A Dynamic Mechanism Design for Scheduling with Different Lengths of Use*

Ryuji Sano†
Institute of Social and Economic Research, Osaka University
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Abstract

A dynamic mechanism design problem is considered. A number of identical perishable goods, such as time slots of a central facility or hotel rooms, are allocated at each period. A number of agents randomly come to a mechanism, and each agent wants to keep winning a good for more than one period to make profits. The seller offers simple long-term contracts that guarantee future allocations to agents. We characterize incentive compatible mechanisms in our domain of mechanisms, and provide a dynamic VCG mechanism that achieves efficient allocations. The seller’s revenue is maximized by virtual valuation maximization under a monotone hazard rate condition. In the revenue-maximizing mechanism, long-stay agents tend to pay some premium per period compared to short-stay agents.

Keywords: dynamic mechanism, online mechanism, dynamic population, dynamic VCG mechanism, optimal auction

JEL classification: C73, D44, D82

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†Institute of Social and Economic Research, Osaka University, 6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan. Telephone: +81-6-68798560. E-mail: r-sano@iser.osaka-u.ac.jp
1 Introduction

This paper considers a dynamic allocation problem where a seller allocates many perishable objects over time. A constant number of identical objects are supplied at each period. A number of agents randomly arrive over time. Agents want to obtain at most one unit of the object at each period. However, they stay for several periods of time, and need to keep obtaining the object during their stay to earn positive profits. We consider dynamic auction designs in such a situation.

A motivating example is a time-slot allocation problem. Suppose that there are a number of facilities such as city halls, meeting rooms, or shared computer servers, and time slots to use such facilities are allocated over time. Potential users (agents) randomly arrive over time, and slots of time are allocated at a time. Agents often want to use the facilities for several periods of time. For example, an academic conference or an exhibition would be held at a hotel or a convention center for several days or a week. A musician wants to hold a concert at a hall for several days. People would like to stay at a hotel for several nights, or a computer job would need a long time to complete on a server. Agents need the object for different periods to be satisfied, and the necessary period is in general private information of each agent along with the valuation. Agents thus have two dimensional type, value and time period, and earn the value only when they use the facilities for the whole of the time they need it for.

In such a case, a seller often offers long-term allocations to the current agents and reserves future slots for the current agents in advance. Although a long-term contract might possibly be contingent on future events, it is frequently hard to make such a complex contract in practical situations. In many real situations, only simple contracts that are not contingent on future events are available.

This paper considers a mechanism in which a seller offers simple long-term contracts to agents. Agents randomly arrive over time, having a two dimensional type: valuation and time period. Each agent wants to obtain one unit of an object (or a slot) at a time. He experiences a value only if he gets the object for the length of the period. A seller or a mechanism designer makes an \( m_i \)-period simple contract or a package contract to an agent when he arrives. A simple contract specifies a sequence of allocations to an agent regardless of future events.

We focus on dynamic direct mechanisms by the revelation principle. We charac-
terize incentive compatible mechanisms, and provide an efficient mechanism and a revenue-maximizing mechanism in our domain of mechanisms. We show that most of the techniques and results can be applied to a dynamic allocation problem of our situation. Given a period type, incentive compatibility requires that the allocation policy for an agent is increasing in his valuation, and that the payment is determined by the well-known “revenue equivalence” formula. In addition, to ensure reporting true period types, the allocation policy needs to satisfy a weak notion of a monotonicity condition on period types. We show that these conditions are also sufficient for incentive compatibility.

We then provide an efficient mechanism and a revenue-maximizing mechanism. The efficient mechanism is provided as an extension of the well-known Vickrey-Clarke-Groves mechanism. Each agent pays the expected externality that he gives to the other agents, who are the other current entrants and future entrants to the mechanism. The revenue-maximizing or optimal mechanism is provided as in Myerson (1981). We show that the optimal allocation policy maximizes the expected sum of the virtual valuations under a regularity condition. The regularity condition is satisfied by imposing a monotone hazard rate condition.\(^1\)

The contribution of the paper is to formulate a model of a dynamic allocation problem, which has not been considered in preceding studies but is often observed in practical situations. Although several studies such as Bergemann and Valimaki (2010), Pai and Vohra (2011), and Pavan et al. (2009) include similar environments, they do not consider long-term contracts investigated in this paper. In addition, we fully characterize incentive compatibility and provide an optimal mechanism in a situation where agents have two dimensional types in a special form. We show that the standard approach in mechanism design is applied. Although the allocation problem is dynamic and complex for the seller, the auction mechanisms are static for agents, and thus we can deal with the problem in a standard manner.

\(^1\)A similar notion of monotone hazard rate condition is introduced by Pai and Vohra (2011).
1.1 Related Literature

In recent years, there have been a number of studies on dynamic mechanism design. Dynamic auction design in an environment where agents strategically arrive and depart is often called online mechanism design, and has been investigated in the fields of computer sciences and operations research. Our model is closely related to such a “model with dynamic populations,” and there are many preceding studies such as Lavi and Nisan (2000), Parkes and Singh (2003), Porter (2004), Hajiaghayi et al. (2005), Gershkov and Moldovanu (2009, 2010), Board and Skrzypacz (2010), Pai and Vohra (2011), and Said (2012).

Parkes and Singh (2003) and Pai and Vohra (2011) consider an allocation problem of durable goods, such as air tickets sales and hotel room reservations. Buyers arrive over time, stay for several periods, and participate in auctions during the stay. They want to buy an object at most once in their stay. A buyer’s type is defined by his arrival time, departure time, and valuation. Parkes and Singh (2003) formulate an incentive compatible efficient mechanism by extending the Vickrey-Clarke-Groves mechanism. Pai and Vohra (2011) characterize incentive compatible and individually rational mechanisms and investigate the optimal mechanism.

Hajiaghayi et al. (2005) and Parkes (2007) consider a perishable goods case such as scheduling of facilities in the presence of strategic arrivals and departures. They consider incentive compatible mechanisms and investigate the efficiency of an allocation policy. Porter (2004) considers a dynamic mechanism in the context of a computer job assignment on a server, and introduce the notion of job length, which corresponds to period type in this paper.

Gershkov and Moldovanu (2009, 2010), Board and Skrzypacz (2010), and Said (2012) consider that agents randomly come to a mechanism. Gershkov and Moldovanu (2009, 2010) consider the efficient and revenue-maximizing durable goods sales in which agents are impatient and short-lived. Board and Skrzypacz (2010) also consider a durable goods sale, but contrary to Gershkov and Moldovanu, agents are patient and stay until the deadline of the sale. Said (2012) considers a perishable goods case. Buyers arrive at random, and stay in the next period with a positive proba-

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2 See Bergemann and Said (2011) for a review. Dynamic mechanism design is called online mechanism design especially in the fields of operations research and computer sciences. See also Parkes (2007).
bility common among buyers. He considers both efficient and revenue-maximizing mechanisms, and shows that the outcomes in the efficient or the optimal mechanisms can be achieved by simple repeated auctions in a perfect Bayesian equilibrium.

Most of these papers consider the situation where agents stay in a mechanism for a long time and want to win an auction at most once. Hence, agents stay until they win an object, and they are supposed to exit once they win. In other words, agents are assumed to evaluate intertemporal objects as perfect substitutes. Conversely, our model considers the case where agents evaluate intertemporal objects as perfect complements.

Bergemann and Valimaki (2010), Pavan et al. (2009), and Kakade et al. (2011) also consider a similar dynamic allocation problem, but they consider fixed populations and dynamic information. Bergemann and Valimaki (2010) consider an infinitely repeated allocation problem with a type of agent being drawn at each period by Markov process. They formulate an incentive compatible efficient mechanism called “dynamic pivot mechanism,” which is an extension of the Vickrey-Clarke-Groves mechanism. Pavan et al. (2009) and Kakade et al. (2011) characterize incentive compatibility and provide revenue equivalence in a dynamic information model. Cavallo et al. (2010) formulate an incentive compatible efficient mechanism in the presence of both dynamic populations and dynamic information, extending Parkes and Singh (2003) and Bergemann and Valimaki (2010).

