\alpha}_0 = \bar{\alpha}_0 = 0 \), and for \( t = 2, \ldots, I \), let \( \bar{\alpha}_{t-1} \) and \( \underline{\alpha}_{t-1} \) be the states at the beginning of period \( t \) when the buyers \( \rho_1, \ldots, \rho_{t-1} \) have all accepted, and when they have all rejected, respectively. It follows that \( \bar{\alpha}_t \) and \( \underline{\alpha}_t \) (\( t = 1, \ldots, I-1 \)) are recursively defined by

\[
\bar{\alpha}_t = \bar{\alpha}_{t-1} + \kappa(z_{t}(\bar{\alpha}_{t-1})), \quad \text{and} \quad \underline{\alpha}_t = \underline{\alpha}_{t-1} + \lambda(z_{t}(\underline{\alpha}_{t-1})).
\]

The probabilities that all buyers accept and that they all reject can then be expressed as

\[
\prod_{t=1}^{I} z_{t}(\bar{\alpha}_{t-1}) \quad \text{and} \quad \prod_{t=1}^{I} (1 - z_{t}(\underline{\alpha}_{t-1})),
\]

respectively. The following lemma evaluates these probabilities when \( \varepsilon \) is small.

**Lemma 7.4.** Suppose that \( c_1, \ldots, c_I \) are given by (16) for \( \hat{c}_1, \ldots, \hat{c}_I \) that satisfy (17). For any buyer ordering \( \rho \), let \( (z_1, \ldots, z_I) \) be the solution to the sequential pricing problem with \((I, \rho, \alpha_0 = 0)\). Then for \( t = 1, \ldots, I - 1 \), the state variables \( \bar{\alpha}_t \) and \( \underline{\alpha}_t \), when the buyers \( \rho_1, \ldots, \rho_t \) have all accepted and all rejected, respectively, are given by

\[
\bar{\alpha}_t = \frac{t}{4}(\Delta - s - c_0) + o(1) \quad \text{and} \quad \underline{\alpha}_t = -\frac{t}{4}(\Delta + s + c_0) + o(1).
\]

Furthermore, the probability that all buyers accept equals

\[
\prod_{t=1}^{I} z_{t}(\bar{\alpha}_{t-1}) = \left(\frac{\Delta + s + c_0}{2\Delta}\right)^I \left\{ 1 + \frac{\Delta - s - c_0}{4(\Delta + s + c_0)} \sum_{t=1}^{I} (t - 1) c_{\rho_t} \right\} + o(\varepsilon),
\]

and the probability that all buyers reject equals

\[
\prod_{t=1}^{I} (1 - z_{t}(\underline{\alpha}_{t-1})) = \left(\frac{\Delta - s - c_0}{2\Delta}\right)^I \left\{ 1 + \frac{\Delta + s + c_0}{4(\Delta - s - c_0)} \sum_{t=1}^{I} (t - 1) c_{\rho_t} \right\} + o(\varepsilon)
\]

**Proof.** See the Appendix.
The following theorem, which follows immediately from the above lemma, states that the probability of the same decision is maximized when the dependence weights are ordered monotonically from the smallest to the largest.

**Theorem 7.5.** Suppose that \(c_1, \ldots, c_I\) are given by (16) for \(\hat{c}_1, \ldots, \hat{c}_I\) such that \(\hat{c}_1 \leq \cdots \leq \hat{c}_I\). Then there exists \(\bar{\varepsilon} > 0\) such that the following holds when \(\varepsilon < \bar{\varepsilon}\):

Among the solutions to the sequential pricing problems \((I, \rho, \alpha_0 = 0)\) with all possible buyer orderings \(\rho\), the probabilities that the buyers all accept and that they all reject are both maximized by \(\rho = (1, \ldots, I)\), i.e., when the seller trades with buyer \(t\) in period \(t\) for every \(t \in I\).

8 Discussions

As emphasized in the Introduction, careful choice of the timing of trades is at the core of the design of a good sales strategy. Our conclusions shed light on the dynamic sales strategies used by many firms in reality as studied extensively in the marketing literature. Although the analysis of this paper focuses attention on the price-posting sales format, other sales situations may be modeled more appropriately by alternative forms of bilateral bargaining such as Nash bargaining. The only change this modification brings about is the functional form of the link between the outcome of each transaction and the associated change in the expected value of the good. As long as this link has a similar form, the qualitative conclusions of the paper should go through.

The most important assumptions of the present model are the independence of private signals and the additive specification of valuation functions. As seen in Section 3, these assumptions imply that a buyer’s valuation in response to the outcome of each transaction changes by a publicly known amount. This property no longer holds when the signals are correlated or when the valuation functions are not additive. For example, if the valuation is a product of the private signals, then the impact of previous transactions on a buyer’s valuation varies with his own signal. The same is true when the signals are correlated across buyers. With correlated signals, the seller also faces the learning problem. For example, if the private signals indicate the underlying common value of the good, it would be in the interest of the seller to engage in experimental pricing against the initial set of buyers. As seen in Bose et al. (2005, 2006), these issues significantly complicate the analysis.

