The Informational Role of Prices and the Essentiality of Money in the Lagos–Wright Model

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Abstract

A major concern in modern monetary theory has been the development of models where money is essential and yet substantive issues can be analyzed in a tractable manner. At the center of this effort is the framework proposed by Lagos and Wright (2005), whose tractability depends on the fact that trade alternates between centralized and decentralized markets. A widespread view is that money is essential in the Lagos–Wright environment as long as agents can only observe prices in the centralized market. In contrast to this view, in this paper we show that the presence of centralized trading allows agents to use prices to sustain a non–monetary equilibrium that implements the first–best.

Key Words: Money, Centralized Markets, Essentiality.

JEL Codes: E40, C73, D82
1 Introduction

Modern monetary theory is based on the notion that one must be explicit about the underlying frictions that render money essential. Two frictions are considered to be particularly relevant for essentiality: limited commitment and limited record-keeping. In fact, it is commonly agreed that limited commitment and limited record-keeping are necessary for the essentiality of money.\(^1\) A less established, but still widespread, view is that as long as one is careful about how the distribution of preferences and technologies in the economy prevents the recurrence of double-coincidences, the absence of both commitment and record-keeping suffices to render money essential. In particular, it is believed that the essentiality of money does not hinge on whether exchange takes place in decentralized or centralized markets. This belief has granted much flexibility in the recent effort towards building models where money is essential and yet substantive issues can be analyzed in a tractable manner. At the center of this effort is Lagos and Wright (2005) (henceforth LW). The main contribution of LW is in constructing an environment where, unlike in the search models of money in the tradition of Kiyotaki and Wright (1989), money is divisible and the distribution of money holdings is degenerate. The key element of LW is that trade alternates between a centralized market and a decentralized market.\(^2\)

In this paper we examine whether in fact centralized trading has no implications for the essentiality of money in LW. We are not the first ones to pose this question. Aliprantis, Camera and Puzzello (2007a) (henceforth ACP) show that money can fail to be essential if individual actions are observable in the centralized market. Lagos and Wright’s response (see Lagos and Wright (2008)) is that in LW agents only observe prices in the centralized market, and thus ACP’s critique does not apply. Unlike ACP, we assume that agents only


\(^2\)An alternative to LW is Shi (1997), who also constructs an environment where the distribution of money holdings is degenerate. Shi considers a setting with a continuum of families, with each family populated by a continuum of agents. Agents search in a decentralized market and at the end of every period distribute their money holdings inside their families.
observe prices in the centralized market and examine whether this indeed guarantees the essentiality of money.

The environment we consider is the same as in LW but for two features. First, while LW assumes a continuum of agents, we consider large finite populations. In any model, the assumption of a continuum of agents is made for tractability and is justifiable if it has no substantive economic implications. To put it differently: “The rationale for using the continuum-of-agents model is that it is a useful idealization of a situation with a large finite number of agents, but if equilibria in the continuum model are radically different from equilibria in the model with a finite number of agents, then this idealization makes little sense” (Levine and Pesendorfer (1995), p. 1160). Thus, we want to ensure that the argument in Lagos and Wright (2008) does not hinge on the continuum population assumption.

Second, while in LW the centralized market is Walrasian, we model the centralized market as a non-cooperative game. More precisely, we model it as a market game along the lines of Shapley and Shubik (1977). This way, we maintain the centralized market as a large anonymous market where all agents observe the same price and prices are the only conduits of information. The main reason for taking the non-cooperative approach is that the assumption of price-taking behavior is understood to be an idealization of behavior in large markets where individual agents have little market power. Market games have been used in the literature to provide non-cooperative foundations for Walrasian markets (see Mas-Colell (1982) for a survey of this strand of research). Another reason for the non-cooperative approach is that if one wants to assess the conditions under which money is essential, one must consider whether agents have the incentive to follow alternative credit-like arrangements. The standard competitive model does not specify how payoffs are defined off the equilibrium path and thus it is ill-suited to check the feasibility of competing mechanisms of exchange.

The main result of the paper is that if agents are patient enough there exists a non-

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monetary equilibrium that implements the first–best. This result holds for any number of agents, and the lower bound on the discount factor does not depend on the population size. The intuition for our result is simple. When the number of agents is finite, individual actions in the centralized market have an impact on prices regardless of the population size. Thus, prices can convey information about individual deviations and this can be used to sustain cooperative behavior. Our main result stands in contrast to Araujo (2004), who shows that in the Kiyotaki–Wright environment (see Kiyotaki and Wright (1993)) autarky is the only non–monetary equilibrium outcome when the population is large enough, no matter how patient agents are. This fundamental difference between the Kiyotaki–Wright framework and the Lagos–Wright framework suggests that the absence of mechanisms that allow agents to coordinate behavior is an important friction for the essentiality of money.

The equilibrium construction in our main result relies on prices reflecting individual behavior in a very precise way. This suggests that the non–essentiality result may not be robust to the introduction of noise in the price formation process. We show that this is not the case; that is, the first–best can still be achieved when prices are noisy. Intuitively, in the presence of noise, the ability of prices to convey information about individual deviations is reduced. This is not enough to destroy the possibility of cooperation in large populations, though. One also needs a large volume of trade in the centralized market, otherwise individual deviations will still have a non–negligible impact on prices. However, exchange in the centralized market in LW is not necessary for the first–best to be implemented, and so there is no efficiency loss if one keeps the volume of trade in the centralized market low enough for prices to remain sensitive to individual behavior. Hence, the robustness of our non–essentiality result to the introduction of noise in prices.

The paper is organized as follows. We introduce our framework in the next section and describe it as an infinitely repeated game in Section 3. We establish our main result in Section 4 and examine its robustness to the introduction of noise in the price formation process in Section 5. The discussion in the previous paragraph raises the question of whether departing from LW and introducing gains from trade in the centralized market, so that now exchange
in the centralized market is needed for the first–best to be implemented, is enough to restore
the essentiality of money. We address this issue in Section 6. Section 7 concludes the paper
and the Appendix contains omitted proofs and details.