The remainder of the paper is as follows. In section 2, we provide a model of the time-slot scheduling problem. We explain agents’ preferences and the domain of dynamic mechanisms. In section 3, we characterize the incentive compatible mechanisms, and show a revenue equivalence result. In section 4, we argue that the efficient allocation policy is monotone, and provide the dynamic VCG mechanism. We show that truthful reporting is a dominant strategy equilibrium in the dynamic VCG mechanism. In section 5, we provide the revenue-maximizing mechanism. We introduce a notion of the monotone hazard rate condition, and show that the revenue-maximizing allocation policy maximizes the sum of virtual valuations under this condition.
2 The Model

We consider an environment with independent and private values in a discrete-time model. \( K \) identical objects, such as hotel rooms and time slots of a city hall or facilities, are supplied at each period \( t = 1, \ldots, T \). Suppose \( T \leq \infty \); time horizon is either finite or infinite. In most of the paper, we consider the infinite horizon. Objects are non-storable and perish at the end of each period. At each period, a finite number of agents enter a mechanism. The set of entrants at \( t \) is denoted by \( N^t \). Each agent at each period is ex ante homogeneous, and the number of entrants \( |N_t| \) is an i.i.d. random variable at each period. Each entrant wants to own at most one unit of the object at a period. The set of agents having entered by \( t \) is denoted by \( N_t \equiv \bigcup_{s \leq t} N_s \). An allocation at \( t \) is denoted by \( a_t = (a_t^i)_{i \in N_t} \). And, \( a_t^i \in [0, 1] \) denotes the probability of obtaining the object at \( t \). An allocation \( a_t \) is said to be feasible at \( t \) if \( \sum a_t^i \leq K \) and \( a_t^i = 0 \) for any \( i \) who is not in the mechanism at \( t \).

Agents and a seller discount future payoffs by a common factor \( \delta \in (0, 1) \). Each agent \( i \) of type \( \theta_i = (V_i, l_i) \) stays in a mechanism for at least \( l_i \) periods. When agent \( i \) of type \( \theta_i \) enters at \( t \), \( i \)'s payoff evaluated at \( t \) is given by

\[
  u_i = \begin{cases} 
  V_i - \sum_{s = t}^{\infty} \delta^{s-t} p^s_i & \text{if } a_t^i = 1 \text{ for } \forall s \in \{t, \ldots, t + l_i - 1\}, \\
  -\sum_{s = t}^{\infty} \delta^{s-t} p^s_i & \text{otherwise},
  \end{cases}
\]

where \( p^s_i \) denotes \( i \)'s payment at \( s \). This means that agent \( i \) earns a total profit \( V_i \) only if he owns the object for \( l_i \) periods of time.\(^4\) Otherwise, agent \( i \) earns nothing. We call \( V_i \), valuation type and \( l_i \), period type. Each agent’s type \( \theta_i \) is independently drawn from an identical distribution \( F \) on \( \Theta_i \). Let \( F(\cdot|l_i) \) be the cumulative distribution function conditional on \( l_i \). Given any \( l_i \), \( F(\cdot|l_i) \) has a density function \( f(\cdot|l_i) \geq 0 \) for all \( V_i \). In addition, \( f(l_i) \equiv \int_0^{\bar{V}} f(V_i, l_i) dV_i \) denotes the probability that an agent’s period type is \( l_i \).

The seller offers a long-term contract for \( i \). Each agent accepts only one contract when he enters a mechanism.\(^5\) There is no renegotiation and a contract is never revised or canceled before expiration. A contract made for \( i \) at \( t \), \( z^t_i \), consists of its term,

\(^3\)For example, some of \( n \) potential agents, whose set is denoted by \( I^t \), enter a mechanism with an identical distribution \( \lambda \) on \( 2^{I^t} \). \( |I^t| = n \) for all \( t \) and \( N^t \subseteq I^t \) is the set of entrants at \( t \).

\(^4\)Total profit \( V_i \) is evaluated at \( t \).

\(^5\)If an agent rejects a contract, then he leaves the mechanism and gets payoff 0.
and a sequence of allocations and payments for \( i \): \( z^t_i = \{ a^t_i, p^t_i \}_{s=t}^{t+m_i-1} \). The contract term is denoted by \( m_i \), which is assumed to be deterministic. Goods allocation \( a^t_i \) is assigned at \( s \) with payment \( p^s_i \). We limit attention to simple package contracts that depend only on current history. Further, goods allocation is probabilistic only in the first period of the contract. When \( a^t_i \in (0,1) \), \( i \) obtains an object with probability \( a^t_i \) at \( t \), and the allocation finally assigned at \( t \) is assigned with probability 1 in the later periods until its expiration. In other words, probability is assigned not on periodic allocations but on a package of \( m_i \)-period allocations \( (a^t_i, \ldots, a^{t+m_i-1}_i) = (1, \ldots, 1) \).

Let \( \bar{a}^t_i \in \{0,1\} \) be ex post allocation for \( i \) at \( t \), which denotes the allocation for \( i \) finally assigned at \( t \).

Note that two contracts \( z^t_i \) and \( \bar{z}^t_i \) with identical terms and allocations are indifferent for \( i \) whenever \( \sum \delta^{s-t} p^s_i = \sum \delta^{s-t} \bar{p}^s_i \). Hence, abusing notation, a contract can be denoted by \( z^t_i = (a_i, m_i, p_i) \), where \( a_i \) denotes the (random) allocation at \( t \), \( m_i \) denotes the contract term, and \( p_i \equiv \sum \delta^{s-t} p^s_i \). The set of enforceable contracts for \( i \) is denoted by \( Z_i \equiv [0,1] \times \{1, \ldots, L\} \times \mathbb{R} \). The set of bundles of contracts at \( t \) is denoted by \( Z^t \equiv \times_{i \in N} Z_i \). For a contract \( z_i^t = (\bar{a}_i, \bar{m}_i, \bar{p}_i) \), we use the following notations for the corresponding components: \( a_i(z^t_i) = \bar{a}_i \), \( m_i(z^t_i) = \bar{m}_i \), and \( p_i(z^t_i) = \bar{p}_i \).

In addition, \( \bar{a}_i(z^t_i) \) denotes the ex post allocation from contract \( z^t_i \).

It is convenient to define the residual periods of \( z^t_i \) at \( t \geq s \), which is denoted by \( r(t,z^s_i) \) and

\[
r(t,z^s_i) = \max\{m_i(z^s_i) + s - t, 0\}.
\]

In addition, the residual periods of objects at \( t \) is \( K \)-dimensional vector \( x^t_i = (x^t_i)_k \), where \( x^t_i_k \) is the \( k \)-th highest order number of \( r(t,z^s_i) \) of all \( i \) and all \( s < t \) such that \( \bar{a}_i(z^s_i) = 1 \). Note that \( \#\{k|x^t_i_k \geq 1\} \) units of the object are kept at \( t \) by some incumbent agents. The supply at \( t \) is denoted by \( K_i = K - \#\{k|x^t_i_k \geq 1\} \). Let \( X \) be the set of \( x^t_i \): \( X = \{ x \in \mathbb{Z}_+^j | 0 \leq x \leq (L-1, \ldots, L-1) \} \).