We have assumed throughout the paper that the good can be produced at no
cost to the seller. When the production cost is positive and must be incurred before each buyer’s decision, we would need to consider the possibility of exit by the seller when he finds the buyers’ valuations to be too low to justify further production. For example, in a model of monopoly pricing against the sequence of buyers, Bose et al. (2006) identify the range of beliefs that such exit takes place. Another implicit assumption of the paper is that the seller cannot gain by limiting the supply of his good. That is, the seller cannot improve his revenue by, for example, supplying only five units to ten buyers.\textsuperscript{18}

We have also assumed that the buyers’ preferences are aligned in the sense that the weight $c_{ij}$ that buyer $i$’s valuation places on $j$’s signal is non-negative. While this is a natural assumption to make in the study of herd behavior in Section 7, it is not required for some of our conclusions. Specifically, the optimality of a sequential scheme is independent of the signs of the weights. When the weights are all negative, the conclusion on the optimal ordering of buyers holds under the alternative monotonicity condition (5) with the reverse inequality. Such a condition is seen to hold under the uniform signal distribution.

It should be noted that the time dimension introduced in this paper is a sorting device for informational events and does not entail discounting and depreciation. With discounting, of course, the optimality of a sequential scheme and other conclusions of the paper would hold with qualifications. It would also lead to issues such as possible delay in transactions associated with the Coase conjecture in durable good monopoly.

As discussed in the Introduction, an alternative interpretation of the present model is through the seller’s information revelation policy. A more direct model of information revelation would be obtained if we assume that each buyer observes only the outcome of his own transaction. In such a setup, the seller’s information policy specifies which past outcomes to reveal to each subsequent buyer as a function of history. In such a model, the seller’s credibility becomes an issue just like in the optimal direct revelation mechanism discussed in Section 3 where the price for one buyer is conditioned on other buyers’ reports.

\textsuperscript{18}One way to justify this is to assume that selling at the minimum price to every buyer is more profitable than selling at the maximum price to any proper subset of them.
Appendix

Proof of Lemma 3.1 Take any \( j \in A(h) = A(h') \). Write
\[
 w_j^\sigma(h_j) = c_{j0} + \sum_{k \in I(h_j)} c_{jk} E[^\sigma[\tilde{s}_k - \mu_k \mid h_j]]
\]
for the part of \( j \)'s valuation under \( \sigma \) that is determined by the history \( h_j \). Likewise, define \( w_j^\sigma'(h_j') \) to be the part of \( j \)'s valuation under \( \sigma' \) that is determined by the history \( h_j' \). Since \( j \)'s valuation is given by \( V_j^\sigma(s_j \mid h_j) = c_{jj}s_j + w_j^\sigma(h_j) \) and \( V_j^\sigma'(s_j \mid h_j') = c_{jj}s_j + w_j^\sigma'(h_j') \), we have by assumption,
\[
z_j^\sigma(h_i) = P(c_{jj}\tilde{s}_j \geq x_j(h_j) - w_j^\sigma(h_j)) = P(c_{jj}\tilde{s}_j \geq x_j'(h_j') - w_j^\sigma'(h_j')) = z_j^\sigma'(h_j').
\]
Hence
\[
 E[^\sigma[\tilde{s}_j - \mu_j \mid h]] = E[\tilde{s}_j - \mu_j \mid V_j^\sigma(\tilde{s}_j \mid h_j) \geq x_j(h_j)]
 = E[\tilde{s}_j - \mu_j \mid c_{jj}\tilde{s}_j \geq x_j(h_j) - w_j^\sigma(h_j)]
 = E[\tilde{s}_j - \mu_j \mid c_{jj}\tilde{s}_j \geq x_j'(h_j') - w_j^\sigma'(h_j')] \quad (19)
 = E[^\sigma'[\tilde{s}_j - \mu_j \mid h']].
\]
Likewise, for any \( j \in B(h) = B(h') \), we have \( E[^\sigma[\tilde{s}_j - \mu_j \mid h]] = E[^\sigma'[\tilde{s}_j - \mu_j \mid h']]. \) Now take any buyer \( i \notin J \) who has not traded along \( h \) or \( h' \). Since for any \( s_i \), \( V_i^\sigma(s_i \mid h) = c_{i0} + \sum_{j \in J} c_{ij} E[^\sigma[\tilde{s}_j - \mu_j \mid h]] \) and \( V_i^\sigma'(s_i \mid h') = c_{i0} + c_{ii}s_i + \sum_{j \in J} c_{ij} E[^\sigma'[\tilde{s}_j - \mu_j \mid h']]. \) we conclude from the above that \( V_i^\sigma(s_i \mid h) = V_i^\sigma'(s_i \mid h'). \)

