Contingent Prices, Money and Inflation*

Richard Dutu‡  Stella Huangfu‡  Benoit Julien§
Deakin University  University of Sydney  University of New South Wales

December 4, 2009

Abstract

We construct a monetary economy where search is directed yet terms of trade are determined ex-post. Instead of using auctions, sellers compete for buyers via schedules of prices, i.e. sellers post prices that are contingent on realized demand. For instance a seller might advertise a higher price should more than one buyer desire the good. We use the model to examine the role played by money and inflation in markets where terms of trade are determined ex post. In contrast with the existing literature, we show that there exists a unique equilibrium rather than a continuum, and highlight the role of money in bringing uniqueness. In this equilibrium sellers post different prices for different demand realizations. In the resulting subgames buyers randomize over which seller to visit and, as long as the cost of money is not zero, over how much money to bring to the seller. While contingent-price posting shares similarities with auctions, we show that they differ in one key dimension: with contingent prices all prices are set strategically whereas in auctions only the reserve price is. We describe how shifts in liquidity costs for buyers–via higher anticipated inflation–impact the posted schedules, and show that small deviations from the Friedman rule enable sellers to raise their reserve price in real terms despite the higher inflation tax.

Keywords: Money, Directed Search, Contingent Prices, Inflation.

JEL Classification: C78; D40; E40

---

*We thank Nejat Anbarci, Ian King, Sephorah Mangin, Chin-Jeng Sun, Serene Tan, Lawrence Uren and Makoto Watanabe for very interesting conversations. We also thank seminar participants at Deakin University, the University of Melbourne and the Chicago FED Summer Workshop on Money, Banking and Payments. All errors are ours.

†Deakin Business School, Deakin University, 221 Burwood Highway, Burwood Victoria 3125, Melbourne, Australia. Tel: +61 3 9244 5134. Email: rdutu@deakin.edu.au

‡Department of Economics, University of Sydney, Australia. Tel: +61 2 9036 9311. Email: s.huangfu@econ.usyd.edu.au

§Australian School of Business, School of Economics, University of New South Wales, Australia. Tel: +61 2 9385 3876. Email: benoit.julien@unsw.edu.au
1 Introduction

We construct a monetary economy where search is directed yet terms of trade are determined ex-post. Instead of using auctions, sellers compete for buyers via schedules of prices, i.e. posting prices that are contingent on realized demand. For instance a seller might advertise a higher price should more than one buyer desire the good. We use the model to examine the role played by money and inflation in markets where terms of trade are determined by realized demand.

Our first step is to characterize the equilibrium of this economy, if any. We show that there exists a unique equilibrium. In this equilibrium sellers post different prices for different demand realizations. In the resulting subgames buyers randomize over which seller to visit and, as long as the cost of money is not zero, over how much money to bring to the seller. This result comes in sharp contrast with Coles and Eeckhout (2003) indeterminacy of equilibrium.

In their model, which is essentially the non-monetary version of our model, each seller posts a price pair \((p_1, p_2)\) where \(p_1\) is the price charged if only one buyer shows up (who gets the good with certainty), while \(p_2\) is the price charged if two or more buyers visit (and the good is then randomly allocated to one of the buyers). They show that in equilibrium sellers post a unique price for when only one buyer arrives (the pairwise price \(p_1\)), but the price that applies if both buyer visit (the multilateral price \(p_2\)) can be anywhere between the seller's reservation value and the buyer's reservation value, hence the indeterminacy.

In our model both \(p_1\) and \(p_2\) are uniquely determined. The reason why money brings down the indeterminacy comes from the internalization by sellers of buyers' liquidity costs. In the non-monetary version, advertising more buyer surplus in some states and offering less in others, that is posting \(p_2 \neq p_1\), can leave expected buyer and seller surplus unchanged. Those changes, however, change the demand elasticities of buyers and therefore the best-response correspondence of competing sellers, hence the continuum of equilibria. If now buyers are asked to choose their money holding but money is free for some reason, buyers should hold \(p_2\) and the indeterminacy goes through since money holding decisions do not impact on payoffs. But if holding money is costly, because of inflation for instance, buyers weight the cost of holding more money against the augmented trading opportunities that it offers. If \(p_2\) is only marginally higher than \(p_1\), then holding \(p_2\) is a better option as the marginal return (being able to trade in multilateral meeting) clearly outweighs the negligible marginal cost. By contrast, if \(p_2\) is much higher than \(p_1\), close to the buyer's reservation value, then the marginal cost outweighs the marginal benefit. Because buyer's expected net surplus decreases monotonically as \(p_2\) rises above \(p_1\), there exists only one multilateral price \(p_2\) within the former continuum that leaves buyers indifferent between holding \(p_2\) or \(p_1\).

The model can be used to highlight the role of money in markets where terms of trade are
determined ex post. In any protocol where money is essential for trade, buyers are to play a subgame in which they must choose their money holding, balancing the costs and benefits of the additional dollar, and taking other buyers’ choices as given. Whether search is directed or not, when the price is unique there is no proper subgame and all buyers simply bring the monetary equivalent of the posted or expected price. This is the case in Rocheteau and Wright (2005)’s fixed-price posting economy. This is also true the random matching and bargaining economy of Lagos and Wright (2005) since buyers are able to anticipate the solution to the bargaining problem. When prices are contingent on realized demand, however, buyers are uncertain about which seller other buyers will visit, then which price will apply, and therefore randomize over the set of relevant money holdings. As a result homogenous buyers hold different amounts of money in equilibrium. A similar result emerges in monetary economies where sellers use auctions as in Galenianos and Kircher (2008), or compete in auctions as in Dutu, Julien and King (2009).

There are several differences with auctions though. A first difference is that in auctions only the reserve price is set strategically, all other prices being determined by the mechanism. Here all prices are linked to each other to maintain the ex-ante indifference condition for buyers between all possible prices. A second difference is that when a seller uses auctions instead of contingent prices, she does not have to figure out the price schedule that would both exploit competition among buyers yet preserve the mixed strategy played by buyers in the monetary subgame. This may explain why price schedules are not frequently observed, while auctions are quite common in many segments of the economy (industry, housing). Another difference, although it is not specific to the role played by money, is that the marginal buyer always influence the price here whereas this will be the case in auction only to the extent that the marginal buyer can afford the good.

Coming back to Coles and Eeckhout (2003)’s original question—why do sellers not try to take advantage of competition among buyers by using auctions instead fixed prices?—our results imply that contingent-price posting and fixed-price posting are not subcases of a more general posting mechanism. Actually, if sellers try to screen ex ante via prices on the basis of ex-post realized demand, there is just one price schedule that enables to do that. And the resulting mechanism produces different incentives for players and yields a different outcome than an auction.

In a final step we study how liquidity costs, via anticipated inflation, impacts on terms of trade. We have two results here. First we show that the uniqueness of equilibrium holds for the entire range of nominal interest rates compatible with existence of a monetary equilibrium (above a certain threshold buyers simply discard money). In particular, one may have expected the indeterminacy to come back at low nominal interest rates as the low cost of holding money
could make it profitable for buyers to always hold the high amount of money. It turns out sellers extract this low cost of liquidity by charging buyers a multilateral price that approaches the buyer’s reservation value as the nominal interest rate approaches zero, thereby supporting the uniqueness. Second, we show that for small deviations from the Friedman rule, sellers increase the pairwise price they charge in real terms, forcing buyers who pick the low money holding to bring more money. As inflation rises it becomes proportionally more costly to hold the large amount of money. Sellers extract this incentive to shift to the low money holding by raising the pairwise price. If one interprets the pairwise price as the reserve price this suggests that small increases in the inflation rate enable sellers to raise their reserve price. This results holds in a large economy as well where competition between sellers usually drives the reserve price down to the seller’s reservation value (McAfee, 1993; Peters and Severinov, 1997; Hernando-Veciana, 2005). As soon as the nominal interest rate is not zero, this result no longer holds. Strategic reserve price posting is different in monetary economy than it is in a non monetary economy. More generally, the main conclusion we derive from this study is that monetary models are not the same as non-monetary models.