### 2.1 Dynamic Direct Mechanisms

We assume that each agent cannot manipulate the arrival time. However, they may manipulate their departures or period types. By the revelation principle, we limit attention to dynamic direct mechanisms. Each agent reports the type to the seller or the mechanism designer at the arrival time. Then, the seller offers a contract to each agent just once at his arrival time.
At each \( t \), entrants and the seller observe the set of entrants \( N^t \), and hence, \( N^t \) is common knowledge. Then, each entrant \( i \in N^t \) makes a report \( \gamma^t_i = (\hat{V}_i, \hat{h}_i) \in \Theta_i \). The profile of types at \( t \) is denoted by \( \theta^t = (\theta_i)_{i \in N^t} \). The seller makes a contract for each \( i \in N^t \) based on the vector of reports \( \gamma^t \in \Theta^t \equiv \prod_{i \in N^t} \Theta_i \) and history up to \( t \):

\[
    h_t = (N^0; \gamma^0, z^0; \gamma^1, z^1, N^2; \ldots; \gamma^{t-1}, z^{t-1}, N^t).
\]

Let \( \mathcal{H}_t \) be the set of possible history at \( t \). A mechanism is denoted by \( \{z^t\}_{t=0}^\infty \), where

\[
    z^t : \Theta^t \times \mathcal{H}_t \rightarrow Z^t.
\]

A mechanism is feasible if \( z^t = \emptyset \) for all \( i \notin N^t \) and if \( \sum_{i \in N^t} a_i(z^t_i) \leq K_i \).

For a profile of reports \( \gamma^t = (\gamma^t_j)_{j \in N^t} \) at \( t \), an entrant \( i \) at \( t \) earns payoff

\[
    u_i(\gamma^t, \theta, h_t) = a_i(z_i^t(\gamma^t, h_t))\mathbb{I}_{\{m_i(z_i^t) \geq l_i\}} V_i - p_i(z_i^t(\gamma^t, h_t)),
\]

where \( \mathbb{I} \) denotes the indicator function that is 1 if the associated condition holds. Let \( U_i(\theta^t, h_t) \equiv u_i((\theta_i, \theta^t_{-i}), \theta_i, h_t) \), which indicates \( i \)’s payoff when every entrant at \( t \) reports true information.

Given a mechanism, a bidder’s strategy is a mapping \( \gamma_i : \Theta_i \times \mathcal{H}_t \rightarrow \Theta_i \). Given that the others report their true types, agent \( i \)’s interim expected payoff is

\[
    \pi_i(\gamma_i^t, \theta, h_t) = \mathbb{E}_i[u_i(\gamma_i^t, \theta_{-i}, \theta_i, h_t)].
\]

Each agent’s expected payoff at \( t \) is also written as

\[
    \pi_i(\gamma_i^t, \theta_i, h_t) = \alpha_i(\gamma_i^t, l_i, h_t)V_i - q_i(\gamma_i^t, h_t),
\]

where

\[
    \alpha_i(\gamma_i^t, l_i, h_t) = \mathbb{E}[a_i(z_i^t(\gamma_i^t, \theta^t_{-i}, h_t))\mathbb{I}_{\{m_i(z_i^t) \geq l_i\}}]
\]

and

\[
    q_i(\gamma_i^t, h_t) = \mathbb{E}[p_i(z_i^t(\gamma_i^t, \theta^t_{-i}, h_t))].
\]

Let \( \Pi_i(\theta_i, h_t) \equiv \pi_i(\theta_i, \theta_i, h_t) \), which denotes the expected payoff when \( i \) reports his true information. Incentive compatibility and individual rationality are defined in a standard manner.

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\(^6\)The expectation is taken conditionally on \( h_t \). Henceforth, we drop subscript \( t \) when no confusion arises.
**Definition 1** A dynamic direct mechanism is *(Bayesian) incentive compatible* if for all $i$, all $t$, all $h_t$, all $\theta_t$, and all $\gamma_t^i$,

$$\Pi_i(\theta_t, h_t) \geq \pi_i(\gamma_t^i, \theta_t, h_t).$$

In addition, a mechanism is *individually rational* if for all $i$, all $t$, all $h_t$, and all $\theta_t$, $\Pi_i(\theta_t, h_t) \geq 0$.

**Definition 2** A dynamic direct mechanism is *dominant strategy incentive compatible* if for all $i$, all $t$, all $h_t$, all $\theta_t$, and all $\gamma_t^i$,

$$U_i(\theta_t^i, h_t) \geq u_i((\gamma_t^i, \theta_t^{e-i}_t), \theta_t, h_t).$$

**Remark 1** In a dynamic environment, it is hard to construct a dominant strategy incentive compatible mechanism in general. This is because future agents might play a strange strategy contingent on history, which prevents truth-telling from being optimal. Most of the related studies consider ex post incentive compatibility or periodic ex post incentive compatibility (Bergemann and Valimaki, 2010). However, in our model, the payoff of an entrant $i$ at $t$ is determined only by the current entrants and past agents, and future events are independent from $i$’s payoff. Thus, we can establish dominant strategy incentive compatible mechanisms as we do later.

### 3 Characterization

We characterize incentive compatible mechanisms. In this section, we restrict the domain of mechanisms such that each contract term is set to agent’s reported period type.

**Assumption 1** Every dynamic direct mechanism satisfies $m_i(z_i(\theta_t^i, h_t)) = l_t$ for all $t$, all $i$, all $\theta_t$, and all $h_t$.

This restriction would be natural.\textsuperscript{7} As we will verify later, it does not reduce the efficiency in an efficient mechanism design or the optimality in revenue maximization.

\textsuperscript{7}In preceding studies such as Parkes and Singh (2003), Hajiahayi et al. (2005), and Pai and Vohra (2011), there is explicitly no assumption corresponding to Assumption 1. In their model, however, agents determine their arrival and departure times *outside of* a mechanism. Thus, we can interpret a similar assumption is implicitly imposed.
Note that since each agent reports his type only once at his unmanipulatable entry period, a dynamic direct mechanism is just static for each agent. Thus, if we suppose that each agent never manipulates his period type, incentive compatibility is characterized in a standard manner (Myerson, 1981). Abusing notation, let $\alpha_i(V_i, l_i, h_t) \equiv \alpha_i((V_i, l_i), l_i, h_t)$.

**Lemma 1 (Myerson, 1981)** Suppose each agent is restricted to report the true period type. Then $\gamma_i(\theta_i, h_t)$ maximizes $i$’s expected payoff if and only if

1. $\alpha_i(V_i, l_i, h_t)$ is weakly increasing in $V_i$ for every $l_i$ and $h_t$, and
2. $\Pi_i(V_i, l_i, h_t) = \Pi_i(0, l_i, h_t) + \int_0^{V_i} \alpha_i(\nu, l_i, h_t) d\nu$ for all $V_i$, $l_i$, and $h_t$.

Now we consider that agents may manipulate both valuations and period types. Lemma 1 implies that even in the presence of the period type, the payment rule in an incentive compatible mechanism is determined by the envelope formula. Incentive compatibility requires several conditions on the allocation rule for different period types. Since each agent earns no profit when he obtains the object for less than a period of his type, we need to take care that agents never report a longer period type: for $l_i < l_i'$, $\Pi_i(V_i, l_i, h_t) \geq \Pi_i(V_i, l_i', h_t)$. That means an agent’s payoff is weakly decreasing in $l_i$. The following theorem is our first main theorem of the paper.

**Theorem 1** Suppose Assumption 1. A dynamic direct mechanism is incentive compatible if and only if for all $i$ and all $h_t$,

1. $\alpha_i(V_i, l_i, h_t)$ is weakly increasing in $V_i$ for every $l_i$,
2. $\int_0^{V_i} \alpha_i(\nu, l_i, h_t) d\nu$ is weakly decreasing in $l_i$ for every $V_i$, and
3. there exists a “worst-case payoff” $\Pi_i(h_t)$ independent from $l_i$, and

$$\Pi_i(V_i, l_i, h_t) = \Pi_i(h_t) + \int_0^{V_i} \alpha_i(\nu, l_i, h_t) d\nu$$

(7)

for all $V_i$ and all $l_i$.