2 The Environment

Time is discrete and indexed by \( t \geq 1 \). There are two stages at each date and preferences
are additively separable across dates and stages. The population consists of an even number
\( N \) of infinitely lived agents indexed by \( j \in \{1, \ldots, N\} \). Agents do not discount payoffs
between stages in a period and have a common discount factor \( \delta \in (0,1) \) across periods. The
two stages of a period differ in terms of the matching process, preferences, and technology.
In the first stage, agents are randomly and anonymously matched in pairs. In the second
stage, trade takes place in a centralized market. It makes no difference for our results if
centralized trading takes place first in each period.

In the decentralized market, agents trade a divisible special good. There are no double–
coincidences and the probability that an agent is a producer in a match is equal to the
probability that he is a consumer. For expositional ease, we assume that this probability is
equal to \( 1/2 \). An agent who consumes \( q \geq 0 \) units of the special good enjoys utility \( u(q) \),
whereas an agent who produces \( q \) units of this good incurs disutility \( c(q) \). We assume that
\( u(0) = c(0) = 0 \), \( u'(0) > c'(0) \), and that \( u \) and \( c \) are strictly increasing and differentiable, with
\( u \) strictly concave and \( c \) (weakly) convex. Moreover, we assume that there exists \( \bar{q} > 0 \) such
that \( u(\bar{q}) = c(\bar{q}) \). Let \( q^* > 0 \) be the unique solution to \( u'(q) = c'(q) \). Surplus is maximized
in a match when the producer transfers \( q^* \) units of the special good to the consumer.

In the centralized market, agents can consume and produce a divisible general good. The
market operates as a trading post, which we describe in the next section. Production is as
follows. Every unit of effort generates one unit of the general good. There exists an upper
bound \( \bar{x} > 0 \) on the amount of effort an agent can exert in a period. An agent who consumes
\( x \geq 0 \) units of the general good obtains utility \( U(x) \), while an agent who produces \( x \) units
of this good incurs disutility $x$. The function $U$ is strictly increasing and differentiable, with $U(0) = 0$. Moreover, $U$ is either strictly concave with $U'(0) > 1$ and $\lim_{x \to -\infty} U'(x) = 0$, or linear with $U(x) = \alpha x$, $\alpha > 1$. If $U$ is strictly concave, let $x^* > 0$ be the unique solution to $U'(x^*) = 1$. If $U(x) = \alpha x$, the maximizer of $U(x) - x$ is $\pi$. Both the special good and the general good are perishable across stages and dates.

3 The Lagos–Wright Framework as a Repeated Game

In this section, we cast our environment as an infinitely repeated game. This is useful when we discuss the robustness of our non-essentiality result to the introduction of noise in the price formation process. We start with a description of the stage game.

3.1 The Stage Game

The stage game is an extensive form game that consists of one round of trade in the decentralized market followed by one round of trade in the centralized market.

Trade in the decentralized market takes place as follows. First, in every match the agents simultaneously and independently choose from \{yes, no\} after learning whether they are consumers or producers. If either agent in a match says no, the match is dissolved with no trade occurring. If both agents in a match say yes, that is, if both agents agree to trade, the producer transfers $q^*$ units of the special good to the consumer, after which the match is dissolved.\(^4\)

In the centralized market, an agent can produce for his own consumption and for exchange in a trading post. The sequence of events is as follows. In every period $t \geq 1$, each agent $j$ simultaneously and independently chooses: \(i\) the quantity $z^j_t$ of the general good he produces for his own consumption; \(ii\) the quantity $y^j_t$ of the general good he produces to the trading post; \(iii\) the bid $0 \leq b^j_t \leq y^j_t$ he submits to the trading post. The assumption

\(^4\)The same results obtain if the producer can choose the quantity $q$ he transfers to the consumer. The approach we follow simplifies the description of strategies considerably.
that $0 \leq b^j_t \leq y^j_t$ implies that an agent cannot demand more than what he contributes to the trading post. This restriction plays a role in the proofs of our non-essentiality results. In the Appendix we show that this restriction can be dropped as long as agents cannot bid more than some finite amount $B \in (\overline{x}, \infty)$. The price of the general good in period $t$ is

$$p_t = \frac{\sum_{j=1}^{N} b^j_t}{\sum_{j=1}^{N} y^j_t},$$

where $p_t = 0$ if $\sum_{j=1}^{N} y^j_t = 0$. The quantity of the general good that $j$ obtains in the trading post in period $t$ is then $x^j_t = b^j_t/p_t$, where we adopt the convention that $0/0 = 0$. Note that $\sum_{j=1}^{N} y^j_t = \sum_{j=1}^{N} b^j_t/p_t = \sum_{j=1}^{N} x^j_t$, so that the aggregate supply in the trading post is always equal to the aggregate demand (on and off the path of play). Prices are publicly observable.

In the stage game, autarky corresponds to the situation in which: (i) no production takes place in the decentralized market; and (ii) agents only produce for themselves in the centralized market, that is, agents do not use the trading post for exchange. When $U$ is strictly concave, the highest payoff in autarky is $U(x^*) - x^*$. When $U(x) = \alpha x$, the highest payoff in autarky is $(\alpha - 1) \overline{x}$. Let $\overline{x} = x^*$ if $U$ is strictly concave and $\overline{x} = \overline{x}$ if $U(x) = \alpha x$. We then have the result in Lemma 1.

**Lemma 1.** The following holds in every Nash equilibrium of the stage game. No production takes place in the decentralized market. Each agent produces and consumes $\overline{x}$ in the centralized market.