The paper is organized as follows. Section 2 describes the players, timing, strategies and payoffs. Section 3 characterizes the unique symmetric equilibrium. Section 4 conducts the comparative statics exercise. Section 5 concludes.

2 The model

2.1 Players and Strategies

There are two sellers indexed by $y \in \{1, 2\}$ and two buyers indexed by $x \in \{1, 2\}$. They all meet first in a Walrasian market where they can produce, trade and consume any quantity of a first good called the general good. Then they enter a second market in which each seller is endowed with one indivisible unit of a second good, the search good, that buyers with identical preferences wish to buy. We call the first market the centralized Walrasian market (CWM) and the second the directed search market (DSM). Agents discount at rate $\beta$ between the CWM and the DSM but not between periods.

The sequence of events is as follows. First, the CWM opens and buyers receive a lump sum money injection from the central bank which is common knowledge. Then sellers simultaneously advertise terms of trade contingent on realized demand for the coming DSM. The terms of trade consist of one unit of the search good and a price schedule: $p_1$ is the price posted by seller 2 that

\footnote{Our results generalize to an $n$-buyer, $m$-seller economy. Our 2-by-2 economy highlights the interdependence between strategies and facilitates comparison with Coles and Eeckhout (2003). We provide results specific to the larger economy in section 3.4.}
applies in case only one buyer shows up (pairwise match), and \( p_2 \) is the price posted by seller 2 that applies if both buyers show up (multilateral match), respectively \( p'_1 \) and \( p'_2 \) for seller 1. Then buyers observe the advertisements and choose a seller to visit in the coming DSM. After choosing a seller to visit, buyers choose how much money to carry by producing and selling some general good in exchange for money on the CWM. Finally, the Walrasian market closes and the frictional market opens, buyers visit the seller of their choosing and trade takes place. The sequence of events is represented on Figure 1.

2.2 Money

To create a role for money in this economy, we make three assumptions. First, as in Rocheteau and Wright (2005), we assume that buyers cannot produce on the DSM so that they cannot compensate sellers by producing on the spot in exchange for the search good. Second, we assume that both goods perish at the end of their subperiod and therefore cannot be used as commodity money. Finally, the cost of setting up a public-record keeping system is extremely high, implying that each agent’s trading history is private information. These assumptions make money essential for trade on the DSM (Kocherlakota, 1998; Wallace, 2001).\(^2\)

\(^2\)Keeping track of each other trading history is probably not that difficult in a 2-by-2 economy and our last assumption may sound extreme in that regard. Correlated equilibria could also be an issue in our small economy since buyers may infer from the equilibrium price of money the other buyer’s choice of money holding. Our interest in analyzing a small economy is expositional as it enables to clearly spell out the interdependence in
Typically, buyers use the CWM to replenish their money holdings by selling some of their general good output to sellers. Symmetrically, because sellers do not need money in the DSM, sellers use the CWM to spend any balances accumulated in the previous DSM. To produce the general good buyers and sellers can exert effort $h$ to produce $h$ units of that good. We assume linearity in effort on the centralized Walrasian market as it erases the influence of trading histories on money demand (see Lagos and Wright, 2005). Note also that money holdings is buyers’ private information.

The price of money in terms of the general good on the CWM is denoted $\phi$, that is 1 unit of money buys $\phi$ units of this good. It adjusts every period to equate the supply of money from sellers to demand of money coming from buyers. The money supply grows at rate $\tau$ via lump sum transfers to buyers by the central bank at the beginning of each CWM. Inflation (noted $\pi$) is perfectly forecasted and both the quantity theory and the Fisher effect apply: if the money supply increases at rate $\tau$, so do prices $\pi = \tau$ and the nominal interest rate is then given by $i = r + \pi$ where $r$ is the real interest rate. Since $\beta = \frac{1}{1+\tau}$ the Fisher equation $(1 + i) = (1 + r)(1 + \pi)$ enables to write the nominal interest rate as $i = (1 - \beta + \tau)/\beta$.

2.3 The Payoffs

If a buyer pays $p_i$ for the good traded in DSM, he consumes it immediately, which provides him utility $Q - p_i$. The seller’s utility for consuming her own production good in this market is normalized to zero. On the CWM we denote $X$ consumption of the good and $U(X)$ the corresponding utility with $U' > 0$ and $U'' < 0$. The instantaneous utility function for a buyer is then given by

$$U(X) - h + \beta Q.$$  \hspace{1cm} (1)

Since sellers are endowed with the search good, their instantaneous utility function is simply

$$U(X) - h.$$  \hspace{1cm} (2)

Because decisions are related through time via the choice of money holdings, the model is one of dynamic optimization. We will focus on stationary allocations, however, where aggregate real variables are constant, including the aggregate real money supply. This implies $\phi_{t+1}M_{t+1} = (1 + \tau)\phi_tM_t$ and therefore $\phi_{t+1} = \frac{\phi_t}{1+\tau}$. From now on we will suppress time subscript and use subscript $+1$ to denote the value of a variable in the next period.

Let $V$ denote the present value of the infinite sequence of payoffs an agent receives from making his choices optimally in the present DSM and in the future. Similarly, let $W$ denote agents’ strategies, and also facilitates comparison with Coles and Eeckhout (2003). In a larger economy, such as the one studied in Appendix A4, none of these problems appear.
the present value of the infinite sequence of payoffs an agent receives from making his choices optimally in the present CWM and in the future. That is $V$ and $W$ are value functions. Let $V^{bx}(m)$ be the value function for buyer $x$ holding $m$ units of money when entering the DSM and $W^{bx}(m)$ the value function when entering the CWM. We have

$$W^{bx}(m) = \max_{X,h,\hat{m}} \left\{ U(X) - h + \beta V^{bx}(\hat{m}) \right\}$$

(3)

s.t. $\phi \hat{m} + X = h + \phi (m + T).$

(4)

A buyer chooses how much of the general good to consume, $X$, how much effort to exert, $h$, and how much money to bring to the DSM, $\hat{m}$. The budget constraint equalizes resources, $h+\phi (m+T)$, to demand, $\phi \hat{m} + X$, where $T$ is how many units of money per buyer are injected by the central bank at the beginning of the CWM. Substituting for $h$, the program for a buyer in the CWM can be rewritten

$$W^{bx}(m) = U(X^*) - X^* + \phi (m + T) + \max_{\hat{m}} \left\{ -\phi \hat{m} + \beta V^{bx}(\hat{m}) \right\}$$

(5)

where $X^*$ is such that $U'(X^*) = 1$.