**Proof.** See Appendix.

Note that $\Pi_i(0, l_i, h_t)$ does not depend on $l_i$. Preceding studies (Hajiaghayi et al., 2005; Pai and Vohra, 2011) also derive a similar condition with ours in a model
of strategic arrival and departure. Hajiaghayi et al. (2005) investigate incentive compatible mechanisms when each agent wants to obtain at most one unit of the object at most one period during the stay.\footnote{Hajiaghayi et al. (2005) in fact mischaracterize the incentive compatibility. A correct result is provided by Parkes (2007).} Pai and Vohra (2011) characterize incentive compatible mechanisms in the case where goods are durable and agents’ preferences are the same as Hajiaghayi et al. (2005). These studies characterize incentive compatibility along with the binding individual rationality; i.e., they assume the expected payoff when value is 0 is 0. It is worthy noting that Theorem 1 does not use the individual rationality.

Preceding studies introduce a stronger notion of monotonicity of the allocation policy, which requires that the allocation for an agent is monotone in both valuation and length of the stay. We introduce a similar concept in our model. Given a mechanism \{z^t\}, an allocation policy is denoted by \(a^t: \Theta^t \times \mathcal{H}_t \rightarrow [0, 1]^{N_t}\) and

\[
a^t_i(\theta^t, h_t) \equiv a^t_i(z^t_i(\theta^t, h_t)).
\]

**Definition 3** An allocation policy for \(i\) is said to be monotone if \(a^t_i(\theta^t, h_t)\) is weakly increasing in \(V_i\) and weakly decreasing in \(l_i\).

Note that incentive compatibility does not require that the allocation policy is monotone in period type. Theorem 1 immediately shows that any monotone allocation policy is implementable.

**Corollary 1** Suppose Assumption 1. If an allocation policy is monotone, then there exists a payment scheme that induces the incentive compatibility.

### 3.1 Dominant Strategy Incentive Compatibility

Bikhchandani et al. (2006) show that in multi-dimensional model, a deterministic allocation rule is implemented in dominant strategy if and only if it is weakly monotone. Indeed, our model can be included by the model of Bikhchandani et al. (2006), and any monotone allocation policy is implemented in dominant strategy when we focus on deterministic allocation policies. The following proposition is provided as a corollary of Bikhchandani et al. (2006), but we provide the proof independently in Appendix.
Proposition 1 A deterministic allocation policy is implemented in dominant strategy if it is monotone.

4 An Efficient Mechanism

In this section, we establish an efficient incentive compatible mechanism, which is an extension of the well-known Vickrey-Clarke-Groves mechanism. We define the dynamic VCG mechanism.

4.1 Social Welfare

We start by defining the socially optimal allocation policy. The state of the world at $t$ is $(\theta^t, h^t)$. The socially optimal policy is given by maximizing the sum of the expected discounted values. The socially optimal welfare at $t$, $W(\theta^t, h^t)$, is written recursively as

$$W(\theta^t, h^t) = \max_{a^t, m^t} \sum_{i \in N^t} a^t_i \mathbb{1}_{\{m^t_i \geq l_i\}} V_i + \delta \mathbb{E} W(\theta^{t+1}, h_{t+1})$$

s.t. $a^t_i \in [0, 1]$, $\forall i \in N^t$

$$\sum_{i \in N^t} a^t_i \leq K_t,$$

$h_{t+1} = (h_t, \theta^t, \bar{a}^t, m^t, N^{t+1})$.

Let $y^* = \{a^{t^*}, m^{t^*}\}_{t^*}^{\infty}$ be a solution of the problem (8).

Note that the efficiency is defined under the assumption that a seller makes irrevocable simple long-term contracts. The social welfare would be better off by introducing renegotiations and history-contingent contracts.

We have several observations on the social optimization problem (8) and its solutions. First, with the assumption of the i.i.d. types and entrants, the past information on entrants $N^\tau$ and their reports $\gamma^\tau$ for $\tau < t$ is independent from the optimization problem. Among the past contracts, only the residual periods of objects $x^t$ matters in the problem. We can reduce the state of the world to $(\theta^t, x^t)$.

Second, there is a socially optimal policy that is deterministic. By assumption on the domain of contracts, the objective function of (8) is linear in $a^t$.

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9 Abusing notations, we assume that the information $N^t$ is included in $\theta^t$. 
Third, in an efficient policy, $m^*_i = l_i$ whenever $a^*_i > 0$. Agent $i$ makes no profit if $a_i = 1$ and $m_i < l_i$. Hence, it is clear that if $a^*_i = 1$, then $m^*_i \geq l_i$. In addition, agent $i$ produces no marginal profit even if an object is assigned for more than $l_i$ periods. If $m^*_i > l_i$ in an efficient policy $y^*$, then the seller achieves at least the same social welfare as $y^*$ by modifying it to $\tilde{y}$ such that $\tilde{m}_i = l_i$, and that the seller keeps an object for $(m^*_i - l_i)$ periods after the expiration of the contract for $i$. Thus, Assumption 1 is redundant when we consider a socially optimal mechanism. These observations are summarized as follows.

**Observation 1** The socially optimal allocation policy at $t$ depends only on $(\theta^t, x^t)$. There is a socially optimal allocation policy $y^*$ that is deterministic and satisfies $m^*_i = l_i$.

Henceforth, we focus on deterministic mechanisms with $m_i^* = l_i$. Then, in the social optimization problem, we need to find a deterministic allocation policy $a^* : \Theta^t \times X \to \{0, 1\}^{N_i}$. Note that given $x^t$, the state of the next period $x^{t+1}$ is determined by the allocation at $t$, $y^t = (a^t, l^t)$. Let $G$ be the state transition function: $x^{t+1} = G(y^t, x^t)$. Then, we can rewrite the social optimization problem as

$$W(\theta^t, x^t) = \max_{y^t} \sum_{i \in N^t} a^t_i V_i + \delta EW(\theta^{t+1}, x^{t+1})$$

s.t. $a^t_i \in \{0, 1\}$, $m^t_i = l_i$, $\sum_{i \in N^t} a^t_i \leq K_i$, $x^{t+1} = G(y^t, x^t)$.

Similarly, we define the social optimization problem without $i \in N^t$. Let $W_{-i}(\theta^t, x^t)$ be the maximized social welfare when $i$ is excluded. The efficient allocation policy at $t$ without $i \in N^t$ is denoted by $\hat{y}^t_{-i}$.

### 4.1.1 Further Observations on the Socially Optimal Policy

We have further observations on an efficient allocation policy. It is obvious that the efficient allocation policy $a^*_i(\theta^t, x^t)$ must be weakly increasing in $V_i$. Also, there exists a efficient policy for $i$ that is weakly decreasing in $l_i$.
Lemma 2 There is an efficient allocation policy that is monotone.

Proof. See Appendix.

4.2 The Dynamic VCG Mechanism

Since the efficient allocation policy is monotone, it is implementable in dominant strategy. The socially optimal allocations are implemented via dynamic Vickrey-Clarke-Groves (VCG) mechanism. As defined in the static VCG mechanism, we introduce the marginal contribution of agents. The marginal contribution of \( i \) with state of the world \((\theta^t, x^t)\) is defined by

\[
C_i(\theta^t, x^t) \equiv W(\theta^t, x^t) - W_{-i}(\theta^t, x^t).
\]

Let \( x^{t+1*} = G(y^{t*}, x^t) \), which indicates the allocation state at \( t + 1 \) given an efficient policy at \( t \). Similarly, let \( \hat{x}^{t+1}_{-i} = G(\hat{y}^{t}_{-i}, x^t) \), which denotes the allocation state at \( t + 1 \) given the efficient policy \( \hat{y}^{t}_{-i} \) when \( i \) is excluded at \( t \). Note that agent \( i \) makes no report after \( t + 1 \). Hence, we have

\[
W_{-i}(\theta^t, x^t) = \sum_{j \in N \setminus \{i\}} \hat{a}_{-i,j} V_j + \delta \mathbb{E}W(\theta^{t+1}_{-i}, \hat{x}^{t+1}_{-i}).
\]

The payment scheme for the dynamic VCG mechanism is defined so that each agent earns his marginal contribution under the current state. Hence, the (total) monetary transfer \( p^*_i(\theta^t, x^t) \) in the dynamic VCG mechanism is defined as follows: for \( i \notin N^t \), \( p^*_i(\theta^t, x^t) = 0 \), and for \( i \in N^t \),

\[
p^*_i(\theta^t, x^t) = \sum_{j \in N \setminus \{i\}} (\hat{a}_{-i,j} - a^*_j) V_j + \delta (\mathbb{E}W(\hat{x}^{t+1}_{-i}) - \mathbb{E}W(x^{t+1*})), \tag{10}
\]

where \( \mathbb{E}W(x) \equiv \mathbb{E}W(\theta^{t+1}, x) \).