**Proof:** First, observe that every agent can guarantee himself a payoff of at least $U(\overline{x}) - \overline{x}$ in the centralized market. Indeed, an agent can always produce $\overline{x}$ for his own consumption. Since the highest aggregate payoff possible in the centralized market is $N[U(\overline{x}) - \overline{x}]$, we can then conclude that in every Nash equilibrium of the stage game all agents obtain $U(\overline{x}) - \overline{x}$

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Footnotes:

5 The existence of an upper bound on bids in the centralized market is natural. Indeed, we also show in the Appendix that without this restriction, the stage game has no Nash equilibrium where trade takes place in the centralized market. Intuitively, this happens because, by bidding high enough, an agent can obtain virtually all the output in the trading post to himself.

6 This convention is consistent with the fact that $p_t = 0$ if all agents produce zero in period $t$, in which case they cannot buy the general good in the market.
in the centralized market regardless of how they behave in the decentralized market. Hence, the producers in the decentralized market have no incentive to say yes, so that no production takes place in this market in any Nash equilibrium of the stage game.

Notice that there are no gains from trade in the centralized market. This is reflected in the fact that in every Nash equilibrium of the stage game, each agent’s payoff is equal to the payoff he obtains if he produces $\bar{x}$ for his own consumption in the centralized market. As discussed in the Introduction, the absence of gains from trade in the centralized market in the LW environment is important when we discuss the robustness of our main result to the introduction of noise in the price formation process.

### 3.2 The Infinitely Repeated Game

Our environment consists of infinite repetitions of the stage game of Subsection 3.1. The history of an agent in every period is the list of: (i) his past actions in both markets; (ii) the actions of his past partners in the decentralized market; (iii) the past prices in the centralized market. A behavior strategy for an agent is a map from the set of all his possible histories into a (mixed) action.

We describe strategies by means of automata.\(^7\) For this, let $A_1 = \{\text{yes, no}\}$ be the action set of an agent in the decentralized market and $A_2 = \{a_2 = (z, y, b) : z + y \leq \bar{x} \text{ and } b \leq y\}$ be the action set of an agent in the centralized market. In our setting, an automaton is a list $(W, w^0, (f_1, f_2), (\tau_1, \tau_2))$ where: (i) $W$ is a set of states; (ii) $w^0 \in W$ is the initial state; (iii) $f_1 : W \rightarrow \Delta(A_1)$ and $f_2 : W \rightarrow \Delta(A_2)$ are decision rules in the decentralized and centralized markets, respectively; (iv) $\tau_1 : W \times A_1^2 \rightarrow W$ and $\tau_2 : W \times A_2 \times \mathbb{R}_+ \rightarrow W$ are transition rules in the decentralized and centralized markets, respectively. A decision rule is a specification of behavior as a function of states. Thus, an agent in state $w$ chooses $f_1(w)$ if he is in the decentralized market and $f_2(w)$ if he is in the centralized market. A transition rule in the decentralized market associates the next state of the automaton with

\(^7\)See Section 2.3 in Mailath and Samuelson (2006) for a discussion of the use of automata to describe behavior in repeated games.
the agent’s current state and the profile of actions in his match: \( \tau_1(w, a_1, a_1') \) is the new state of an agent who enters the decentralized market in state \( w \) if he chooses \( a_1 \) and his partner chooses \( a_1' \). Similarly, a transition rule in the centralized market associates the next state of the automaton with the agent’s current state, his action, and the observed price. We restrict attention to strategy profiles \( \sigma \) where all agents behave according to the same automaton.

Given a strategy profile \( \sigma \), a profile of states for an agent is a map \( \pi : W \rightarrow \{1, \ldots, N-1\} \) such that \( \pi(w) \) is the number of other agents in the population who are in state \( w \) (notice that \( \sum_{w \in W} \pi(w) = N - 1 \)). Denote the set of all state profiles by \( \Pi \). A belief for an agent is a map \( p : \Pi \rightarrow [0, 1] \) such that \( \sum_{\pi \in \Pi} p(\pi) = 1 \), where \( p(\pi) \) is the probability the agent assigns to the event that the profile of states is \( \pi \). Let \( \Delta \) be the set of all possible beliefs. A belief system for an agent is a map \( \mu : W \rightarrow \Delta \). In an abuse of notation, we use \( \mu \) to denote the profile of belief systems where all agents have the same belief system \( \mu \).

We consider sequential equilibria of the repeated game. Note that the repetition of Nash equilibria of the stage game is an equilibrium outcome. The first–best is achieved when in every period trade takes place in all meetings in the decentralized market and all agents consume and produce \( \bar{x} \) in the centralized market. In the next section, we prove that there exists \( \delta \in (0, 1) \) independent of the population size such that the first–best is also an equilibrium outcome as long as \( \delta \geq \delta \).

4 The Non–essentiality of Money

In this section, we construct an equilibrium that sustains the first–best if agents are patient enough. In what follows, \( C \) stands for cooperation, \( D \) stands for deviation, and \( A \) stands for autarky.

Let \( \sigma^* \) be the strategy profile where all agents behave according to the following automaton. The set of states is \( W = \{C, D, A\} \) and the initial state is \( C \). The decision rules \( f_1 \) and
are given by

\[ f_1(C) = f_1(D) = \text{yes}, \quad f_1(A) = \text{no} \]

\[ f_2(C) = (0, \bar{x}, \bar{x}), \quad f_2(D) = (0, \bar{x}, 0), \quad f_2(A) = (\bar{x}, 0, 0); \]

recall that \( \bar{x} = x^* \) if \( U \) is strictly concave and \( \bar{x} = \bar{x} \) if \( U \) is linear. For instance, an agent in state \( C \) agrees to trade in the decentralized market and chooses \((0, \bar{x}, \bar{x})\) in the centralized market. Let

\[ p_D = \frac{(N - 2)\bar{x}}{(N - 2)\bar{x} + 2\bar{x}}. \]

The transition rules \( \tau_1 \) and \( \tau_2 \) are such that

\[
\tau_1(C, a_1, a'_1) = \begin{cases} 
C & \text{if } (a_1, a'_1) = (\text{yes, yes}) \\
D & \text{if } (a_1, a'_1) \neq (\text{yes, yes})
\end{cases}
\]

\[
\tau_2(w, a_2, p) = \begin{cases} 
C & \text{if } w \in \{C, D\} \text{ and } p \in \{p_D, 1\} \\
A & \text{if } w \in \{C, D\} \text{ and } p \notin \{p_D, 1\}
\end{cases}
\]

For instance, an agent in state \( C \) in the decentralized market remains in \( C \) only if there is trade in his match, otherwise he moves to state \( D \). Likewise, an agent in state \( C \) in the centralized market stays in \( C \) if the price he observes is either 1 or \( p_D \), otherwise he moves to state \( A \). Notice that \( \sigma^* \) implements the first–best.