Since sellers have no reason to carry money into the DSM and only buyers receive a lump sum money transfer, the program for seller 2 is

$$W^{s2}(m) = \max_{X,h,p_1,p_2} \left\{ U(X) - h + \beta V^{s2} \right\}$$

(6)

s.t. $X = h + \phi m$

(7)

Seller 2’s problem is to choose consumption in the Walrasian market $X$, effort $h$ and a pair of prices $(p_1,p_2)$ for the DSM. This implies

$$W^{s1}(m) = U(X^*) - X^* + \phi m + \max_{p_1,p_2} \beta V^{s1}$$

(8)

$$W^{s2}(m) = U(X^*) - X^* + \phi m + \max_{p_1,p_2} \beta V^{s2}.$$ 

(9)

3 The Equilibrium

We define a monetary strategy as the probability with which a buyer chooses to carry a particular amount of money. Notice that since each seller advertises only two prices, $(p'_1,p'_2)$ for seller 1 and $(p_1,p_2)$ for seller 2, if a buyer decides to visit seller 1 for instance, then bringing an amount of money equal to either $p'_1$ or $p'_2$ strictly dominates any other amount. To see this assume buyer 1 visits seller 1 and brings $p_1$. If he is alone, either $p_1 > p'_1$ in which case he brought too much money, or $p_1 < p'_1$ in which case he did not bring enough. The same is true if the
other buyer is there: either \( p_1 > p'_2 \) in which case he brought too much money or \( p_1 < p'_2 \) in which case he did not bring enough. As a result, given there can be four prices at most and idle balances are costly, money holdings will be of four types: \( p_1, p_2, p'_1 \) or \( p'_2 \).

Correspondingly, we define \( \theta'_x \) as the probability with which buyer \( x \in \{1, 2\} \) chooses to hold \( p'_1 \) *given* that he has decided to visit seller 1. Then \( 1 - \theta'_x \) is the probability that buyer \( x \) chooses to hold \( p'_2 \) given that he has decided to visit seller 1. Similarly, \( \theta_x \) denotes the probability that buyer \( x \) chooses to hold \( p_1 \) given that he visits seller 2 and \( 1 - \theta_x \) the probability that buyer \( x \) chooses to hold \( p_2 \) given that he visits seller 2.

We define a *visit strategy* for buyer \( x \) as the probability with which a buyer chooses to visit a particular buyer. We denote by \( \sigma_x \) the probability that buyer \( x \in \{1, 2\} \) chooses to visit seller 1. With probability \( 1 - \sigma_x \) he chooses to visit seller 2.

Finally, we define a *pricing strategy* for seller 1 as posting a price pair \((p'_1, p'_2)\), and a pricing strategy for seller 2 as posting a price pair \((p_1, p_2)\). We restrict our attention to sellers using pure pricing strategies and strictly positive prices. By contrast to buyers, since sellers move first, a seller’s strategy space is equivalent to his action space.

The resulting game is a three-stage dynamic game of complete but imperfect information due to the simultaneity of moves for sellers and buyers. The Nash equilibrium is solved by backward induction. First, taking prices and visit strategies as given, we solve for the equilibrium of the monetary subgame in which buyers choose their money holdings simultaneously. Second, given the equilibrium strategies in the monetary subgame and the prices posted by sellers, we solve for the equilibrium of the visit subgame in which buyers choose a seller simultaneously. Finally, given the equilibrium visit and monetary strategies in the two successive subgames, we solve for the Nash equilibrium in the price posting game between the two sellers. As for the price of money on the CWM, \( \phi \), it equalizes money supply and money demand consistent with equilibrium strategies.

**Definition 1** An equilibrium is a list of value functions \((W^{b1}, W^{b2}, W^{s1}, W^{s2}, V^{b1}, V^{b2}, V^{s1}, V^{s2})\), a list of prices \((p_1, p_2, p'_1, p'_2)\) and a list of visit and monetary strategies \((\sigma_1, \sigma_2, \theta'_1, \theta'_2, \theta_1, \theta_2)\) such that:

1. **Best response by buyers:** (i) given \((p_1, p_2, p'_1, p'_2)\) and \((\sigma_1, \sigma_2)\), then \( \theta'_x \) and \( \theta_x \) describe the Nash equilibrium in the monetary subgame for each buyer \( x \); (ii) given \((p_1, p_2, p'_1, p'_2)\) and \((\theta'_1, \theta'_2, \theta_1, \theta_2)\), then \( \sigma_x \) describe the Nash equilibrium in the visit subgame for each buyer \( x \);
2. **Best response by sellers:** given the subgame visit and monetary strategies \((\sigma_1, \theta'_1, \theta_1)\) and \((\sigma_2, \theta'_2, \theta_2)\), then \((p_1, p_2)\) and \((p'_1, p'_2)\) describe a Nash equilibrium in pricing strategies for the two sellers;
3. **Individual Rationality:** \(-\phi p_j + \beta V^{bi}(p_j) \geq 0\) and \(-\phi p'_j + \beta V^{bi}(p'_j) \geq 0\), \( \forall j = 1, 2, \ldots \)
\( \forall i = 1, 2; \)

4. **Money market clearing and stationarity:** \( M^D = M^S \) and \( \phi = (1 + \tau) \phi_{+1} \).

The inequalities in point 3 of the definition guarantee that buyers’ expected return from holding money is positive. Note that there also exists a non-monetary equilibrium in which money is not valued and no economic activity takes place in the DSM. From now on we will focus on the monetary equilibrium. We start with the case where \( p_1 < p_2 \) and show in the appendix that no equilibrium exists when \( p_1 > p_2 \).

### 3.1 Monetary strategies

We consider a buyer’s choice of money holding taking both the prices posted by sellers and buyers’ visit strategies as given. Assume buyer 1 decides to visit seller 1 who posts \((p'_1, p'_2)\). Holding a money level of \( p'_1 \), buyer 1’s payoff is \(-\phi p'_1 + \beta V^{b_1}(p'_1)\) with

\[
V^{b_1}(p'_1) = (1 - \sigma_2) \left[ Q + W_{+1}^{b_1} (p'_1 - p'_1) \right] + \sigma_2 W_{+1}^{b_1} (p'_1). 
\]

If buyer 1 holding \( p'_1 \) is alone, which happens with probability \( 1 - \sigma_2 \), he purchases the good, enjoys utility \( Q \) and proceeds to the centralized market with no money. If he is not alone, he cannot purchase the good since he does not have enough money.

Holding \( p'_2 \) his payoff is \(-\phi p'_2 + \beta V^{b_1}(p'_2)\) with

\[
V^{b_1}(p'_2) = \sigma_2 \left( \frac{\theta'_2}{2} - \frac{\theta'_2}{2^2} \right) \left[ Q + W_{+1}^{b_1} (p'_2 - p'_2) \right] + (1 - \sigma_2) \left[ Q + W_{+1}^{b_1} (p'_2 - p'_1) \right] 
\]

\[
+ \sigma_2 \left[ 1 - \frac{\theta'_2}{2} W_{+1}^{b_1} (p'_2) \right]. 
\]

With probability \( \sigma_2 \) buyer 1 is not alone, in which case he purchases and consumes the good if buyer 2 holds the low amount of money, which happens with probability \( \theta'_2 \), or if buyer 2 holds the high amount of money yet buyer 1 wins the draw, which happens with probability \( \frac{1-\theta'_2}{2} \). Note that if two buyers holding the low amount of money show up at the same seller, no trade takes place as none of the buyers is able to pay the posted price.\(^3\) If buyer 1 is alone, which happens with probability \( 1 - \sigma_2 \), he consumes the good but pays only \( p'_1 \). Finally, with probability \( \sigma_2 \frac{1-\theta'_2}{2} \) buyer 2 is there too holding the high amount of money, but buyer 1 does not win the draw and then proceeds to the centralized market with the same amount of money as when he entered the DSM.