Definition 4 A dynamic direct mechanism \( \{z^*_t\} = \{y^*_t, p^*_t\}_{t=0}^\infty \) is said to be the dynamic VCG mechanism if \( a^* \) is the efficient allocation policy, \( m^*_i = l_i \) for all \( i \), and if the payment is determined by (10).

Theorem 2 The dynamic VCG mechanism \( \{z^*_t\} \) is dominant strategy incentive compatible and individually rational. The equilibrium payoff of \( i \) entering at \( t \) is \( U_i(\theta^t, h_i) = C_i(\theta^t, x^t) \).
Proof. Suppose \( i \in N^t \). For any \( \theta_i \) and reports of the others \( \theta_{-i}^t \), \( i \)'s ex post payoff given a report \( \gamma_i \) is

\[
u_i((\gamma_i, \theta_{-i}^t), \theta_i, h_t) = a_i^*(\gamma_i, \theta_{-i}^t, x^t)V_i - p_i^*(\gamma_i, \theta_{-i}^t, x^t)
\leq \sum_{N^t} a_j(\gamma_i, \theta_{-i}^t, x^t)V_j + \delta E W(G(y^*(\gamma_i, \theta_{-i}^t, x^t), x^t)) - W_{-i}(\theta^t, x^t)
\leq \sum_{N^t} a_j^*(\theta^t, x^t)V_j + \delta E W(G(y^*(\theta^t, x^t), x^t)) - W_{-i}(\theta^t, x^t)
= U_i(\theta^t, h_t).
\]

(11)

Therefore, truth-telling is optimal, and the associated payoff is \( W(\theta^t, x^t) - W_{-i}(\theta^t, x^t) = C_i(\theta^t, x^t) \). Since \( W(\theta^t, x^t) \geq W_{-i}(\theta^t, x^t) \) by definition, \( \{z^*\} \) is ex post individually rational. ■

Note that Bergemann and Valimaki (2010) and Parkes and Singh (2003) provide similar mechanisms that extend the VCG mechanism to dynamic environments. Bergemann and Valimaki (2010) can include agents’ preferences in our model. In their “dynamic pivot mechanism,” an allocation at each period is determined by the current profile of types, and the set of available allocations does not change over time. However, in our model, the set of available allocations is constrained by the past contracts as in the case of durable goods. If we allow complete contracts contingent on history, the dynamic pivot mechanism achieve the full efficiency.

It is also worthy noting that the dynamic VCG mechanism is dominant strategy incentive compatible. The dynamic pivot mechanism and Parkes and Singh’s (2003) mechanism are not dominant strategy incentive compatible but periodic ex post incentive compatible. This is because the payoff of a current entrant is affected by the behavior of future agents, who might play a strange history-contingent strategy. On the other hand, an agent’s payoff is determined only by the past and current agents in our case. Therefore, the dynamic VCG mechanism of this paper satisfies a stronger notion of incentive compatibility.

4.3 Single Object Case

We consider a simple case of \( K = 1 \). The set of the state of the world is \( X = \{0, 1, \ldots, L - 1\} \). For any \( t \) such that \( x^t \geq 1 \), we have \( K_t = 0 \) and \( a_t^* = 0 \) for all
\[ i \in N^t. \text{ Therefore, for } x^t = r \geq 1, \]
\[ W(\theta^t, r) = \delta^r E W(\theta^{t+r}, 0) \equiv \delta^r \bar{W}. \]

An allocation problem is considered only when \( x^t = 0. \) The social optimization problem for \( x^t = 0 \) is described as
\[ W(\theta^t, 0) = \max_{i \in N^t \cup \{0\}} V_i + \delta^i \bar{W}, \tag{12} \]
where agent 0 is a dummy agent (or the seller), whose type is \( \theta_0 = (0, 1). \) The agent who maximizes the value \( V_i + \delta^i \bar{W} \) win the object, or the seller assigns nothing to each agent and waits for the next period.

The efficient policy is expressed in a per-period representation. Let \( v_i \) be \( i's \) value per period; that is,
\[ v_i \equiv \frac{1}{1 + \delta + \ldots + \delta_{i-1}} V_i = \frac{1 - \delta}{1 - \delta^i} V_i. \]
The average welfare is denoted by \( \bar{w} \equiv (1 - \delta) \bar{W}. \) Then, for each period \( t \) such that \( x^t = 0, \) the efficient policy solves the problem
\[ \max_{i \in N^t \cup \{0\}} (1 - \delta^i) v_i + \delta^i \bar{w}, \tag{13} \]
Suppose agent \( i \) wins an object at \( t, \) and that \( j \in N^t \cup \{0\} \) is the second-highest agent who maximizes the social welfare. Then \( i's \) payment in the dynamic VCG mechanism is
\[ p^*_i(\theta^t) = V_j + (\delta^j - \delta^i) \bar{W}. \]

To fully characterize the efficient allocation policy, we need to compute the expected social welfare \( \bar{W} \) or \( \bar{w}. \) It is derived by a standard manner of dynamic programming such as value function iteration.

4.4 Multi-Object Case

When there are multiple objects, it becomes difficult to obtain the efficient allocation policy. In this subsection, we suppose that \( K = 2 \) and \( L = 2 \) and consider the efficient allocation policy. Agents stay for at most 2 periods. Then, the set of possible residual objects is given by \( X = \{(0,0), (1,0), (1,1)\}. \) Instead of that, we can simply let \( x^t \) denote the number of objects reserved for incumbent agents: \( x^t \in \{0,1,2\}. \)
First, consider the case of the 2-period horizon. Suppose that all agents are short-stay agents at period 2. At period 1, both long- and short-stay agents may exist, and agent $i$'s type is denoted by $\theta_i = (v_i, 1)$ or $((1 + \delta)v_i, 2)$, where $v_i$ is the value per period. Let $v^{(j)}_S$ indicate the $j$-th highest order statistics among agents with $l_i = 1$ at the current period. In addition, let $v^{(j)}_L$ be the $j$-th highest order statistics per period among agents with $l_i = 2$ at the current period.

At period 2, the efficient allocation policy is simply such that non-reserved objects are assigned to the agents with the highest values. Hence, the social welfare at period 2 is given by

$$W_2(\theta_2, 0) = v^{(1)}_S + \delta W(0), \quad W_2(\theta_2, 1) = v^{(1)}_S + \delta(v^{(1)}_L + W(1)), \quad W_2(\theta_2, 2) = 0.$$ 

Consider period 1 with $x_1 = 0$. Given the social welfare at period 2, the efficient allocation policy at period 1 solves the following:

$$\max \begin{cases} v^{(1)}_S + v^{(2)}_S + \delta E[v^{(1)}_S + v^{(2)}_S], \\ v^{(1)}_S + v^{(1)}_L + \delta(v^{(1)}_L + E[v^{(1)}_S]), \\ (1 + \delta)(v^{(1)}_L + v^{(2)}_L) \end{cases}.$$

Hence, a long-stay agent $i$ wins if and only if

$$(1 + \delta)v_i \geq \begin{cases} v^{(2)}_S + \delta E[v^{(2)}_S] & \text{if } v_i = v^{(1)}_L, \\ v^{(1)}_S + \delta E[v^{(1)}_S] & \text{if } v_i = v^{(2)}_L. \end{cases}$$

Conversely, a short-stay agent $i$ at period 1 wins if and only if

$$v_i \geq \begin{cases} (1 + \delta)v^{(2)}_L - \delta E[v^{(1)}_S] & \text{if } v_i = v^{(1)}_S, \\ (1 + \delta)v^{(1)}_L - \delta E[v^{(2)}_S] & \text{if } v_i = v^{(2)}_S. \end{cases}$$

Note that the continuation value for short-stay agents depends on the rank among all the short-stay agents. Therefore, the efficient allocation policy becomes complicated as the time horizon gets long.