Now consider the belief system \( \mu^* \) where: (i) an agent in state \( C \) believes that all other agents are in state \( C \); (ii) an agent in state \( A \) believes that all other agents are in state \( A \); (iii) an agent in state \( D \) believes that there is one other agent in state \( D \) and the remaining agents are in state \( C \). Clearly, \((\sigma^*, \mu^*)\) is a consistent assessment.

In the remainder of this section, we assume that agents behave according to \( \sigma^* \) and form beliefs according to \( \mu^* \). Let \( V_{DM}^* \) and \( V_{CM}^* \) be the (discounted) lifetime payoffs to an agent in state \( C \) before he enters the decentralized and centralized markets, respectively. Then,

\[ V_{DM}^* = \frac{1}{1 - \delta} \left\{ \frac{1}{2} [u(q^*) - c(q^*)] + U(\bar{x}) - \bar{x} \right\} \quad \text{and} \quad V_{CM}^* = U(\bar{x}) - \bar{x} + \delta V_{DM}^*. \]

Now, let \( V_A^* \) be the lifetime payoff to an agent in state \( A \). Since there is no discounting between stages in a period,

\[ V_A^* = \frac{1}{1 - \delta} [U(\bar{x}) - \bar{x}]. \]
Finally, let $V_D^*$ be the lifetime payoff to an agent in state $D$ before he enters the centralized market. In this case, the agent’s action in the centralized market is $(0, \bar{x}, 0)$. Since he believes that one other agent chooses the same action and the remaining agents choose $(0, \bar{x}, \bar{x})$, he expects the price in the centralized market to be $p_D$. Thus,

$$V_D^* = -\bar{x} + \delta V_{DM}^*. $$

Observe that no agent is ever in state $D$ before he enters the decentralized market (on and off the path of play).

**Proposition 1.** Suppose that $\bar{x} \geq c(q^*)$. There exists $\delta \in (0, 1)$ independent of $N$ such that $(\sigma^*, \mu^*)$ is a sequential equilibrium for all $\delta \geq \delta$.

**Proof:** It is immediate to see that no one–shot deviation is profitable in state $A$. Let us begin with incentives in state $C$ then. An agent in the decentralized market has no profitable one–shot deviation if

$$-c(q^*) + U(\bar{x}) - \bar{x} + \delta V_{DM}^* \geq -\bar{x} + \delta V_{DM}^*, $$

which is satisfied since $\bar{x} \geq c(q^*)$. Consider now an agent in the centralized market and suppose he chooses $(Z, Y, B) \neq (0, \bar{x}, \bar{x})$. In this case, his flow payoff is $U(B/p + Z) - (Z + Y)$, where

$$p = \frac{(N - 1)\bar{x} + B}{(N - 1)\bar{x} + Y}. $$

It is easy to see that $B/p$ is increasing in $B$. Since $B \leq Y$, the highest flow payoff the agent can obtain is $U(Y + Z) - (Y + Z)$, which is not greater than $U(\bar{x}) - \bar{x}$. Because $V_{DM}^*$ is the highest continuation payoff possible for the agent, we can then conclude that he has no profitable one–shot deviation.

We now check incentives in state $D$. As observed previously, no agent is ever in state $D$ in the decentralized market. Consider then an agent in the centralized market. Under $\sigma^*$, the price in the centralized market is $p_D$. Note that any deviation in state $D$ induces a price $p \notin \{1, p_D\}$. Indeed, if the agent chooses $(Z, Y, B) \neq (0, \bar{x}, 0)$, the price is

$$p = \frac{(N - 2)\bar{x} + B}{(N - 2)\bar{x} + \bar{x} + Y}. $$

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Since $B \leq Y$, the agent can never behave in a way that $p = 1$. Moreover, since $Y \leq \bar{y}$, $p = p_D$ only if $Z = 0$, $Y = \bar{y}$, and $B = 0$. Thus, a one-shot deviation is not profitable if

$$-\bar{y} + \delta V^*_{DM} \geq U\left(\frac{B}{p} + Z\right) - (Y + Z) + \delta V^*_A,$$

which is equivalent to

$$-\bar{y} + \frac{\delta}{2(1-\delta)} \left[u(q^*) - c(q^*)\right] \geq U\left(\frac{B}{p} + Z\right) - (Y + Z).$$

Since $B/p$ is increasing in $B$ and $B \leq Y$, we have that

$$U\left(\frac{B}{p} + Z\right) \leq U\left(Y \left(\frac{N-2}{N-2}\bar{x} + \frac{\bar{x} + Y}{(N-2)\bar{x} + Y} + Z\right) \leq U\left(Y + Z + \frac{Y \bar{y}}{(N-2)\bar{x} + Y}\right).$$

Since the right-hand side of the last inequality is bounded above by $U(2\bar{y})$, we can then conclude that there exists $\delta \in (0,1)$ independent of the population size such that a one-shot deviation is not profitable if $\delta \geq \delta$. \hfill \Box

We should point out that our strategy of proof is quite different from the strategy of proof in ACP. Their environment is very much like a repeated prisoner’s dilemma in the sense that communicating a defection to the population in the centralized market involves taking an action that is myopically optimal. In our setting, communicating a defection is costly in terms of flow payoffs. What sustains the threat of punishment is that if an agent deviates off the path of play, this leads to an even greater punishment (permanent autarky).