\(^3\)It is a central feature of directed search that sellers are somehow committed to the posted terms of trade, even though in this case there is an incentive for sellers to allocate the good randomly between the two buyers for the low amount of money. The same is true in directed search with fixed prices where there is an incentive for a seller facing two or more buyers to switch to an auction.
Lemma 1 Assume a buyer decides to visit seller $y \in \{1, 2\}$. Given the price pair posted by seller $y$ and the visit and monetary strategies played by the other buyer, there exists a unique mixed-strategy Nash equilibrium in the monetary subgame.

Proof. See Appendix. ■

A buyer is uncertain about which seller the other buyer decides to visit and how much money he holds. As a result, once a visit decision is made, a buyer randomizes between the low and the high amount of money.

3.2 Visit strategies

Given the unique subgame outcome in the money holding subgame for given $(p_1, p_2, p'_1, p'_2)$ and $(\sigma_1, \sigma_2)$, we now consider buyers’ best response in the visit subgame.

Lemma 2 There exists a unique symmetric mixed visit strategy for buyers given by

$$
\sigma = \sigma(p'_1, p_1) = \frac{i\phi_{+1}(p_1 - p'_1) + Q - \phi_{+1}p'_1}{(Q - \phi_{+1}p'_1) + (Q - \phi_{+1}p_1)}
$$

(12)

Proof. See Appendix. ■

When deciding which seller to visit, a buyer internalizes the other buyer’s indifference between the low and the high money holding (Lemma 2). This has two effects: first, neither $p_2$ nor $p'_2$ appear as an argument for $\sigma$, and second, buyer’s best response depends only on the visit strategy of the other buyer and not on monetary strategies. In absence of any coordination between buyers, there exists a unique mixed strategy equilibrium in the visit subgame as well: buyers randomize between the two sellers. As for the opportunity cost of money (measured by $i$) it can be seen from (12) that it influences buyers’ visit strategy only if sellers post different pairs of prices. But if they post identical price pairs, then $\sigma = 1/2$.

Combining Lemmas 1 and 2, we show that if buyers play the same mixed visit strategy, they also play the same mixed monetary strategies.

Lemma 3 Let us note $i^*$ and $i'$ two lower bounds on the nominal interest rate, and $\bar{i}$ and $\bar{i}'$ two upper bounds. Given $\sigma$, $p'_1 < p'_2$ and $p_1 < p_2$, the best-responses by buyers in the two monetary subgames are unique and defined as:

$$
\theta'_1 = \theta'_2 = \theta = \frac{2i\phi_{+1}(p'_2 - p'_1)}{\sigma(Q - \phi_{+1}p'_2)} - 1 \quad \text{if } i \in [i^*, i']
$$

(13)

$$
\theta_1 = \theta_2 = \theta = \frac{2i\phi_{+1}(p_2 - p_1)}{(1 - \sigma)(Q - \phi_{+1}p_2)} - 1 \quad \text{if } i \in [\bar{i}, \bar{i}]
$$

(14)

where $\bar{i} = \frac{\sigma(Q - \phi_{+1}p_2)}{2\phi_{+1}(p_2 - p_1)}$, $\bar{i}' = \frac{\sigma(Q - \phi_{+1}p'_2)}{2\phi_{+1}(p'_2 - p'_1)}$, $2i = \bar{i}$ and $2i' = \bar{i}'$.  10
The proof is straightforward. Inserting \( \sigma_1 = \sigma_2 = \sigma \) from Lemma 2 into (24) and (25) and into (26) and (27) yield \( \theta'_1 = \theta'_2 = \theta' \) and \( \theta_1 = \theta_2 = \theta \).

From Lemma 1 buyers play mixed strategies in each monetary subgame. Given the expressions for \( \theta' \) and \( \theta \) in (13) and (14) this imposes bounds on the value of the nominal interest rate, i.e. \( i \in [i', \bar{i}] \) and \( i \in [i, \bar{i}] \) respectively. In particular for low enough \( \theta \) we have \( \theta < 0 \) and \( \theta' < 0 \). This suggests that for low nominal interest rates buyers always opt for the high money holding, destroying the equilibrium in contingent prices. However, note that those bounds are endogenous. It turns out that the bounds adjust to shifts in \( \theta \) in a way that makes sure \( \theta \) always lies within the intervals that support the mixed strategies. Finally, note that symmetric price posting by sellers \( (p'_1 = p_1 \text{ and } p'_2 = p_2) \) implies \( \theta' = \bar{i} \) and \( \bar{i} = i' \) in addition to \( \sigma = 1/2 \).

### 3.3 Pricing strategies

Now using the set of possible subgame outcomes for given \( (p_1, p_2) \) and \( (p'_1, p'_2) \), we derive sellers’ best response to each other. Seller 1’s value function upon entering the DSM is given by

\[
V^{s1} = 2\sigma (1 - \sigma) W^{s1}_{+1} (p'_1) + \sigma^2 \left[ 1 - (\theta')^2 \right] W^{s1}_{+1} (p'_2) + \left[ \sigma^2 (\theta')^2 + (1 - \sigma)^2 \right] W^{s1}_{+1} (0). \tag{15}
\]

It says that if buyer 1 is the only buyer or buyer 2 is the only buyer, which happens with probability \( 2\sigma(1 - \sigma) \) given the symmetry in visit strategies, then seller 1 proceeds to the CWM with \( p'_1 \). If both buyers are present and at least one of them holds the high amount of money, which happens with probability \( \sigma^2 \left[ 1 - (\theta')^2 \right] \), the seller 1 enters the next CWM with \( p'_2 \). In all other circumstances he could not sell his good. Similarly, for seller 2 we have

\[
V^{s2} = 2\sigma (1 - \sigma) W^{s2}_{+1} (p_1) + (1 - \sigma)^2 \left( 1 - \theta^2 \right) W^{s2}_{+1} (p_2) + \left[ (1 - \sigma)^2 \theta^2 + \sigma^2 \right] W^{s2}_{+1} (0). \tag{16}
\]

Using the linearity of the \( W \) function once again and getting rid of the constant \( W^{sy}_{+1}(0) \) term, the two value functions simplify into the following expressions

\[
\Pi^1 = 2\sigma (1 - \sigma) \phi_{+1} p'_1 + \sigma^2 \left[ 1 - (\theta')^2 \right] \phi_{+1} p'_2 \tag{17}
\]

\[
\Pi^2 = 2\sigma (1 - \sigma) \phi_{+1} p_1 + (1 - \sigma)^2 \left( 1 - \theta^2 \right) \phi_{+1} p_2 \tag{18}
\]

which we call profit for simplicity.

We are now in a position to characterize the symmetric equilibrium in price posting pairs. We focus on the symmetric posting equilibrium as it is the natural equilibrium as the economy becomes large and coordination impossible. Note however that symmetry in visit strategies \( (\sigma_1 = \sigma_2 = \sigma) \) and symmetry in monetary strategies \( (\theta'_1 = \theta'_2 = \theta' \text{ and } \theta_1 = \theta_2 = \theta) \) were obtained without assuming symmetric price posting.
Proposition 1  Denote $i_s$ an upper bound on the nominal interest rate. For $i \in (0, i_s]$, there exists a unique symmetric equilibrium where each seller posts $p'_2 > p'_1$, and in the resulting subgames each buyer visits either seller with probability $\sigma = \frac{1}{2}$ and chooses to hold the low amount of money with probability $\theta = \theta' = \frac{4i\phi_{+1}(p'_2 - p'_1)}{(Q-\phi_{+1}p'_2)} - 1$. For $i = 0$ the indeterminacy of equilibria remains valid. For $i > i_s$ there is no monetary equilibrium.