Proof of Theorem 4.1 Fix any sales scheme \( \sigma \) that is not sequential. That is, \( \sigma \) induces a history \( h \in H_{n-1} \) \( (n \geq 1) \) such that \( r(h) = \{m\} \cup J \) for some \( m \in I \) and \( J \neq \emptyset \). In other words, according to \( \sigma \), the seller trades with buyer \( m \) and at least one other buyer in period \( n \) at history \( h \). We will construct an alternative scheme \( \sigma^* \) that raises the same expected revenue as \( \sigma \) as follows: The sales scheme \( \sigma^* \) operates in the same way as \( \sigma \) does except when \( h \) arises. At history \( h \), \( \sigma^* \) trades only with buyer \( m \) with the same offer price as under the original scheme. For simplicity, denote the outcome \( y_n \in Y(\{m\}) \) in period \( n \) from buyer \( m \) under \( \sigma^* \) by either 0 or 1: 1 represents the outcome \((\{m\}, \emptyset)\) that buyer \( m \) accepts the seller’s offer, and 0 represents the outcome \((\emptyset, \{m\})\) that he rejects it. In period \( n + 1 \) at either \((h, 1)\) or \((h, 0)\), \( \sigma^* \) trades with the buyers in \( J \) with the offer prices adjusted
according to the outcome in period \( n \). In any subsequent period, the set of buyers and prices specified by \( \sigma^* \) along any history \((h, y_n, \ldots, y_{t-1}) \in H_{t-1}\) are the same as those specified by \( \sigma \) along the history \((h, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1}) \in H_{t-2}\), where \( y_n \cup y_{n+1} = (A_n \cup A_{n+1}, B_n \cup B_{n+1}) \) is the “union” of two outcomes \( y_n \) and \( y_{n+1} \): Those who accept under \( y_n \cup y_{n+1} \) are the union of those who accept under \( y_n \) and \( y_{n+1} \), and those who reject under \( y_n \cup y_{n+1} \) are the union of those who reject under \( y_n \) and \( y_{n+1} \). In other words, \( \sigma^* \) operates just as \( \sigma \) by assuming that the outcomes in periods \( n \) and \( n+1 \) came from the same period. A formal description of \( \sigma^* \) is given as follows:

\[
\begin{align*}
  r^*(h) &= \begin{cases} 
    \{m\} & \text{if } h = h, \\
    J & \text{if } h = (h, 1) \text{ or } (h, 0), \\
    r(h, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1}) & \text{if } h = (h, y_n, \ldots, y_{t-1}) \text{ for some } y_n, \ldots, y_{t-1} (t \geq n + 2), \\
    r(h) & \text{otherwise.}
  \end{cases}
\end{align*}
\]  

(20)

and for any \( i \in I \),

\[
\begin{align*}
x_i^*(h) &= \begin{cases} 
    x_i(h) + c_{im} \kappa_m(h) & \text{if } h = (h, 1) \\
    x_i(h) + c_{im} \lambda_m(h) & \text{if } h = (h, 0) \\
    x_i(h, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1}) & \text{if } h = (h, y_n, \ldots, y_{t-1}) \text{ for some } y_n, \ldots, y_{t-1} (t \geq n + 2), \\
    x_i(h) & \text{otherwise.}
  \end{cases}
\end{align*}
\]  

(21)

In what follows, we will show that \( \sigma^* \) yields the same expected revenue as \( \sigma \). Since \( \sigma \) is an arbitrary non-sequential scheme, repeated application of this argument shows that for any scheme \( \sigma \) that is not sequential, there exists a sequential scheme that yields the same expected payoff as \( \sigma \). The desired conclusion would then follow.

For simplicity, denote

\[
V_i(s_i \mid h) = V_i^\sigma(s_i \mid h), \quad V_i^*(s_i \mid h) = V_i^{\sigma^*}(s_i \mid h),
\]

\[
\kappa_i(h) = \kappa_i^\sigma(h), \quad \text{and} \quad \lambda_i(h) = \lambda_i^\sigma(h).
\]