Consider the case of the infinite horizon. For $x \in \{0, 1, 2\}$, let $W(x) \equiv EW(\theta^{t+1}, x)$. Note that $W(2) = \delta W(0)$. The efficient allocation policy solves the following problems:

$$W(\theta^t, 0) = \max \begin{cases} v^{(1)}_S + v^{(2)}_S + \delta W(0), \\ v^{(1)}_S + v^{(1)}_L + \delta(v^{(1)}_L + W(1)), \\ (1 + \delta)(v^{(1)}_L + v^{(2)}_L) + \delta^2 W(0) \end{cases},$$

and

$$W(\theta^t, 1) = \max \begin{cases} v^{(1)}_S + \delta W(0), \\ (1 + \delta)v^{(1)}_L + \delta W(1) \end{cases}.$$
Revenue Maximization

In this section, we provide a revenue maximizing mechanism in a similar manner with Myerson (1981). We introduce the virtual valuation, which is denoted by $\phi(\theta_i)$, and

$$\phi(\theta_i) = V_i - \frac{1 - F(V_i|l_i)}{f(V_i|l_i)}.$$ (14)

From Theorem 1 and some calculations, we have the expected revenue from agent $i$, $E_t[q_i(\theta_i, h_t)]$ as follows:

$$E_t[q_i(\theta_i, h_t)] = -\Pi_i(h_t) + \int_{\theta_i} \phi(\theta_i) \alpha_i(\theta_i, h_t) f(\theta_i) d\theta_i.$$ 

Hence, the expected revenue raised at $t$ is given by

$$E_t \left[ \sum_{i \in N^t} q_i(\theta_i, h_t) \right] = - \sum_{i \in N^t} \Pi_i(h_t) + \int_{\theta^t} \sum_{i \in N^t} [\phi(\theta_i) a_i(\theta^t, h_t)] f^t(\theta^t) d\theta^t,$$

where $f^t(\theta^t) = \prod_{i \in N^t} f(\theta_i)$. Therefore, we have the revenue maximizing problem as follows.

**Proposition 2** Suppose Assumption 1. A revenue maximizing policy is a solution of the following problem:

$$\max_{\{a^s\}_{s=t}^\infty} E_t \left[ \sum_{s=t}^\infty \delta^{s-t} \left[ \sum_{i \in N^s} \left( -\Pi_i(h_s) \right) + \sum_{i \in N^s} a_i^s \phi(\theta_i) \right] \right]$$

s.t. $\alpha_i(\theta_i, h_s)$ is weakly increasing in $V_i$,

$$\int_0^V \alpha_i(\nu, l_i, h_s) d\nu \text{ is weakly decreasing in } l_i,$$

$$\Pi_i(h_s) \geq 0,$$

$$a_i^s \in [0, 1],$$

$$\sum_{i \in N^s} a_i^s \leq K_s.$$ (15)

Obviously, we have $\Pi_i(h_s) = 0$ for all $i$ and all $h_s$ in an optimal mechanism.
Then, let us consider a relaxed problem (in a recursive form) below:

\[
R(\theta^t, h_t) = \max_{a^t} \sum_{i \in N^t} a^t_i \phi(\theta_i) + \delta E R(\theta^{t+1}, h_{t+1}) \\
\text{s.t. } a^t_i \in [0, 1], \sum_{i \in N^t} a^t_i \leq K_t, h_{t+1} = (h_t, \theta^t, \bar{a}^t, m^t, N^{t+1}),
\]

(16)

where \( R(\theta^t, h_t) \) denotes the maximized total virtual value function. This problem is the same as the social optimization problem (8), except that the valuation types are replaced with the virtual valuations. Consider a solution of the virtual social optimization problem (16). From Lemma 2, there is a deterministic solution \( a^{**} \) that is weakly increasing in \( \phi \).

In order that a solution of the relaxed problem (16) also solves the original problem (15), we need a regularity condition. A sufficient condition is that the virtual valuation \( \phi \) is increasing in \( V_i \) and weakly decreasing in \( l_i \). We impose the following monotone hazard rate condition.

**Assumption 2** The conditional hazard rate \( \frac{f(V_i|l_i)}{1 - F(V_i|l_i)} \) is weakly increasing in \( V_i \) and weakly decreasing in \( l_i \).\(^{10}\)

Roughly speaking, Assumption 2 requires that the longer the period type gets, the higher valuation an agent has. Indeed, when the assumption holds, the distribution \( F(\cdot|l_i) \) stochastically dominates \( F(\cdot|l'_i) \) for \( l'_i < l_i \). This would likely be the cases in real situations. An example of a distribution structure satisfying Assumption 2 is following: The value per period \( v_i \) and the period type \( l_i \) are independently drawn, and the distribution of \( v_i \) satisfies increasing monotone hazard rate.

**Definition 5** A type distribution satisfies the independent value per period (IVPP) if a pair of random variables \((v_i, l_i)\) are independently drawn from a density \( f(v_i, l_i) = f^v(v_i)f^l(l_i) \) on \([0, \bar{v}] \times \{1, \ldots, L\}\), and type is generated by \( \theta_i = (1 - \frac{\delta_i}{\delta}, v_i, l_i) \).

Lemma 2 shows that an efficient allocation policy \( a^*_t(\theta^t, x^t) \) is weakly decreasing in \( l_i \). Under Assumption 2, in a solution of the relaxed problem, \( \alpha_i \) is decreasing in \( l_i \). This is because when \( l_i \) gets shorter with \( V_i \) constant, then the virtual value

\(^{10}\)Pai and Vohra (2011) impose a similar hazard rate condition.
\( \phi \) weakly increases. Since the efficient policy is weakly increasing in valuation type and weakly decreasing in period type, the solution for (16) must be monotone.

**Theorem 3** Under Assumption 2, the allocation policy derived by a relaxed problem (16) is monotone and maximizes the expected revenue for the seller.

**Remark 2** The revenue-maximizing allocation policy \( a^{**} \) is in general very different from the efficient allocation policy \( a^* \) even if type distribution is i.i.d. In the standard static optimal auction design, the optimal allocation policy is constrained efficient in the sense that the agent who is awarded an object has the highest value. However, in our model, the virtual value \( \phi \) depends on both valuation type and period type.

### 5.1 Single-Object Case

Now we consider the case of \( K = 1 \). Similarly to the efficient mechanism, an allocation problem is considered only when \( x_t = 0 \). Since

\[
R(\theta_t, r) = \delta^r \mathbb{E} R(\theta_{t+r}, 0) \equiv \delta^r \bar{R},
\]

the virtual social optimization problem is expressed as

\[
R(\theta_t, 0) = \max_{i \in N_t \cup \{0\}} \phi(\theta_i^t) + \delta_i \bar{R}, \tag{17}
\]

where 0 denotes a dummy agent, who has the *reservation type* \( \theta_0 = (V, 1) \). The reservation value \( V \in (0, \bar{V}) \) is determined by \( \phi(V, 1) = 0 \).