To conclude this section, notice that Proposition 1 is true without the restriction that $\bar{y} \geq c(q^*)$. In our candidate equilibrium, we assume that the only punishment for an agent who defects in the decentralized market is his payoff loss in the subsequent round of trading in the centralized market. This happens because cooperation is restored after agents observe a price of $p_D$. The condition $\bar{y} \geq c(q^*)$ can be dropped if a defection in the decentralized market were to lead to a greater punishment, as it would be the case if a price of $p_D$ led to a number of periods of autarky.\footnote{For instance, if a price of $p_D$ were to trigger $T$ periods of autarky, equation (1) in the proof of Proposition 1 would become

$$\bar{y} - c(q^*) + \frac{\delta(1-\delta^T)[u(q^*) - c(q^*)]}{2(1-\delta)} \geq U(\bar{x}) - \bar{x},$$

which is always satisfied when $\delta$ is close enough to one as long as $T$ is large enough.}
5 Noisy Prices

The equilibrium construction in the proof of Proposition 1 relies on the fact that prices in the centralized market are sensitive to changes in individual behavior. This raises the question of whether Proposition 1 is true when prices are noisy. When prices in the centralized market are a random function of individual behavior, our framework becomes a repeated game with noisy observations. Results from Green (1980), Sabourian (1990), and Al–Najjar and Smorodinsky (2001) suggest that when the population is large enough, the only possible equilibrium outcomes are the ones where in each period all agents play a Nash equilibrium of the stage game.\(^9\) By Lemma 1, this translates into autarky being the only non–monetary outcome when the number of agents is large. It turns out that this is not true in our case. In what follows, we show that the first–best remains an equilibrium outcome in large populations even when prices are noisy.

The environment is the same as before except that now in every period \(t\) the total effort \(Y_t = \sum_{j=1}^{N} y^j_t\) directed to production for exchange in the trading post yields \(\theta_t Y_t\) units of the general good, where \(\theta_t\) is a stochastic shock to production in period \(t\). Thus, the price \(p_t\) in the centralized market in period \(t\) is now

\[
p_t = \frac{\sum_{j=1}^{N} b^j_t}{\theta_t \sum_{j=1}^{N} y^j_t}.
\]

Since \(\sum_{j=1}^{N} b^j_t/p_t = \theta_t \sum_{j=1}^{N} y^j_t\), we still have that aggregate demand is always equal to aggregate supply (on and off the path of play).

We assume that the \(\theta_t\) are i.i.d. with \(E[\theta_t] = 1\) and continuously distributed in some interval \([\theta_{\min}, \theta_{\max}]\) with \(0 < \theta_{\min} < 1 < \theta_{\max} < \infty\); it is straightforward to establish a non–essentiality result if the support of the \(\theta_t\) is finite. We denote the p.d.f. of \(\theta_t\) by \(\omega\) and assume there exists \(\Lambda > 0\) such that \(\omega(\theta) \geq \Lambda\) for all \(\theta \in [\theta_{\min}, \theta_{\max}]\). This assumption, which can be dropped (see Footnote 11 below), is satisfied, for instance, if the \(\theta_t\) are uniformly distributed.

Given that production for own consumption is not subject to shocks, agent \(j\)’s period–

expected payoff from consumption in the centralized market if his action is \((z^j_t, y^j_t, b^j_t)\) is

\[
E \left[ U \left( \frac{b^j_t}{p_t} + z^j_t \right) \right] = E \left[ U \left( \theta_t \frac{\sum_{j=1}^N y^j_t}{\sum_{j=1}^N b^j_t} + z^j_t \right) \right].
\]

Thus, even though \(E[\theta_t] \equiv 1\),

\[
E \left[ U \left( \theta_t \frac{\sum_{j=1}^N y^j_t}{\sum_{j=1}^N b^j_t} + z^j_t \right) \right] \neq U \left( \frac{b^j_t}{p_t} + z^j_t \right)
\]

when agents are risk-averse. In other words, when agents are risk-averse, the same profile of actions in the centralized market leads to different payoffs depending on whether prices are noisy or not. Since the purpose of our robustness exercise is to check whether the introduction of noise affects the informational role of prices, we assume that \(U(x) = \alpha x\), so that agents are risk-neutral. This way, we guarantee that risk considerations do not play a role in agents’ decisions in the centralized market.

Consider the strategy profile \(\sigma^{**}\) where all agents behave according to the following automaton. Once more, the set of states is \(W = \{A, C, D\}\) and the initial state is \(C\). The decision rules \(f_1\) and \(f_2\) are given by

\[
f_1(C) = f_1(D) = \text{yes}, \quad f_1(A) = \text{no}
\]

\[
f_2(C) = (\bar{\pi} - \varepsilon, \varepsilon, \varepsilon), \quad f_2(D) = (0, \bar{\pi}, 0), \quad \text{and} \quad f_2(A) = (0, 0, 0),
\]

where \(0 < \varepsilon < \bar{\pi}\) is such that

\[
\frac{1}{\theta_{\min}} \frac{(N-2)\varepsilon}{(N-2)\varepsilon + 2\pi} = \frac{p_D}{\theta_{\max}} < \frac{1}{\theta_{\min}}.
\]

(2)

In order to define the transition rules \(\tau_1\) and \(\tau_2\), let

\[
S = S_D \cup S_C = \left[ \frac{p_D}{\theta_{\max}}, \frac{p_D}{\theta_{\min}} \right] \cup \left[ \frac{1}{\theta_{\max}}, \frac{1}{\theta_{\min}} \right].
\]