Let also \( w_i(h) \) be defined by

\[
w_i(h) = c_{i0} + \sum_{j \in I_{n-1}} c_{ij} E^\sigma[\hat{s}_j - \mu_j \mid h].
\]
Note that
\[ V_i(s_i \mid \mathbf{h}) = V_i^*(s_i \mid \mathbf{h}) = c_{ii}s_i + w_i(\mathbf{h}), \tag{22} \]
and for any outcome \( y_n \in Y(\{m\}) = \{0, 1\} \) from buyer \( m \) in period \( n \),
\[ V_i^*(s_i \mid \mathbf{h}, y_n) = \begin{cases} 
    c_{ii}s_i + w_i(\mathbf{h}) + c_{im}\kappa_m(\mathbf{h}) & \text{if } y_n = 1, \\
    c_{ii}s_i + w_i(\mathbf{h}) + c_{im}\lambda_m(\mathbf{h}) & \text{if } y_n = 0.
\end{cases} \tag{23} \]
It hence follows from (21) that for any \( i \in J \),
\[
z_i^*(\mathbf{h}, y_n) = P\left(V_i^*(\tilde{s}_i \mid \mathbf{h}, y_n) \geq x_i^*(\tilde{s}_i \mid \mathbf{h}, y_n)\right)
= P\left(c_{ii}\tilde{s}_i + w_i(\mathbf{h}) \geq x_i(\mathbf{h})\right)
= P\left(V_i(\tilde{s}_i \mid \mathbf{h}) \geq x_i(\mathbf{h})\right)
= z_i(\mathbf{h}). \tag{24} \]
It then follows from Lemma 3.1 that
\[ V_i^*(s_i \mid \mathbf{h}, y_n, y_{n+1}) = V_i(s_i \mid \mathbf{h}, y_n \cup y_{n+1}). \]
For any \( t \geq n + 2 \) and any sequence of outcomes \( y_n, \ldots, y_{t-1} \) in periods \( n, \ldots, t-1 \) under \( \sigma^* \), we will show that a buyer’s valuation function \( V_i^*(\cdot \mid \mathbf{h}, y_n, \ldots, y_{t-1}) \) in period \( t \) induced by \( \sigma^* \) is the same as the valuation function \( V_i(\cdot \mid \mathbf{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1}) \) in period \( t-1 \) induced by \( \sigma \). As an induction hypothesis, suppose that
\[ V_i^*(s_i \mid \mathbf{h}, y_n, \ldots, y_{t-1}) = V_i(s_i \mid \mathbf{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1}) \]
for some \( t \geq n + 2 \). Since
\[ x_i^*(\mathbf{h}, y_n, \ldots, y_{t-1}) = x_i(\mathbf{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1}) \]
by definition, we have
\[ z_i^*(\mathbf{h}, y_n, \ldots, y_{t-1}) = z_i(\mathbf{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_{t-1}). \]
Hence, Lemma 3.1 implies that
\[ V_i^*(s_i \mid \mathbf{h}, y_n, \ldots, y_t) = V_i(s_i \mid \mathbf{h}, y_n \cup y_{n+1}, y_{n+2}, \ldots, y_t). \]
For any \( h \in H_{t-1} \), let \( \pi_t(h) \) denote the seller’s expected revenue in periods \( t, \ldots , I \) at history \( h \) when he employs the sales scheme \( \sigma \). Define \( \pi^*_t(h) \) similarly for \( \sigma^* \). Given the equality of the valuation functions induced by the two schemes as seen above, we have

\[
\pi^*_{n+2}(h, y_n, y_{n+1}) = \pi_{n+1}(h, y_n \cup y_{n+1})
\]

for any sequence of outcomes \( (y_n, y_{n+1}) \) in periods \( n \) and \( n + 1 \) under \( \sigma^* \). On the other hand,

\[
\pi^*_n(h) = \sum_{y_n \in Y\{m\}} \sum_{y_{n+1} \in Y(J)} P^\sigma(y_n \mid h) P^\sigma(y_{n+1} \mid h, y_n) \cdot \left\{ \sum_{i \in A_n} x^*_i(h) + \sum_{i \in A_{n+1}} x^*_i(h, y_n) + \pi^*_{n+2}(h, y_n, y_{n+1}) \right\}. \tag{25}
\]

Likewise, the expected revenue in period \( n \) under \( \sigma \) conditional on \( h \) can be expressed using \( Y\{m\} \) and \( Y(J) \) as:

\[
\pi_n(h) = \sum_{y_n \in Y\{m\}} \sum_{y_{n+1} \in Y(J)} P(y_n \mid h) P(y_{n+1} \mid h, y_n) \cdot \left\{ \sum_{i \in A_n} x_i(h) + \sum_{i \in A_{n+1}} x_i(h, y_n) + \pi_{n+1}(h, y_n \cup y_{n+1}) \right\}. \tag{26}
\]

Since \( \sigma \) and \( \sigma^* \) are identical up to and including period \( n - 1 \), we have for \( y_n \in Y\{m\} = \{0, 1\} \),

\[
P^\sigma(y_n \mid h) = P^\sigma(y_n \mid h). \tag{27}
\]

By (24), we also have for any \( y_{n+1} \in Y(J) \),

\[
P^\sigma(y_{n+1} \mid h, y_n) = \prod_{i \in A_{n+1}} z^*_i(h, y_n) \prod_{i \in B_{n+1}} (1 - z^*_i(h, y_n))
\]

\[= \prod_{i \in A_{n+1}} z_i(h) \prod_{i \in B_{n+1}} (1 - z_i(h)) \tag{28}\]

\[= P^\sigma(y_{n+1} \mid h). \]

Using (26), (28) and (27), and substituting the definitions of \( x^*_i(h) \) and \( x^*_i(h, y_n) \), we can rewrite (25) as:

\[
\pi^*_n(h) = \pi_n(h) + \sum_{y_{n+1} \in Y(J)} P^\sigma(y_{n+1} \mid h) \cdot \sum_{i \in A_{n+1}} \left[ \sum_{y_n \in Y\{m\}} P^\sigma(y_n \mid h) \left\{ \sum_{j \in A_n} c_{ij} \kappa_j(h) + \sum_{j \in B_n} c_{ij} \lambda_j(h) \right\} \right], \tag{29}
\]

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where the order of the summations in the second term is reversed since their ranges are independent of each other. Since \( Y(\{m\}) = \{\emptyset, \{m\}, \{\{m\}, \emptyset\} \}, \) the quantity in the square brackets on the right-hand side of (29) equals

\[
\sum_{y_n \in Y(\{m\})} P(y_n \mid h) \left\{ \sum_{j \in A_n} c_{ij} \kappa_j(h) + \sum_{j \in B_n} c_{ij} \lambda_j(h) \right\}
\]

\[
= c_{im} \left\{ z_m(h) \kappa_m(h) + (1 - z_m(h)) \lambda_m(h) \right\}
\]

\[= 0. \]

This completes the proof of the theorem. 