Proof. See Appendix. ■

The indeterminacy found in directed search with contingent prices does not hold once monetary strategies are introduced into the model. This result comes from the internalization by sellers of an indifference condition for buyers between the two money holdings. This additional constraint destroys the degree of freedom in choosing two prices that sellers had in the non-monetary version.

One way to understand this result is to look at Figure 2. In the non-monetary version of this model buyers are assumed to be able to pay any posted price so that monetary strategies are irrelevant. When money is costly, holding $p_2$ is clearly preferred to holding $p_1$ when $p_2$ is only slightly greater than $p_1$ (the marginal cost is small compared to the additional trading opportunities). But a buyer is clearly better off holding $p_1$ if $p_2$ is significantly greater than $p_1$ (the marginal cost more than cancels the marginal benefit). Since the value of holding $p_2$ decreases linearly with $p_2$ there exists only one multilateral price that makes a buyer indifferent between holding the low or the high amount of money.
Uniqueness of equilibria contrasts with Coles and Eeckhout’s indeterminacy. Models with money have then different implications than models without money. The reason is that the use of money adds a subgame to be played by buyers in which buyers are to choose their money holding taking other buyers’ choices as given. In a model where prices are determined ex ante, the subgame reduces to a single-person maximization problem. In a model where prices are determined ex post as here, since a buyer is uncertain about what the other buyers will do, he randomizes over the set of relevant money holdings, i.e., posted prices. The nominal interest rate plays a key role in the monetary subgame by determining which money holdings are payoff equivalent once both their corresponding opportunity cost and expected gross payoff are taken into account.

4 Comparative Statics

We examine how shifts in the inflation rate impact the price pair posted by sellers in the symmetric equilibrium and the distribution of money holdings. Recall from the proof of Proposition 1 that in equilibrium

\[
\phi p_1' = \phi p_1 = \frac{Q (1 + \theta) [2i - \theta (1 + 6i) - \theta^2]}{2i [1 + 2\theta + 8i\theta + \theta^2]}
\]

\[
\phi p_2' = \phi p_2 = \frac{Q (1 - \theta^2)}{1 + \theta (2 + 8i + \theta)}.
\]

In particular, setting \(Q = 1\) and \(i = 0.1\) one obtains \(\theta' = 0.052\), \(\phi p_1 = 0.518\) and \(\phi p_2 = 0.867\). Figure 3 represents the equilibrium real values of \(p_1\) and \(p_2\) as the nominal interest rate increases away from the Friedman rule. The blue lines represent the prices in the 2-by-2 game while the red lines represent the prices in the \(n\)-by-\(n\) game.\(^4\) Inserting \(\phi p_1\) into the RHS of (29) the individual rationality constraint imposes \(-\phi p_1 + \beta V^{bi}(p_1) \geq 0\) which transforms into \(i < i_s\) with \(i_s \simeq 63\%\).

First, one may have expected the indeterminacy to survive for low values of the nominal interest rate. As the opportunity cost of holding money becomes small, holding even the high amount of money does not cost much yet offers to trade in multilateral matches. Actually, sellers extract this low cost by charging a real price \(\phi p_2\) that gets closer to \(Q\) (the buyer’s reservation value) as the nominal interest rate approaches zero, leaving very little gains from trade to buyers. Because sellers charge such a high price in multilateral meetings, buyers still have an incentive to carry the low amount of money despite the low opportunity cost of money. At the Friedman rule, since holding money is free, a (weakly) dominant strategy is to hold \(p_2\).

\(^4\)Details on the \(n\)-by-\(n\) game are given in Appendix A4.
This suppresses the monetary subgame and we are back to Coles and Eeckhout’s economy and its indeterminacy: there exists a continuum of symmetric equilibria indexed by $\alpha \in (0, Q]$ in which each seller charges $p_1 = Q/2$ in pairwise meetings and $p_2 = \alpha$ in multilateral meetings.

Second, we show that for small deviations from the Friedman rule sellers increase the pairwise price they charge in real terms, forcing buyers who pick the low money holding to bring more money. This is surprising since one would expect greater inflation to reduce real balances and therefore real posted prices. But as inflation rises it becomes proportionally more costly to hold the large amount of money. Sellers extract this incentive for buyers to shift to the low money holding by raising the price they charge in pairwise meetings. If one interprets the pairwise price as the reserve price this suggests that sellers perceive small increases in the inflation rate as a green light to raise their reserve price.

5 Conclusion

The model highlights several properties of monetary economies in which terms of trade are determined ex post. First, we emphasized the subgame in money holdings played by buyers in those environments, whether auctions or contingent prices. A major difficulty with contingent prices is that sellers must make sure buyers are indifferent between holding the monetary equivalent of any of the posted prices. If holding one amount strictly dominates all other then sellers’ ability to extract surplus from buyers competing against each other unravels. With auctions buyers will still randomize over how much money they bring along, but it is not the seller’s
problem to determine which prices make buyers ex ante indifferent. It is the buyer’s problem. Second, with contingent prices all prices in the schedule are set strategically and related to each other. In auctions, by contrast, the reserve price is the only strategic price as other prices are determined ex post by the mechanism. Finally, we have shown how liquidity costs impact on the reserve price set by sellers. In particular inflation makes large money holdings relatively more expensive which enables sellers to raise the low prices, especially the reserve price.

An interesting extension would be to allow prices to be contingent on every single possible realization of the matching function, with sellers advertising \( \{p_j\}_{j \in \mathbb{N}} \) where \( j \) is the number of buyers visiting a seller. We conjecture that there would still be a unique equilibrium in which buyers randomize over all the \( p_j \) and that the resulting distribution of money holdings would have similarities with that obtained by Galenianos and Kircher (2008) and Dutu Julien King (2009) in an auction environment. We leave this issue open for future research.
Appendix

A1. Proof of Lemma 1. Using the fact that $W_{x+1}^b (m - p) = W_{x+1}^b (m) - \phi_{x+1} p$, rewrite (10) as

$$V_{b1}^b (p'_1) = (1 - \sigma_2) [Q - \phi_{x+1} p'_1] + W_{x+1}^b (p'_1)$$

(21)

and (11) as

$$V_{b1}^b (p'_2) = \sigma_2 \left( \theta'_2 + \frac{1 - \theta'_2}{2} \right) [Q - \phi_{x+1} p'_2] + (1 - \sigma_2) [Q - \phi_{x+1} p'_1] + W_{x+1}^b (p'_2).$$

(22)

Buyer 1 strictly prefers holding $p'_1$ to $p'_2$ if $-\phi_{x+1} + \beta V_{b1}^b (p'_1) > -\phi_{x+1} + \beta V_{b1}^b (p'_2)$. Plugging (21) and (22) into this inequality, using $\phi_{x+1} (1 + \tau) = \phi$, dividing by $\beta$ and recalling that $i = \frac{1 - \beta + \tau}{\beta}$ it simplifies into

$$i \phi_{x+1} (p'_2 - p'_1) > \left( \theta'_2 + \frac{1 - \theta'_2}{2} \right) \sigma_2 [Q - \phi_{x+1} p'_2]$$