The transition rules \(\tau_1\) and \(\tau_2\) are such that

\[
\tau_1(C, a_1, a'_1) = \begin{cases} 
C & \text{if } (a_1, a'_1) = (\text{yes}, \text{yes}) \\
D & \text{if } (a_1, a'_1) \neq (\text{yes}, \text{yes})
\end{cases}, \quad \tau_1(D, a_1, a'_1) \equiv D, \quad \tau_1(A, a_1, a'_1) \equiv A,
\]

\[
\tau_2(w, a_2, p) = \begin{cases} 
C & \text{if } w \in \{C, D\} \text{ and } p \in S \\
A & \text{if } w \in \{C, D\} \text{ and } p \notin S
\end{cases}, \quad \text{and} \quad \tau_2(A, a_2, p) \equiv A.
\]
Then $p_t = 1/\theta_t$ if in period $t$ all agents are in state $C$ in the centralized market and $p_t = p_D/\theta_t$ if in period $t$ two agents are in state $D$ and the remaining agents are in state $C$. Note that $\sigma^{**}$ implements the first–best.

Now, let $\mu^{**}$ be the belief system used in the proof of Proposition 1. Thus, an agent in state $w \in \{C, A\}$ believes that all other agents are in the same state, while an agent in state $D$ believes that one other agent is in state $D$ and everyone else is in state $C$. It is easy to see that $(\sigma^{**}, \mu^{**})$ is consistent. We have the following result. As in Section 4, the assumption that $\pi \geq c(q^*)$ can be dropped if we change $\sigma^{**}$ so that a price in $S_D$ leads to a number of periods in autarky.

**Proposition 2.** Suppose that $\pi \geq c(q^*)$. There exists $\delta \in (0, 1)$ independent of the population size such that if $\delta \geq \delta$, then $(\sigma^{**}, \mu^{**})$ is a sequential equilibrium.

**Proof:** Let $V_{DM}^{**}$ and $V_{CM}^{**}$ be the lifetime payoffs to an agent in state $C$ before he enters the decentralized market and the centralized market, respectively. Then,

$$V_{DM}^{**} = \frac{1}{1-\delta} \left\{ \frac{1}{2} [u(q^*) - c(q^*)] + (\alpha - 1) \pi \right\} \quad \text{and} \quad V_{CM}^{**} = (\alpha - 1) \pi + \delta V_{DM}^{**}.$$

Now let $V_{D}^{**}$ be the lifetime payoff to an agent in state $D$ before he enters the centralized market. Since such an agent believes that $p_t = p_D/\theta_t$, we have that

$$V_{D}^{**} = -\pi + \delta V_{DM}^{**}.$$

As in the equilibrium of Section 4, an agent is never in state $D$ when he is in the decentralized market. Finally, let $V_{A}^{**} = (1 - \delta)^{-1} (\alpha - 1) \pi$ be the lifetime payoff to an agent in state $A$.

It is immediate to see that no one–shot deviation is profitable in state $A$. Let us start with incentives in state $C$ then. As in Section 4, an agent in the decentralized market has no profitable one–shot deviation since $\pi \geq c(q^*)$. Consider then an agent in the centralized market. Under $\sigma^{**}$, his expected flow payoff is $(\alpha - 1) \pi$ and his continuation payoff is $V_{DM}^{**}$. Suppose the agent chooses $(Z, Y, B) \neq (\pi - \varepsilon, \varepsilon, \varepsilon)$. His flow payoff gain is

$$\pi(Z, Y, B) = \alpha \left[ B \frac{Y + (N - 1)\varepsilon}{B + (N - 1)\varepsilon} + Z \right] - (Y + Z) - (\alpha - 1) \pi,$$
which is increasing in $B$. Since $B \leq Y$ and $Y + Z \leq \bar{\tau}$, we then have that $\pi(Z,Y,B) \leq (\alpha - 1)(Y + Z - \bar{\tau}) \leq 0$. Moreover, $V_{DM}^{**}$ is the highest continuation payoff possible for the agent. Hence, the agent has no profitable one-shot deviation.

Consider now the incentives for an agent in state $D$. A one-shot deviation $(Z,Y,B) \neq (0,\bar{\tau},0)$ in some period $t$ implies that $p_t = \tilde{p}_D/\theta_t$, where

$$\tilde{p}_D = \frac{(N - 2)\varepsilon + B}{(N - 2)\varepsilon + \bar{\tau} + Y}.\$$

The flow payoff gain in this case is

$$\pi(Z,Y,B) = \alpha \left( \frac{B}{\tilde{p}_D} + Z \right) - (Y + Z) + \bar{\tau}$$

$$= (\alpha - 1)(Y + Z - \bar{\tau}) + \alpha \left( \frac{B}{\tilde{p}_D} + \bar{\tau} - Y \right) \leq \alpha \left( \frac{B}{\tilde{p}_D} + \bar{\tau} - Y \right),$$

given that $Y + Z \leq \bar{\tau}$. Since $B \leq Y$, we have that $p_D < \tilde{p}_D$ for all $(Z,Y,B) \neq (0,\bar{\tau},0)$. Let then $\tilde{p}_D = \gamma p_D$, with $\gamma > 1$. Since $\tilde{p}_D < 1$, the supremum $\gamma$ of $\gamma$ is such that $\gamma p_D < 1$. Hence,

$$\left( \frac{\gamma p_D}{\theta_{\text{max}}}, \frac{\gamma p_D}{\theta_{\text{min}}} \right) \cap \left( \frac{p_D}{\theta_{\text{min}}}, \frac{1}{\theta_{\text{max}}} \right) \neq \emptyset$$