**Proof of Lemma 5.1** Define \( \sigma^* = (r^*, x^*) \) as follows:

\[
r^*(h) = \begin{cases} 
  i & \text{if } h = \underline{h}, \\
  j & \text{if } h = (\underline{h}, 0) \text{ or } (\underline{h}, 1), \\
  r(h) & \text{otherwise},
\end{cases}
\]

\[
x^*_i(h) = \begin{cases} 
  x_j(h) - (c_j - c_i) \alpha^*(h) & \text{if } h = \underline{h}, \\
  x_i(h) & \text{otherwise},
\end{cases}
\]

\[
x^*_j(h) = \begin{cases} 
  x_i(h, 0) + (c_j - c_i) \{ \alpha^*(h) + \lambda_j(h) \} & \text{if } h = (\underline{h}, 0), \\
  x_i(h, 1) + (c_j - c_i) \{ \alpha^*(h) + \kappa_j(h) \} & \text{if } h = (\underline{h}, 1), \\
  x_j(h) & \text{otherwise},
\end{cases}
\]

and \( x^*_k(h) = x_k(h) \) for any \( h \in H \) and \( k \neq i, j \). Just as in the proof of Theorem 4.1, for any history \( h \in H \), denote

\[
V^*_i(s_i \mid h) = V^*_{i\sigma^*}(s_i \mid h), \quad \text{and} \quad z^*_i(h) = z^*_{i\sigma^*}(h).
\]

It is clear that \( r^* \) is non-contingent after \( \underline{h} \). It can also be verified that \( z^*_i(h) = z_j(h) \) since \( \alpha^*(h) = \alpha^*_{\sigma^*}(h) \) and hence

\[
z^*_i(h) = P \left( V^*_i(\bar{s}_i \mid h) \geq x^*_i(h) \right)
\]

\[
= P \left( c_0 + \bar{s}_i + c_i \alpha^*_{\sigma^*}(h) \geq x_j(h) - (c_j - c_i) \alpha^*(h) \right)
\]

\[
= P \left( c_0 + \bar{s}_i + c_j \alpha^*(h) \geq x_j(h) \right)
\]

\[
= P \left( V_j(\bar{s}_j \mid h) \geq x_j(h) \right)
\]

\[
= z_j(h).
\]
Since $\lambda_i(h) = \lambda_j(h)$ and $\kappa_i(h) = \kappa_j(h)$, it can also be verified that $z^*_i(h, 0) = z_i(h, 0)$ and $z^*_j(h, 1) = z_i(h, 1)$. Lemma 3.1 then implies that for any $k \neq i, j$ and any sequence of outcomes $(y_n, y_{n+1})$ in periods $n$ and $n+1$, $V_k^\ast(\cdot \mid h, y_n, y_{n+1}) = V_k(\cdot \mid h, y_n, y_{n+1})$. It follows from this and $x_k^\ast(h) = x_k(h)$ for any $k \neq i, j$ and $h \in H$ that for any $k \neq i, j$ and any sequence of outcomes $(y_n, \ldots, y_{t-1})$ in periods $n, \ldots, t-1$,

$$z_k^\ast(h, y_n, \ldots, y_{t-1}) = z_k(h, y_n, \ldots, y_{t-1}), \quad (31)$$

and

$$V_k^\ast(\cdot \mid h, y_n, \ldots, y_{t-1}) = V_k(\cdot \mid h, y_n, \ldots, y_{t-1}), \quad (32)$$

(31) in particular shows that $\sigma^\ast \in \Sigma^0$.

Now for any history $h \in H_{t-1}$ and $t \in I$, let $\pi_i(h)$ denote the seller’s expected revenue over periods $t, \ldots, I$ at history $h$ under $\sigma$. Likewise, let $\pi^\ast_i(h)$ denote his expected revenue over periods $t, \ldots, I$ at history $h$ under $\sigma^\ast$. By (32), $\pi^\ast_{n+2}(h, y_n, y_{n+1}) = \pi_{n+2}(h, y_n, y_{n+1})$ for any sequence of outcomes $(y_n, y_{n+1})$ in periods $n$ and $n+1$. It hence follows from the definition of $\sigma^\ast$ and the above observation that

$$\pi^\ast_n(h) = z^\ast_i(h) x^\ast_i(h)$$

$$+ z^\ast_i(h) \left\{ x^\ast_j(h, 1) \left[ x^\ast_j(h, 1) + \pi_{n+2}(h, 1, 1) \right] + (1 - z^\ast_j(h, 1)) \pi_{n+2}(h, 1, 0) \right\}$$

$$+ (1 - z^\ast_i(h)) \left\{ z^\ast_j(h, 0) \left[ x^\ast_j(h, 0) + \pi_{n+2}(h, 0, 1) \right] + (1 - z^\ast_j(h, 0)) \pi_{n+2}(h, 0, 0) \right\}$$

$$= \pi_n(h) + (c_j - c_i) \left\{ z_j(h) z_i(h, 1) + (1 - z_j(h)) z_i(h, 0) - z_j(h) \right\}$$

$$+ \left\{ z_i(h, 1) z_j(h) \kappa^\ast_j(h) + z_i(h, 0) \left( 1 - z_j(h) \right) \lambda^\ast_j(h) \right\}. \quad (33)$$