(23)

where the left-hand side is the marginal cost of holding $p'_2$ instead of $p'_1$ and the right-hand side is the marginal benefit. From (23) buyer 1’s best-response correspondence is given by

$$\theta'_1 = \begin{cases} 
0 & \text{if } \theta'_2 < \frac{2i \phi_{x+1} (p'_2 - p'_1)}{\sigma_2 (Q - \phi_{x+1} p'_2)} - 1 \\
(0, 1) & \text{if } \theta'_2 = \frac{2i \phi_{x+1} (p'_2 - p'_1)}{\sigma_2 (Q - \phi_{x+1} p'_2)} - 1 \\
1 & \text{if } \theta'_2 > \frac{2i \phi_{x+1} (p'_2 - p'_1)}{\sigma_2 (Q - \phi_{x+1} p'_2)} - 1 
\end{cases}$$

(24)

Similarly the best-response correspondence of buyer 2 in the monetary subgame is

$$\theta'_2 = \begin{cases} 
0 & \text{if } \theta'_1 < \frac{2i \phi_{x+1} (p'_2 - p'_1)}{\sigma_1 (Q - \phi_{x+1} p'_2)} - 1 \\
(0, 1) & \text{if } \theta'_1 = \frac{2i \phi_{x+1} (p'_2 - p'_1)}{\sigma_1 (Q - \phi_{x+1} p'_2)} - 1 \\
1 & \text{if } \theta'_1 > \frac{2i \phi_{x+1} (p'_2 - p'_1)}{\sigma_1 (Q - \phi_{x+1} p'_2)} - 1 
\end{cases}$$

(25)

Assuming $\sigma_1 \in (0, 1)$ and $\sigma_2 \in (0, 1)$, which is shown to hold in equilibrium below, then both correspondences intersect only once: if a buyer decides to visit seller 1, then there exists a unique mixed-strategy equilibrium in the monetary subgame given the monetary and visit strategies played by the other buyer.

Similarly, if a buyer decides to visit seller 2 then there exists a unique mixed-strategy equilibrium in the monetary subgame given the monetary and visit strategies played by the
other buyer, with the best-response correspondences given by

\[
\theta_1 = \begin{cases} 
0 & \text{if } \theta_2 < \frac{2i\phi_{+1}(p_2-p_1)}{(1-\sigma_2)(Q-\phi_{+1}p_2)} - 1 \\
(0,1) & \text{if } \theta_2 = \frac{2i\phi_{+1}(p_2-p_1)}{(1-\sigma_2)(Q-\phi_{+1}p_2)} - 1 \\
1 & \text{if } \theta_2 > \frac{2i\phi_{+1}(p_2-p_1)}{(1-\sigma_2)(Q-\phi_{+1}p_2)} - 1
\end{cases}
\]  

(26)

and

\[
\theta_2 = \begin{cases} 
0 & \text{if } \theta_1 < \frac{2i\phi_{+1}(p_2-p_1)}{(1-\sigma_1)(Q-\phi_{+1}p_2)} - 1 \\
(0,1) & \text{if } \theta_1 = \frac{2i\phi_{+1}(p_2-p_1)}{(1-\sigma_1)(Q-\phi_{+1}p_2)} - 1 \\
1 & \text{if } \theta_1 > \frac{2i\phi_{+1}(p_2-p_1)}{(1-\sigma_1)(Q-\phi_{+1}p_2)} - 1
\end{cases}
\]  

(27)

for buyer 1 and buyer 2 respectively.

**A2. Proof of Lemma 2.**

For a buyer, visiting seller 1 can be done holding either \( p_1' \) or \( p_2' \). The corresponding expected payoff is given by

\[
\theta_1' \left[ -\phi p_1' + \beta V^{bl}(p_1') \right] + (1 - \theta_1') \left[ -\phi p_2' + \beta V^{bl}(p_2') \right].
\]  

(28)

From Lemma 2 there is a unique Nash equilibrium in the monetary subgame. In this equilibrium buyers randomize between the two money holdings because the expected payoff to holding the low amount or the high amount of money while visiting one seller are equal. For buyer 1 who visits seller 1, this means that \( -\phi p_1' + \beta V^{bl}(p_1') = -\phi p_2' + \beta V^{bl}(p_2') \) so that (28) is simply equal to \( -\phi p_1' + \beta V^{bl}(p_1') \). Similarly, buyer 1’s expected payment from visiting seller 2 is simply \( -\phi p_1 + \beta V^{bl}(p_1) \). Therefore buyer 1 strictly prefers visiting seller 1 to visiting seller 2 if \( -\phi p_1' + \beta V^{bl}(p_1') > -\phi p_1 + \beta V^{bl}(p_1) \). Using the same simplification techniques used in the Proof of Lemma 1, this transforms into

\[
-i\phi_{+1}p_1' + (1-\sigma_2)\left[ Q - \phi_{+1}p_1' \right] > -i\phi_{+1}p_1 + \sigma_2 \left[ Q - \phi_{+1}p_1 \right].
\]  

(29)

Buyer 1’s best response correspondence is then given by

\[
\sigma_1 = \begin{cases} 
0 & \text{if } \sigma_2 < \frac{i\phi_{+1}(p_1-p_1') + Q - \phi_{+1}p_1'}{(Q-\phi_{+1}p_1') + (Q-\phi_{+1}p_1)} \\
(0,1) & \text{if } \sigma_2 = \frac{i\phi_{+1}(p_1-p_1') + Q - \phi_{+1}p_1'}{(Q-\phi_{+1}p_1') + (Q-\phi_{+1}p_1)} \\
1 & \text{if } \sigma_2 > \frac{i\phi_{+1}(p_1-p_1') + Q - \phi_{+1}p_1'}{(Q-\phi_{+1}p_1') + (Q-\phi_{+1}p_1)}
\end{cases}
\]  

(30)
Similarly buyer 2’s best response correspondence is given by

\[
\sigma_2 = \begin{cases} 
0 & \text{if } \sigma_1 < \frac{i \phi_1(p_1 - p'_1) + Q - \phi_1 p'_1}{(Q - \phi_1 p'_1) + (Q - \phi_1 p_1)} \\
(0, 1) & \text{if } \sigma_1 = \frac{i \phi_1(p_1 - p'_1) + Q - \phi_1 p'_1}{(Q - \phi_1 p'_1) + (Q - \phi_1 p_1)} \\
1 & \text{if } \sigma_1 > \frac{i \phi_1(p_1 - p'_1) + Q - \phi_1 p'_1}{(Q - \phi_1 p'_1) + (Q - \phi_1 p_1)}
\end{cases}
\]  

(31)

Both best-response correspondences intersect only once at \( \sigma_1 = \sigma_2 = \sigma \) with \( \sigma \) given in (12). There exists a unique mixed-strategy equilibrium in the visit subgame given the Nash equilibrium played by buyers in the monetary subgame and the prices posted by sellers.