for all $\gamma \in (1,\bar{\gamma})$, and so every one-shot deviation with $\gamma > 1$ has a positive chance of triggering permanent autarky. For simplicity, assume that $\gamma p_D/\theta_{\text{min}} \leq 1/\theta_{\text{max}}$, in which case $\tilde{p}_D/\theta_t \in S_A$ if, and only if, $\theta_t \in (\theta_{\text{min}}, \gamma \theta_{\text{min}})$; the case where $\gamma p_D/\theta_{\text{min}} > 1/\theta_{\text{max}}$ can be dealt with in a similar way.\footnote{The key observation is that both the flow gain from a one-shot deviation and the probability that this deviation leads to permanent autarky are commensurate with the change in the support of the price distribution.} Thus, the loss in continuation payoff due to a one-shot deviation with $\gamma > 1$ is

$$\frac{1}{2(1-\delta)}[u(q^*) - c(q^*)] \int_{\theta_{\text{min}}}^{\gamma \theta_{\text{min}}} \omega(\theta) d\theta \geq \frac{1}{2(1-\delta)}[u(q^*) - c(q^*)] \Delta (\gamma - 1) \theta_{\text{min}},$$

and so the gain from such a one-shot deviation is bounded above by

$$\Delta = \alpha \left( \frac{B}{\tilde{p}_D} + \bar{\tau} - Y \right) - \frac{1}{2(1-\delta)}[u(q^*) - c(q^*)] \Delta (\gamma - 1) \theta_{\text{min}}.$$
Straightforward algebra (see the Appendix for the omitted details) shows that
\[
\alpha \left( \frac{B}{\bar{p}_D} + \bar{\pi} - Y \right) = \alpha \frac{\gamma - 1}{\gamma} [(N - 2)\varepsilon + 2\bar{\pi}] \leq (\gamma - 1) \alpha \frac{\varepsilon + 2\bar{\pi}}{\kappa},
\]
and so
\[
\Delta \leq (\gamma - 1) \left[ \kappa - \frac{1}{2(1 - \delta)} [u(q^*) - c(q^*)] \lambda \theta_{\min} \right].
\]
Since \(\kappa\) is independent of \(\gamma\), we can then conclude that there exists \(\delta \in (0, 1)\) independent of the population size such that the agent has no profitable one-shot deviation if \(\delta \geq \delta.\)

The intuition behind Proposition 2 is simple. Since there are no gains from trade in the centralized market, there is no loss of efficiency if agents use the centralized market in a way that the volume of trade on the path of play is always small. This ensures that prices are informative about individual actions and thus can be used to coordinate behavior. Condition (2) implies precisely this, as it implies that the amount \(\varepsilon\) that each agent produces in the centralized market in each period is bounded above by \(2\bar{\pi}\theta_{\min}/(N - 2)(\theta_{\max} - \theta_{\min})\), and so total production \(N\varepsilon\) is bounded above by a quantity that is independent of population size.

6 Noisy Prices and Gains from Trade

The discussion at the end of the previous section suggests that if there are gains from trade in the centralized market, then it may no longer be possible to sustain the first-best with noisy prices when the population is large. The intuition for this is that when the population is large, efficiency requires a large volume of trade in the centralized market, which destroys the ability of prices to signal individual deviations. A detailed discussion of environments in which there are gains from trade in the centralized market is beyond the scope of this paper. In what follows we present a simple example to show that the introduction of gains

\[\text{We can accommodate the case where } \inf\{\omega(\theta) : \theta \in [\theta_{\min}, \theta_{\max}]\} = 0 \text{ as follows. Since } \omega \text{ is continuous and its image contains positive numbers, there exists } \Lambda > 0 \text{ such that } \{\theta : \omega(\theta) > \Lambda\} \text{ is a non-empty open set. Therefore, there exists an open interval } (\theta'_{\min}, \theta'_{\max}) \text{ such that } \omega(\theta) \geq \Lambda \text{ for all } \theta \text{ in this interval. Take the set } S \text{ in the definition of } \tau_2 \text{ in } \sigma^{**} \text{ to be such that } S = [p_D/\theta_{\max}, p_D/\theta'_{\min}] \cup S_C, \text{ so that now a price in the interval } (p_D/\theta'_{\min}, p_D/\theta_{\min}] \text{ triggers permanent autarky.}\]
from trade in the centralized market need not destroy the ability of agents to use prices to coordinate behavior and implement the first-best.

We introduce gains from trade in the centralized market by assuming that an agent only derives utility from goods produced for exchange in a trading post. We also assume that there are two general goods, say apples and coconuts, and that an agent can only produce and bid on one type of general good at each point in time. The disutility of producing \( x \) units of the general good is \( x \) and, as in Section 5, the utility \( U(x) \) of consuming \( x \) units of the general good is linear in \( x \). However, this utility is smaller for coconuts than for apples. More precisely, the utility of consuming \( x \) units of apples is \( U(x) = \alpha_a x \), with \( \alpha_a > 1 \), while the utility of consuming \( x \) units of coconuts is \( U(x) = \alpha_c x \), with \( \alpha_c < \alpha_a < 2\alpha_c \).

Now an action for an agent in the centralized market is a triple \((\xi, y, b)\), where \( \xi \in \{\text{apple, coconut}\} \) is the type of good he supplies to the trading post, \( y \) is how much he supplies of this good, and \( b \leq y \) is his bid for this good. Prices are determined as before. We let \( p_a \) denote the price of apples and \( p_c \) denote the price of coconuts. The first-best is achieved if: (i) in every single coincidence meeting in the decentralized market, the producer transfers \( q^* \) units of the special good to his partner; (ii) all agents supply \( \bar{x} \) apples to the trading post and bid this amount back.