By the monotonicity condition (5), the quantity in the square brackets on the far right-hand side of (33) is $\geq 0$. We hence obtain the desired conclusion that $\pi(\sigma^\ast) \geq \pi(\sigma)$.

**Proof of Theorem 5.2** Let $\sigma \in \Sigma^0$ be an arbitrary scheme that is optimal within $\Sigma^0$. For any $h \in H_{t-2}$, since $U(h, 0) = U(h, 1) = \{ i \}$ for some $i \in I$, we must have $r(h, 0) = r(h, 1) = i$. If $r(h) = j > i$, then Lemma 5.1 implies that $\sigma$ is weakly dominated by an alternative scheme in $\sigma^0$ which, after history $h$, trades with buyer

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Proof of Theorem 7.1 Suppose for simplicity that \( n = I \) and that \( \rho_t = t \) for every \( t \in I \). Note that \( g(z) = z \left( \frac{s}{z} + \Delta(1 - z) \right) \) for the given uniform distribution \( F \).

Since

\[
\frac{\partial \pi_l}{\partial z_l}(z_l, \alpha_{l-1}) = \hat{s} - 2\Delta z_l + c_l \alpha_{l-1} + c_0
\]

\( i \) in period \( I - 1 \) and buyer \( j \) in period \( I \). Therefore, we conclude that there exists a revenue maximizing scheme \( \sigma \in \Sigma^0 \) that satisfies for any \( h \in H_{I-2} \), if \( U(h) = \{i_1, i_2\} \) for some \( i_1 < i_2 \), then

\[
 r(h) = i_1 \quad \text{and} \quad r(h, 0) = r(h, 1) = i_2.
\]

As an induction hypothesis, given \( t \) \((2 \leq t \leq I - 1)\), suppose that there exists a revenue maximizing scheme \( \sigma \in \Sigma^0 \) that satisfies for any \( h \in H_{I-t} \), if \( U(h) = \{i_1, \ldots, i_t\} \) for \( i_1 < \cdots < i_t \), then

\[
 r(h) = i_1, r(h, 0) = r(h, 1) = i_2,
\]

\[
 \ldots, r(h,0,\ldots,0) = \cdots = r(h,1,\ldots,1) = i_t. \tag{34}
\]

Take any \( h \in H_{I-t-1} \). Since \( U(h,0) = U(h,1) = \{i_1, \ldots, i_t\} \) for some \( i_1 < \cdots < i_t \), it follows from the induction hypothesis that

\[
 r(h,0) = r(h,0) = i_1, r(h,0,0) = \cdots = r(h,1,1) = i_2,
\]

\[
 \ldots, r(h,0,\ldots,0) = \cdots = r(h,1,\ldots,1) = i_t. \tag{35}
\]

Hence, if \( r(h) = j \) for some \( j > i_1 \), then Lemma 5.1 implies that \( \sigma \) is weakly dominated by an alternative scheme in \( \Sigma^0 \) which, after history \( h \), trades with buyer \( i_1 \) in period \( t - 1 \) and buyer \( j \) in period \( t \). If \( j > i_2 \), then the latter scheme is further dominated by a scheme that offers buyer \( i_2 \) in period \( t \) and buyer \( j \) in period \( t + 1 \). Repeating this argument, we can conclude that there exists a revenue maximizing scheme \( \sigma \in \Sigma^0 \) that satisfies for any \( h \in H_{I-t-1} \), if \( U(h) = \{i_1, \ldots, i_{t+1}\} \) for some \( i_1 < \cdots < i_{t+1} \), then

\[
 r(h) = i_1, r(h,0) = r(h,1) = i_2,
\]

\[
 \ldots, r(h,0,\ldots,0) = \cdots = r(h,1,\ldots,1) = i_{t+1}. \tag{36}
\]