In an incentive compatible mechanism, winning agents pay the threshold value in order to win given the others’ types. Suppose agent \( i \) wins at \( t \) and agent \( j \) is the second highest: \( j \in \text{arg max}_{j' \in N_t \cup \{0\}} \phi(\theta_{j'}) + \delta^{j'} \bar{R} \). Let \( \phi^{-1}(\cdot | l_i) \) be the inverse function of \( \phi(\cdot | l_i) \) given \( l_i \) fixed. Then agent \( i \) pays the total amount of

\[
p^{**}_i(\theta^t_i) = \phi^{-1}(\phi(\theta_j) + (\delta_j - \delta_i) \bar{R} | l_i).
\]

### 5.2 IVPP and Volume Premium

In this section, we investigate the revenue-maximizing (or optimal) allocation policy and pricing rule. We assume IVPP and the value per period \( v_i \) is drawn from an i.i.d. density function \( f > 0 \) on \([0, 1]\). The distribution of \( v_i \) satisfies the monotone hazard rate condition. For simplicity, we suppose \( K = 1 \) and \(|N^t| \leq 1 \) for all \( t \). That
is, at each period, at most one agent comes to a mechanism. Consider that an agent enters with probability \( \lambda \in (0, 1] \) at each period.

Under the assumption of IVPP, we have the virtual valuation by a simple calculation

\[
\phi(\theta_i) = \frac{1 - \delta^l_i}{1 - \delta} \tilde{\phi}(v_i),
\]

where \( \tilde{\phi}(v_i) = v_i - \frac{1 - F(v_i)}{f(v_i)} \). Hence, as we have seen in the previous subsection, agent \( i \) is awarded (given \( K_t = 1 \)) if and only if

\[
\tilde{\phi}(v_i) \geq \delta^l_i \frac{1 - \delta^l_i - 1}{1 - \delta^l_i} \bar{r},
\]

where \( \bar{r} \equiv (1 - \delta) \bar{R} \) is the average revenue per period.

Let \( \hat{\phi}(l_i) \) be the threshold value per period, that satisfies (18) with equality. Note that \( \hat{\phi}(1) \) is simply the reservation value that satisfies \( \tilde{\phi}(\hat{\phi}(1)) = 0 \). For \( l_i \geq 2 \), the RHS of (18) is positive and increasing in \( l_i \). Therefore, the threshold value \( \hat{\phi}(l_i) \) is increasing in \( l_i \). An optimal incentive compatible mechanism is such that when \( i \) is awarded for \( l_i \) periods, then he pays \( \hat{\phi}(l_i) \) per period during the stay. This means that long-stay agents need to pay more than short-stay agents.

Since an agent with a high virtual value might come in a near future, a long-stay agent is required to have a strictly positive virtual value. Hence, a long-stay agent needs to have a higher value than a short-stay agent in order to be awarded. This seems a little contrary to “long-stay discount” often observed in hotel reservations. Although the threshold value is decreasing in the arrival probability \( \lambda \), higher prices for long-stay agents hold even if \( \lambda \) is very small.

Importantly, in the optimal mechanism, a long-stay agent would not like to accept a long term contract but make a short-term contract at each period, which is excluded in our model. This implies that the result would change if agents are allowed to make a new contract after the expiration of a contract. We need to consider a stronger notion of incentive compatibility to deal with this kind of deviations.\(^{11}\)

6 Conclusion

We formulate a model of the dynamic allocation problem in which agents want to obtain an object for periods of time. We characterize incentive compatible mecha-\(^{11}\)If current outcome is determined by all the history in a mechanism, this kind of deviation can easily be overcome.
nisms, and construct an efficient mechanism and a revenue-maximizing mechanism in a domain where the seller offers simple long-term contracts. Any allocation policy that is increasing in valuation type and decreasing in period type is implementable. The socially optimal allocation policy satisfies the monotonicity of both valuation and period types, and the dynamic VCG mechanism achieves dynamic efficiency in a dominant strategy equilibrium. With a monotone hazard rate condition, the revenue maximizing allocation policy maximizes the total virtual valuation, similarly to Myerson (1981).

There are several avenues for future research. First, we need to investigate the efficient or optimal allocation policy in detail. Although we focus on simple contracts, it is difficult to specify the allocation policy when there are multiple objects. Second, it would be interesting and important to consider the case of fixed population. In this paper, we focus on the case where agents come and report a message only once. However, in real applications, agents often make new contracts after expiration, as considered by Bergemann and Valimaki (2010), Pavan et al. (2009), and Kakade et al. (2011). We would need to introduce a dynamic structure to both the seller and agents in order consider various practical situations.

A Proofs

A.1 Proof of Theorem 1

(Only if part.) Suppose a mechanism is incentive compatible. Then, we have

\[ \alpha_i(V_i, l_i, h_t)V_i - q_i(V_i, l_i, h_t) \geq \alpha_i(V_i', l_i, h_t)V_i - q_i(V_i', l_i, h_t), \]

hence,

\[ (\alpha_i(V_i, l_i, h_t) - \alpha_i(V_i', l_i, h_t))V_i \geq q_i(V_i, l_i, h_t) - q_i(V_i', l_i, h_t). \]

Similarly, we have

\[ (\alpha_i(V_i, l_i, h_t) - \alpha_i(V_i', l_i, h_t))V_i' \leq q_i(V_i, l_i, h_t) - q_i(V_i', l_i, h_t). \]

Therefore,

\[ (\alpha_i(V_i, l_i, h_t) - \alpha_i(V_i', l_i, h_t))V_i' \leq (\alpha_i(V_i, l_i, h_t) - \alpha_i(V_i', l_i, h_t))V_i. \]

Therefore, \( V_i' < V_i \) implies \( \alpha_i(V_i', l_i, h_t) \leq \alpha_i(V_i, l_i, h_t). \)
From the standard argument of the envelope theorem (Milgrom and Segal, 2002), if \( V_i \in \arg \max_{\nu \in [0,\bar{V}]} \alpha_i(\nu, l_i, h_t) V_i - q_i(\nu, l_i, h_t) \), then

\[
\frac{\partial \Pi_i(V_i, l_i, h_t)}{\partial V_i} = \alpha_i(V_i, l_i, h_t)
\]

almost everywhere, and

\[
\Pi_i(V_i, l_i, h_t) - \Pi_i(V_i', l_i, h_t) = \int_{V_i}^{V_i'} \alpha_i(\nu, l_i, h_t) d\nu.
\]  

(19)

Suppose \( l_i' > l_i \). Then, \( \alpha_i((V_i', l_i'), l_i, h_t) = \mathbb{E}[a_i(V_i', l_i', \theta_i[l_i' - 1], h_t) I_{\{m_i(\varepsilon_i') = l_i' \geq l_i\}}] = \alpha_i(V_i', l_i', h_t) \). Then incentive compatibility implies for all \( V_i \),

\[
\Pi_i(V_i, l_i, h_t) \geq \alpha_i((V_i', l_i'), l_i, h_t) V_i - q_i(V_i, l_i', h_t)
\]

\[
= \alpha_i(V_i, l_i', h_t) V_i - q_i(V_i, l_i', h_t)
\]

\[
= \Pi_i(V_i, l_i', h_t).
\]

(20)

From the envelope formula (19), (20) yields

\[
\Pi_i(0, l_i, h_t) + \int_0^{V_i} \alpha_i(\nu, l_i, h_t) d\nu \geq \Pi_i(0, l_i', h_t) + \int_0^{V_i} \alpha_i(\nu, l_i', h_t) d\nu.
\]

(21)

For \( V_i = 0 \), it also holds that \( \Pi_i(0, l_i, h_t) = -q_i(0, l_i, h_t) \). Incentive compatibility requires \(-q_i(0, l_i, h_t) \geq -q_i(0, l_i', h_t)\) for any \( l_i \) and \( l_i' \), thus that \(-q_i(0, l_i, h_t)\) does not depend on \( l_i \). Hence, \( \Pi_i(0, l_i, h_t) = \Pi_i(h_t) \) for all \( l_i \).