**A.3. Proof of Proposition 1**

The proof is a classic fixed point argument. Using (13) to substitute \( p'_2 \) into (17) and then (12) to substitute \( p'_1 \) we obtain

\[
\Pi^1(\sigma, \theta'; p_1) = 2\sigma(1 - \sigma) K_1 + \sigma^2 \left[ 1 - (\theta')^2 \right] \frac{Q \sigma (1 + \theta') + 2i K_1}{2i + \sigma (1 + \theta')}
\]

(32)

with

\[
K_1 = \frac{Q (1 - 2\sigma) + (i + \sigma) \phi_1 p_1}{1 + i - \sigma}
\]

(33)

Similarly,

\[
\Pi^2(\sigma, \theta; p'_1) = 2\sigma(1 - \sigma) K_2 + (1 - \sigma)^2 (1 - \theta^2) \frac{Q (1 + \theta) (1 - \sigma) + 2i K_2}{2i + (1 + \theta) (1 - \sigma)}
\]

(34)

with

\[
K_2 = \frac{Q (2\sigma - 1) + (1 + i - \sigma) \phi_1 p'_1}{i + \sigma}
\]

(35)

Given \((p_1, p_2)\) we define \( \sigma_1^* \) as the visiting strategy by buyers that maximizes seller 1’s profit, that is \( \sigma_1^* = \arg \max_{\sigma \in [0, 1]} \Pi^1(\sigma, \theta'; i, p_1) \). We define \( \theta'_1^* \) as the monetary strategy by buyers that maximizes seller 1’s profit, that is \( \theta'_1^* = \arg \max_{\theta' \in [0, 1]} \Pi^1(\sigma, \theta'; i, p_1) \). Similarly we have \( \sigma_2^* = \arg \max_{\sigma \in [0, 1]} \Pi^2(\sigma, \theta; i, p'_1) \) and \( \theta'_2^* = \arg \max_{\theta \in [0, 1]} \Pi^2(\sigma, \theta; i, p'_1) \) for seller 2. Seller 1’s best-response correspondence is then given by

\[
\theta' = \arg \max_{\theta' \in [0, 1]} \Pi^1(\sigma, \theta'; i, p_1),
\]

(36)

\[
\sigma = \arg \max_{\sigma \in [0, 1]} \Pi^1(\sigma, \theta'; i, p_1).
\]

(37)
Similarly, seller 2’s best-response correspondence is given by

$$ \theta = \arg \max_{\theta \in [0,1]} \Pi^2 \left( \sigma, \theta; i, p'_1 \right) $$

$$ \sigma = \arg \max_{\theta \in [0,1]} \Pi^2 \left( \sigma, \theta; i, p'_1 \right). $$

It is easy to see that all four conditions for Kakutani’s fixed point theorem are met. To show there is only one fixed point in the symmetric equilibrium, take the first-order condition of \( \Pi^2 \) with respect to \( \sigma \) and insert \( \sigma = 1/2 \). This enables to extract

$$ \phi p'_1 = \phi p_1 = \frac{Q(1 + \theta)}{2i(1 + 2\theta + 8i\theta + \theta^2)} \cdot $$

Then take the first order condition of \( \Pi^2 \) with respect to \( \sigma \), insert \( \phi p'_1 \) from (40) and set \( \sigma = 1/2 \) to obtain one equation in one unknown, \( \theta \), parameterized by \( i \), of the form \( g(\theta; i) = 0 \). It is easy to show that the function \( g \) is strictly increasing and that \( g(0; i) < 0 \) and \( g(1; i) > 0 \) so that there exists a unique \( \theta \) that maximizes seller 2’s profit given \( i, p'_1 \) and \( p'_2 \). Finally, inserting \( p'_1 \) from (40) into (13) and setting \( \theta' = \theta \) one obtains

$$ \phi p'_2 = \phi p_2 = \frac{Q(1 - \theta^2)}{1 + \theta(2 + 8i\theta + \theta^2)}. $$

To check that \( i \in I = [i, \bar{i}] \) simply insert \( p'_2 \) from (41) and \( p'_1 \) from (40) into \( i = \frac{\sigma(Q - \phi_{i+1}p'_2)}{2\phi_{i+1}(p'_2 - p'_1)} \) and \( \bar{i} = \frac{\sigma(Q - \phi_{i+1}p'_2)}{2\phi_{i+1}(p'_2 - p'_1)} \). This yields \( i = \frac{i}{1+i} < i \) and \( \bar{i} = \frac{2i}{1+i} > i \).

**A.4. No Nash Equilibrium with Contingent Prices exists when \( p_1 > p_2 \)**

Buyer 1 is indifferent between holding \( p'_1 \) and \( p'_2 \) if

$$ -\phi p'_1 + \beta V^{b1}(p'_1) = -\phi p'_2 + \beta V^{b1}(p'_2) $$

with

$$ V^{b1}(p'_1) = (1 - \sigma_2) \left[ Q + W^{b1}_+ (p'_1 - p'_1) \right] + \frac{\sigma_2}{2} \left[ Q + W^{b1}_+ (p'_1 - p'_2) \right] + \frac{\sigma_2}{2} W^{b1}_+ (p'_1) $$

and

$$ V^{b1}(p'_2) = (1 - \sigma_2) W^{b1}_+ (p'_2) + \frac{\sigma_2}{2} \left[ Q + W^{b1}_+ (p'_2 - p'_2) \right] + \frac{\sigma_2}{2} W^{b1}_+ (p'_2). $$

Note that buyer 2’s money holding does not impact on buyer 1’s monetary strategy. Equation (42) can be simplified into

$$ i\phi_{i+1} (p'_1 - p'_2) = (1 - \sigma_2) \left[ Q - \phi_{i+1}p'_1 \right] $$
which equalizes the marginal cost of holding \( p'_1 \) instead of \( p'_2 \) to its marginal return. That buyer 1 is indifferent between holding \( p_1 \) and \( p_2 \) similarly implies

\[
i \phi_{+1} (p_1 - p_2) = \sigma_2 [Q - \phi_{+1} p_1]. \tag{46}
\]

For buyer 2 one finds

\[
i \phi_{+1} (p'_1 - p'_2) = (1 - \sigma_1) [Q - \phi_{+1} p'_1] \tag{47}
\]
\[
i \phi_{+1} (p_1 - p_2) = \sigma_1 [Q - \phi_{+1} p_1]. \tag{48}
\]

From (45) and (47) one extracts \( \sigma_1 = \sigma_2 = \sigma \) with

\[
\sigma = 1 - \frac{i \phi_{+1} (p'_1 - p'_2)}{Q - \phi_{+1} p'_1} \tag{49}
\]

and from (46) and (48) one gets

\[
\sigma = \frac{i \phi_{+1} (p_1 - p_2)}{Q - \phi_{+1} p_1}. \tag{50}
\]

As in the \( p_2 > p_1 \) case, the visit indifference condition can be written

\[
-i \phi_{+1} p'_2 + \frac{\sigma}{2} [Q - \phi_{+1} p'_2] = -i \phi_{+1} p_2 + \frac{1 - \sigma}{2} [Q - \phi_{+1} p_2] \tag{51}
\]

which gives

\[
\sigma = \frac{2i \phi_{+1} (p'_2 - p_2) + Q - \phi_{+1} p_2}{Q - \phi_{+1} p'_2 + Q - \phi_{+1} p_2}. \tag{52}
\]

The sellers’ profits are given by

\[
\Pi^1 = 2\sigma (1 - \sigma) \theta' \phi_{+1} p'_1 + \sigma^2 \phi_{+1} p'_2 \tag{53}
\]
\[
\Pi^2 = 2\sigma (1 - \sigma) \theta \phi_{+1} p_1 + (1 - \sigma)^2 \phi_{+1} p_2. \tag{54}
\]

Seller 1’s profit \( \Pi^1 \) is a function of \( \sigma, \theta', p'_1 \) and \( p'_2 \). Using (49) to substitute \( p'_1 \) and (51) to substitute \( p'_2 \), we obtain \( \Pi^1 = \Pi^1 (\sigma, \theta'; p_2) \). Computing \( \partial \Pi^1 / \partial \theta' = 0 \) yields \( \phi p_2 = -\frac{Q(i + \sigma (1 - \sigma))}{1 + 2i - \sigma} < 0 \).