Therefore, we have advanced the induction step and established that among the optimal schemes within \( \Sigma^0 \), there exists a non-contingent scheme \( \sigma \in \Sigma^0 \) which trades with buyer \( t \) in period \( t \).
is decreasing in $z_I$, the first-order condition yields the optimal solution $z_I(\alpha_{I-1}) = \frac{1}{2\Delta}(s + c_0 + c_I\alpha_{I-1})$. The envelope theorem also implies that
\[
\frac{\partial \pi^*_I}{\partial \alpha_{I-1}}(\alpha_{I-1}) = c_I z_I(\alpha_{I-1}).
\]
As an induction hypothesis, suppose now that (11) holds for $i+1, \ldots, I$ ($i \leq I - 1$) and that
\[
\frac{\partial \pi^*_i}{\partial \alpha_i}(\alpha_i) = \sum_{j=i+1}^I c_j z_j(\alpha_i).
\]
The expected revenue function for periods $i, \ldots, I$ can be written as
\[
\pi_i(z_i, \alpha_{i-1}) = g(z_i) + z_i c_i \alpha_{i-1} + c_0 z_i + f_{i+1}(z_i, \alpha_{i-1}),
\]
where
\[
f_{i+1}(z_i, \alpha_{i-1}) = z_i \pi^*_i + (\alpha_{i-1} + \kappa(z_i)) + (1 - z_i) \pi^*_i (\alpha_{i-1} + \lambda(z_i)).
\]
It follows that
\[
\frac{\partial \pi_i}{\partial z_i}(z_i, \alpha_{i-1}) = s - 2\Delta z_i + c_0 + c_i \alpha_{i-1}
\]
\[+ \sum_{j=i+1}^I c_j \int_{\alpha_{i-1} + \kappa(z_i)}^{\alpha_i + \kappa(z_i)} z_j(\alpha_i) \, d\alpha_i
\]
\[- \frac{\Delta}{2} \sum_{j=i+1}^I c_j \left\{ z_i z_j(\alpha_{i-1} + \kappa(z_i)) + (1 - z_i) z_j(\alpha_{i-1} + \lambda(z_i)) \right\}
\]
\[= s - 2\Delta z_i + c_0 + c_i \alpha_{i-1}
\]
\[+ \sum_{j=i+1}^I c_j \int_{\alpha_{i-1} + \kappa(z_i)}^{\alpha_i + \kappa(z_i)} z_j(\alpha_i) \, d\alpha_i - \frac{\Delta}{2} \sum_{j=i+1}^I c_j z_j(\alpha_{i-1})
\]
\[+ \sum_{j=i+1}^I c_j \left\{ \int_{\alpha_{i-1} + \lambda(z_i)}^{\alpha_i + \lambda(z_i)} z_j(\alpha_i) \, d\alpha_i - \frac{\Delta}{2} z_j(\alpha_{i-1}) \right\}
\]
\[= s - 2\Delta z_i + c_0 + c_i \alpha_{i-1} + \sum_{j=i+1}^I \frac{\Delta}{16} b_j c_j (1 - 2z_i),
\]
where the second equality follows since \( z_j \) (\( j = i + 1, \ldots, I \)) is by the induction hypothesis an affine function and since \( z_i \kappa(z_i) + (1 - z_i) \lambda(z_i) = 0: \\
\begin{align*}
z_i z_j (\alpha_{i-1} + \kappa(z_i)) + (1 - z_i) z_j (\alpha_{i-1} + \lambda(z_i)) \\
= z_j \left( z_i (\alpha_{i-1} + \kappa(z_i)) + (1 - z_i) (\alpha_{i-1} + \lambda(z_i)) \right) \\
= z_j (\alpha_{i-1} ).
\end{align*}

Since \( \frac{\partial \pi_i}{\partial z_i} \) is decreasing in \( z_i \), the first-order condition yields the optimal solution \\
\begin{align*}
z_i (\alpha_{i-1}) &= \frac{1}{2\Delta} \frac{\tilde{s} + c_0 + c_i \alpha_{i-1} + \sum_{j=i+1}^{I} \Delta b_j c_j / 16}{1 + \sum_{j=i+1}^{I} b_j c_j / 16} = \frac{1}{2\Delta} (a_i + b_i \alpha_{i-1}).
\end{align*}

Furthermore, using (38) again, we see that \\
\begin{align*}
\frac{\partial f_{i+1}}{\partial \alpha_{i-1}} (z_i, \alpha_{i-1}) &= z_i \frac{\partial \pi^*_{i+1}}{\partial \alpha_i} (\alpha_{i-1} + \kappa(z_i)) + (1 - z_i) \frac{\partial \pi^*_{i+1}}{\partial \alpha_i} (\alpha_{i-1} + \lambda(z_i)) \\
= \sum_{j=i+1}^{I} c_j \left( z_i z_j (\alpha_{i-1} + \kappa(z_i)) + (1 - z_i) z_j (\alpha_{i-1} + \lambda(z_i)) \right) \\
= \sum_{j=i+1}^{I} c_j z_j (\alpha_{i-1} + z_i \kappa(z_i) + (1 - z_i) \lambda(z_i)) \\
= \frac{\partial \pi^*_{i+1}}{\partial \alpha_i} (\alpha_{i-1}).
\end{align*}

Hence, the envelope theorem implies that \\
\begin{align*}
\frac{\partial \pi^*_{i}}{\partial \alpha_{i-1}} (\alpha_{i-1}) &= c_i z_i (\alpha_{i-1}) + \frac{\partial \pi^*_{i+1}}{\partial \alpha_i} (\alpha_{i-1}) = \sum_{j=i}^{I} c_j z_j (\alpha_{j-1}).
\end{align*}

This advances the induction step and completes the proof. 