(If part.) Since \( \alpha_i \) is monotone, each agent’s expected payoff \( \pi_i( (\nu, l_i), (V_i, l_i), h_t) \) satisfies the single crossing condition on \((V_i, \nu)\). Given \( l_i \) and \( h_t \), a standard argument and Lemma 1 implies for all \( V_i' \),

\[
\alpha_i(V_i, l_i, h_t) V_i - q_i(V_i, l_i, h_t) \geq \alpha_i(V_i', l_i, h_t) V_i - q_i(V_i', l_i, h_t).
\]

(22)

Suppose \( l_i' > l_i \). Note that \( \alpha_i((V_i', l_i'), l_i, h_t) = \alpha_i(V_i', l_i', h_t) \). Hence, for any \( V_i' \),

\[
\alpha_i((V_i', l_i'), l_i, h_t) V_i - q_i(V_i', l_i', h_t)
\]

\[
= \alpha_i(V_i', l_i', h_t) V_i - q_i(V_i', l_i', h_t)
\]

\[
\leq \Pi_i(V_i, l_i', h_t)
\]

\[
\leq \Pi_i(V_i, l_i, h_t).
\]

(23)

The last inequality holds from the conditions 2 and 3.

The envelope condition (7) implies

\[
-q_i(V_i, l_i, h_t) = \Pi_i(h_t) + \int_0^{V_i} [\alpha_i(\nu, l_i, h_t) - \alpha_i(V_i, l_i, h_t)] d\nu
\]

\[
\leq \Pi_i(0, l_i, h_t),
\]

23
where inequality comes from the monotonicity of $\alpha_i$. Then, for all $V'_i$ and all $l'_i < l_i$,

$$\begin{align*}
\alpha_i((V'_i, l'_i), l_i, h_t) & = -q_i(V'_i, l'_i, h_t) \\
& \leq \Pi_i(0, l'_i, h_t) \\
& = \Pi_i(0, l_i, h_t) \\
& \leq \Pi_i(V_i, l_i, h_t). 
\end{align*}$$

The second equality comes from condition 3, and the last inequality is from the monotonicity and (7). □

### A.2 Proof of Proposition 1

Given a monotone policy $\{a^t\}$ with Assumption 1 and any period type $l_i$, define a threshold type $\hat{\theta}_i(l_i, h_t)$ as $\hat{\theta}_i = l_i$ and

$$\hat{\theta}_i = \inf\{\tilde{V}_i | a_i((\tilde{V}_i, l_i), \theta_{-i}, h_t) = 1\}. \quad (25)$$

Let $\hat{V}_i(\theta^L_{-i}, h_t) = \infty$ if such $\tilde{V}_i$ does not exist. Fix any $\theta^L_{-i}$ and $h_t$. Abusing notation, we drop $(\theta^L_{-i}, h_t)$ and use the notation $\hat{V}_i(l_i)$.

Note that $\hat{V}_i(l_i)$ is weakly increasing in $l_i$. If not, $\hat{V}_i(l_i) < \hat{V}_i(l'_i)$ for some $l_i > l'_i$. Suppose $\theta_{-i} = (\hat{V}_i(l_i) + \epsilon, l_i)$ and $\theta^L_{-i} = (\hat{V}_i(l_i) + \epsilon, l'_i)$. Then, by definition of $\hat{V}_i$, $a_i(\theta_{-i}) = 1$ for all $\epsilon > 0$ and $a_i(\theta^L_{-i}) = 0$ for $\epsilon \in (0, \hat{V}_i(l'_i) - \hat{V}_i(l_i))$, which is a contradiction.

Now, consider a payment policy, which is defined as

$$p_i(\theta^t, h_t) = \begin{cases} 
\hat{V}_i(l_i, \theta^L_{-i}, h_t) & \text{if } a_i = 1 \\
0 & \text{otherwise} 
\end{cases}$$

**Case 1:** $V_i < \hat{V}_i(l_i)$.

If $i$ reports the truth, he loses and the payoff is 0. When $i$ reports $\theta^t_i$ such that $l'_i \geq l_i$ and wins, then his payoff is

$$V_i - \hat{V}_i(l'_i) \leq V_i - \hat{V}_i(l_i) < 0.$$ 

Hence, it is not profitable. When $i$ reports $\theta^L_i$ such that $l'_i < l_i$ and wins, then his payoff is $-\hat{V}_i(l'_i) \leq 0$. Therefore, truth-telling is optimal.

**Case 2:** $V_i \geq \hat{V}_i(l_i)$.
When \( i \) reports truthfully, then he wins with a payoff \( V_i - \hat{V}_i(l_i) \geq 0 \). When \( i \) reports \( \theta_i' \) such that \( l_i' \geq l_i \) and wins, then his payoff is

\[
V_i - \hat{V}_i(l_i') \leq V_i - \hat{V}_i(l_i).
\]

When \( i \) reports \( \theta_i' \) such that \( l_i' < l_i \) and wins, then his payoff is \(-\hat{V}_i(l_i') \leq 0\). Therefore, truth-telling is optimal. \( \blacksquare \)

**A.3 Proof of Lemma 2**

Suppose that \( a_i^*(\theta^t, x^t) = 1 \) for some \( i \in N^t \) of \( \theta^t = (V_i, l_i) \) under \( (\theta^t, x^t) \). First, consider \( \theta_i' = (V_i', l_i) \) where \( V_i' > V_i \). Note that the maximum social welfare without \( i \) is independent from \( \theta_i \); \( W_{-i}(\theta'_i, \theta_{-i}', x^t) = W_{-i}(\theta^t, x^t) \). On the other hand, the maximized social welfare is at least the same as the value in the case where \( a^*(\theta^t, x^t) \) is assigned at \( t \). Hence,

\[
W(\theta'_i, \theta_{-i}', x^t) \geq V_i' + \sum_{j \in N^t \setminus \{i\}} a_j^*(\theta^t, x^t)V_j + \delta E W(\theta^{t+1}, G(y^*(\theta^t, x^t), x^t)) \\
> W(\theta^t, x^t) \\
\geq W_{-i}(\theta^t, x^t).
\]

Therefore, we have \( a_i^*(\theta'_i, \theta_{-i}', x^t) = 1 \).

Now consider \( l_i \geq 2 \) and \( \theta_i' = (V_i', l_i') \) where \( l_i' < l_i \). Suppose that the allocation at \( t \) is determined by \( a^*(\theta^t, x^t) \) and that the mechanism designer limits the supply of the objects to \( K_s - 1 \) from period \( t + l_i' \) to \( t + l_i - 1 \). Then, the social welfare must be the same as in the case of \( (\theta^t, x^t) \). Hence,

\[
W(\theta'_i, \theta_{-i}', x^t) > \max_{\{y^s\}_{t+l_i'}^{t+l_i-1}} \left\{ \sum_{j \in N^t} a_j^*(\theta^t, x^t)V_j + E \left[ \sum_{s=t+1}^{t+l_i-1} \delta^{s-t-1} \sum_{j \in N^s} a_j^s V_j + \delta^{t+l_i} E W(\theta^{t+l_i}, x^{t+l_i}) \right] | \Omega \right\} \\
= W(\theta^t, x^t) \\
\geq W_{-i}(\theta^t, x^t) = W_{-i}(\theta'_i, \theta_{-i}', x^t),
\]

where \( \Omega \) denotes the condition on the maximization. And,

\[
\Omega = \{(t+1 \leq s \leq t+l_i'-1) \sum_{j \in N^s} a_j^s \leq K_s, \}
\]

\[
(t+l_i' \leq s \leq t+l_i-1) \sum_{j \in N^s} a_j^s \leq K_s - 1,
\]

\[
x^{t+l_i} = G(y^{t+l_i-1}, x^{t+l_i-1}).
\]
The strict inequality must hold because there is a positive probability of arrival of an agent $j$ at $s \in [t + l'_i, t + l_i - 1]$ having a type $\theta_j = (V_j, 1)$. Therefore, we have $a^*_t(\theta', \theta_{t-1}, x^t) = 1$. ■

References