### A.4. The large economy

Suppose there is an infinite yet countable number \( n \) of buyers and an infinite yet countable number \( m = n \) of sellers.\(^5\) The value function for a buyer holding \( p_1 \) on the DSM is given by

\[
V^b(p_1) = \psi_p \left[ Q + W^b_{+1} (p_1 - p_1) \right] + (1 - \psi_p) W^b_{+1} (p_1) \tag{55}
\]

\(^5\)Having the same number of buyers and sellers facilitates comparison with the 2-by-2 game for the comparative statics. Extending the model to any numbers is straightforward.
where \( \psi_p \) is a probability of a pairwise match. Noting \( \psi_m \) is the probability of a multilateral match in which the buyer wins the good, the value function of a buyer holding \( p_2 \) in the directed search market is given by

\[
V^b(p_2) = \psi_p \left[ Q + W^b_{p_1} (p_2 - p_1) \right] + \psi_m \left[ Q + W^b_{p_1} (p_2 - p_2) \right] + (1 - \psi_p - \psi_m) W^b_{p_1}(p_2). \tag{56}
\]

In the symmetric equilibrium, since buyers use identical visit strategies, the matching technology corresponds to a standard urn-ball process. The probability for a buyer to face \( n \) other buyers at a seller’s shop is then given by

\[
\text{Pr}[X = n] = \frac{n^n}{n!} e^{-\gamma}.
\]

The probability of a pairwise match for a buyer is then equal to the probability that no other buyer shows up, that is \( \text{Pr}[X = 0] = e^{-\gamma} \). The value of \( \psi_m \) is more demanding. To win the good in a multilateral match, the buyer must hold the high amount of money \( p_2 \) and be selected among buyers who brought \( p_2 \). If we note \( k \) the number of buyers among the \( n \) other buyers present at the seller’s who also brought \( p_2 \) then

\[
\psi_m = \sum_{n \in \mathbb{N}^*} \text{Pr}[X = n] \left\{ \sum_{k=0}^{n} \binom{k}{n} \theta^k (1 - \theta)^{n-k} \frac{1}{n-k+1} \right\}. \tag{57}
\]

To understand the terms inside the curly brackets, consider the case in which the buyer faces 2 other buyers so that \( n = 2 \). The buyer wins the good if he is the only one holding \( p_2 \) which happens with probability \( \theta^2 \), or if one of the two other buyers holds \( p_2 \), which happens with probability \( \binom{1}{2} \theta (1 - \theta) \) in which case he wins the good with probability \( \frac{1}{2} \), or if both of them hold \( p_2 \), which happens with probability \( (1 - \theta)^2 \) in which case he wins the good with probability \( \frac{1}{3} \). This probability simplifies according to

\[
\sum_{n \in \mathbb{N}^*} \frac{\gamma^n}{n!} e^{-\gamma} \left\{ \sum_{k=0}^{n} \binom{k}{n} \theta^k (1 - \theta)^{n-k} \frac{1}{n-k+1} \right\} \tag{58}
\]

\[
= \sum_{n \in \mathbb{N}^*} \frac{\gamma^n}{(n+1)!} e^{-\gamma} \left\{ \sum_{k=0}^{n} \binom{k}{n+1} \theta^k (1 - \theta)^{n-k} \right\} \tag{59}
\]

\[
= \frac{1}{\gamma} \sum_{n \in \mathbb{N}^*} \frac{\gamma^{n+1}}{(n+1)!} e^{-\gamma} \left\{ \sum_{k=0}^{n'-1} \binom{k}{n'} \theta^k (1 - \theta)^{n'-k} \right\} \tag{60}
\]

\[
= \frac{1}{\gamma (1 - \theta)} \sum_{n \in \mathbb{N}^*} \frac{\gamma^{n+1}}{(n+1)!} e^{-\gamma} \left\{ \sum_{k=0}^{n'-1} \binom{k}{n'} \theta^k (1 - \theta)^{n'-k} - \theta^n \right\} \tag{61}
\]

\[
= \frac{1}{\gamma (1 - \theta)} \sum_{n \in \mathbb{N}^*} \frac{\gamma^{n+1}}{(n+1)!} e^{-\gamma} \left\{ \sum_{k=0}^{n'-1} \binom{k}{n'} \theta^k (1 - \theta)^{n'-k} - \theta^n \right\} \tag{62}
\]
The value functions for a buyer on the centralized market is given by

\[
W^b_x(m) = \phi(m + T) + \max_{\hat{m} \in \{p_1, p_2\}} \left\{ -\phi\hat{m} + \beta V^b_x(\hat{m}) \right\}.
\]

(67)

Inserting (55) into (67) and noting \( z_1 = \hat{\phi}_1 p_1 \), a buyer’s net utility from holding \( p_1 \) is given by

\[
-iz_1 + e^{-\gamma} (Q - z_1).
\]

(68)

Inserting (56) into (67) and noting \( z_2 = \hat{\phi}_2 p_2 \), a buyer’s net utility from holding \( p_2 \) is given by

\[
-iz_2 + e^{-\gamma} (Q - z_1) + \frac{1 - \gamma (1 - \theta) e^{-\gamma} - e^{-\gamma(1-\theta)}}{\gamma (1 - \theta)} (Q - z_2).
\]

(69)

As for sellers, their value function on the directed search market is given by

\[
V^s = \xi_p W^s_{+1}(p_1) + \xi_m W^s_{+1}(p_2) + (1 - \xi_p - \xi_m) W^s_{+1}(0)
\]

(70)

where \( \xi_p \) is the probability of a pairwise match and \( \xi_m \) is the probability of a multilateral match in which at least one of the buyer holds \( p_2 \). We have

\[
\xi_p = \Pr[X = 1] = \gamma e^{-\gamma}
\]

(71)

and

\[
\xi_m = \sum_{n \in \mathbb{N}^*} \Pr[X = n] (1 - \theta^n)
\]

(72)

which using similar techniques as for simplifying \( \psi_m \) gives

\[
1 - \gamma (1 - \theta) e^{-\gamma} - e^{-\gamma(1-\theta)}.
\]

(73)
To compute the value function of a seller on the centralized market, we use a method developed by Montgomery (1991). The key to this method is to assume that sellers must provide buyers with a certain level of expected utility $U$, given by the market, which is later determined endogenously. Sellers are then thought of offering a combination of prices and a probability to trade that yields $U$ to buyers. Suppose a seller chooses a particular $(p_1, p_2)$ and buyers respond by visiting him with probability $\sigma$ and holding $p_1$ with probability $\theta$, implying probabilities to trade equal to $\psi_p(\gamma)$ and $\psi_m(\gamma)$. In a competitive economy no seller can beat the market by posting a different combination of prices and probabilities to trade. Then the combination of $(p_1, p_2)$, $\sigma$ and $\theta$ must provide buyers with the same $U$. Given that $\sigma = \sigma(\gamma)$ the seller’s value function in the centralized market is then

$$W^s(m) = \phi m + \max_{p_1, p_2, \gamma, \theta} \beta V^s.$$ (74)

Inserting (70) into (74) and simplifying, the seller’s problem is

$$\max_{z_1, z_2, \gamma, \theta} e^{-\epsilon z_1} z_1 + \left[ 1 - \gamma (1 - \theta) e^{-\gamma} - e^{-\gamma(1 - \theta)} \right] z_2$$

s.t. $-i z_1 + e^{-\gamma} (Q - z_1) = U,$ (75)

$$-i z_2 + e^{-\gamma} (Q - z_1) + \frac{1 - \gamma (1 - \theta) e^{-\gamma} - e^{-\gamma(1 - \theta)}}{\gamma (1 - \theta)} (Q - z_2) = U.$$ (77)

This is the constrained maximization program that we use to derive terms of trade in the $n$-by-$n$ game.

---

References