**Proof of Lemma 7.2** Since the left-hand side of (13) is quadratic in \( \alpha \), the inequality holds for any \( \alpha \in C_{t-1} \) if either \\
1) \( 16(b_{t+1} - b_t) - b_t^2 b_{t+1} > 0 \), and \\
\begin{align*}
\{8(a_{t+1} - a_t) + b_t b_{t+1} (\Delta - a_t)\}^2 &\leq a_t b_{t+1} (2\Delta - a_t)\{16(b_{t+1} - b_t) - b_t^2 b_{t+1}\},
\end{align*}

or 

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2) \(16(b_{t+1} - b_t) - b_t^2b_{t+1} \leq 0\), and

\[
\{16(b_{t+1} - b_t) - b_t^2b_{t+1}\} \frac{1}{4} (t-1)^2 + \frac{1}{\Delta^2} a_t(2\Delta - a_t) b_{t+1} \\
\geq \frac{1}{\Delta} (t-1) \left| 8(a_{t+1} - a_t) + b_t b_{t+1}(\Delta - a_t) \right|.
\]  

(41)

Suppose now that \(c_0 + \bar{s} = 0\). In this case, we have \(a_t = \Delta\) for any \(t\) by definition. Hence, if \(16(b_{t+1} - b_t) - b_t^2b_{t+1} > 0\), then (40) holds, and otherwise, (41) holds under (14). Suppose next that we let \(\Delta \to \infty\) while keeping fixed \(\bar{s}, c_0, c_1, \ldots , c_I\). In this case, \(\frac{a_t}{\Delta} \to 1\) so that the left-hand side of (40) is held constant while the right-hand side \(\to \infty\) if \(16(b_{t+1} - b_t) - b_t^2b_{t+1} > 0\). Furthermore, the left-hand side of (41) converges to a strictly positive constant under (14) while the right-hand side \(\to 0\). It follows that under (14), either (40) or (41) holds asymptotically. 

\[\square\]

**Proof of Proposition 7.4** For \(t = 1\), we have

\[
\bar{\alpha}_1 = \kappa(z_1(\bar{\alpha}_0)) = \frac{\Delta}{2} \left( 1 - \frac{a_1}{2\Delta} \right) = \frac{\Delta - \bar{s} - c_0}{4} + o(1).
\]

As an induction hypothesis, suppose that for \(t = 2, \ldots , I - 1\),

\[
\bar{\alpha}_{t-1} = \frac{t-1}{4} (\Delta - \bar{s} - c_0) + o(1).
\]

Then

\[
\bar{\alpha}_t = \bar{\alpha}_{t-1} + \kappa(z_t(\bar{\alpha}_{t-1})) \\
= \bar{\alpha}_{t-1} + \frac{\Delta}{2} \left\{ 1 - \frac{1}{2\Delta} (a_t + b_t \bar{\alpha}_{t-1}) \right\} \\
= \bar{\alpha}_{t-1} + \frac{1}{4} \left\{ \Delta - \bar{s} - c_0 - \bar{\alpha}_{t-1} c_{\rho_t} + o(\varepsilon) \right\} \\
= \frac{t}{4} (\Delta - \bar{s} - c_0) + o(1),
\]

and hence the induction step is advanced. The proof for \(\alpha_t\) is similar. It follows that when the buyers \(\rho_1, \ldots , \rho_{t-1}\) have all accepted, the probability that buyer \(\rho_t\) also accepts is given by

\[
z_t(\bar{\alpha}_{t-1}) = \frac{1}{2\Delta} (a_t + b_t \bar{\alpha}_{t-1}) \\
= \frac{1}{2\Delta} \left[ \Delta + \bar{s} + c_0 + o(\varepsilon) + \frac{c_{\rho_t}}{4} (t-1)(\Delta - \bar{s} - c_0) + o(\varepsilon) \right] \\
= \frac{\Delta + \bar{s} + c_0}{2\Delta} \left[ 1 + \frac{\Delta - \bar{s} - c_0}{\Delta + \bar{s} + c_0} \frac{c_{\rho_t}}{4} (t-1) \right] + o(\varepsilon).
\]

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Hence, the probability that every buyer accepts can be computed as

\[
\prod_{t=1}^{l} z_t(\bar{\alpha}_{t-1}) = \left(\frac{\Delta + \bar{s} + c_0}{2\Delta}\right)^l \prod_{t=1}^{l} \left\{1 + \frac{\Delta - \bar{s} - c_0}{\Delta + \bar{s} + c_0} \frac{c_{pt}}{4} (t-1)\right\} + o(\varepsilon)
\]

\[
= \left(\frac{\Delta + \bar{s} + c_0}{2\Delta}\right)^l \left\{1 + \frac{\Delta - \bar{s} - c_0}{\Delta + \bar{s} + c_0} \sum_{t=1}^{l} \frac{c_{pt}}{4} (t-1)\right\} + o(\varepsilon).
\]

The probability that every buyer rejects can be computed in a similar manner. ■

References