Consider the strategy profile \( \sigma^{**} \) where all agents behave according to the following automaton. The set of states is \( W = \{C, D_b, D_g, A\} \) and the initial state is \( C \). The decision rules \( f_1 \) and \( f_2 \) are given by

\[
\begin{align*}
\sigma_1(C) &= \sigma_1(D_b) = \sigma_1(D_g) = \text{yes}, \quad \sigma_1(A) = \text{no}, \\
\sigma_2(C) &= \sigma_2(A) = (\text{apples, } \bar{x}, \bar{x}), \quad \sigma_2(D_b) = (\text{coconuts, } \bar{x}, 0), \quad \sigma_2(D_g) = (\text{coconuts, } \bar{x}, \bar{x}).
\end{align*}
\]

The transition rule \( \tau_1 \) in the decentralized market is such that

\[
\tau_1(C, a_1, a_1') = \begin{cases} 
C & \text{if } (a_1, a_1') \in \{(\text{yes, yes}), (\text{no, no})\} \\
D_b & \text{if } (a_1, a_1') = (\text{no, yes}) \\
D_g & \text{if } (a_1, a_1') = (\text{yes, no})
\end{cases}, \quad \text{and}
\]

\[
\tau_1(w, a_1, a_1') \equiv w \text{ for all } w \in \{D_g, D_b, A\}.
\]
In turn, the decision rule $\tau_2$ in the centralized market is such that

$$\tau_2(w, a_2, p) = \begin{cases} C & \text{if } w \in \{C, D_b, D_g\}, p_a \in S_1 \text{ and } p_c \in \{0\} \cup S_{1/2} \\ A & \text{if } w \in \{C, D_b, D_g\}, p_a \notin S_1 \text{ or } p_c \notin \{0\} \cup S_{1/2} \end{cases},$$

and

$$\tau_2(A, a_2, p) \equiv A,$$

where $S_k = [k/\theta_{\max}, k/\theta_{\min}], k \in \{1/2, 1\}$. Note that $\sigma^{***}$ implements the first-best. Unlike the strategy profile in the proof of Proposition 2, an agent who deviates in the decentralized market is treated differently from an agent who suffers a deviation in the decentralized market. Now an agent who deviates in the decentralized market is punished with zero consumption in the next round of centralized trading, while the agent who suffers the deviation is rewarded with a large consumption of coconuts in the same round of centralized trading.

It is easy to show that the strategy profile described above is part of an equilibrium. The proof is similar to the proof of Proposition 2. Intuitively, as long as the volume of trade for one of the two goods that can be exchanged in the centralized market is sufficiently small, the price of this good can be used to sustain cooperation in the decentralized market.

7 Concluding Remarks

In this paper we show that money can fail to be essential in the LW environment even if agents can only observe prices in the centralized market, and that this holds even if there exists noise in the price formation process. The robustness of our non-essentiality result to noise in prices follows from the fact that in LW there are no gains from trade in the centralized market, and so there is no efficiency loss if agents use the centralized market just to coordinate behavior. If there are gains from trade in the centralized market, the first-best can still be achieved in the presence of noise as long as there are at least two types of general good and the use of one of these types of general good for coordination purposes does not reduce efficiency. To summarize, the assumption of centralized trading has substantive implications for the essentiality of money.
We consider an environment where exchange in the centralized market is mediated by a Shapley–Shubik trading post. One could wonder whether our results survive if we model trade in the centralized market in a different way. We believe that as long as agents have some market power, no matter how small, our non-essentiality result goes through regardless of the details of the trading mechanism. For instance, we believe that our result goes through if we use a notion of Walrasian equilibrium with rationing. In this case, if at a given equilibrium price (that is, a price that clears the market), an agent deviates and demands a larger amount of some particular good, then all agents that demand this good are rationed. As a result, a deviation by an individual agent would be indirectly “observed” by other agents. That is, agents could use quantity signals rather than prices to coordinate behavior.\footnote{Another trading mechanism would be a double auction. Large double auctions have also been used to provide non–cooperative foundations for competitive markets. See Rustichini et. al. (1994) and Cripps and Swinkels (2006). A limitation of double auction is that they require indivisible goods.}

To finish, notice that if we introduce unbounded population growth in our environment, we can restore the essentiality of money in LW using an approach similar to the one used in Aliprantis et al. (2007b) to restore the essentiality of money in their framework. Precisely, we could assume that in period $t$ the population size is $N_t = N^t$ and there are $N^{t-1}$ centralized markets of size $N$. Then, a simple modification of the block recursive construction proposed by Aliprantis et al. (2007b) delivers an economy with strongly anonymous markets, so that autarky is the only Nash equilibrium outcome. This construction, however, requires that the total number of agents in the economy is infinite. Thus, it is subject to the criticism we make in the Introduction that an infinite agent economy is an adequate idealization only if it can be approximated by a large finite economy.

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8 Appendix

Omitted Details from the proof of Proposition 2

First notice that
\[ \gamma = \frac{[\varepsilon + B][\varepsilon + 2\pi]}{[\varepsilon + \pi + Y]}, \]
and so
\[ \gamma - 1 = \frac{1}{[\varepsilon + \pi + Y]} \left\{ \int_{Y}^{Y+N-2\varepsilon} \frac{1}{[\varepsilon + \pi + Y]}. \right\} \]
This implies that
\[ \pi - Y = (\gamma - 1) [\varepsilon + \pi + Y] - \frac{B [(N-2)\varepsilon + 2\pi]}{[\varepsilon + \pi + Y]}, \]
from which we obtain the desired result.

Bidding Restrictions

Here we show that Propositions 1 and 2 remain valid if we drop the restriction that agents cannot bid more than their output (effort) in the centralized market as long as there exists \( \pi \leq \overline{B} < \infty \) such that no agent can bid more than \( \overline{B} \). For simplicity, we assume that \( \pi \geq c(q^*) \); see the discussion at the end of the Appendix.

Before we start, notice that the restriction that there exists an upper bound of how much the agents can bid in the centralized market is natural in our context. Indeed, suppose there is no such upper bound and consider the stage game. In order for the profile \( \{ (z^j, y^j, b^j) \} \) of actions in the centralized market to be part of a Nash equilibrium, it must be that
\[ U \left( b^j \sum_{i=1}^{N} \frac{y^i}{b^i} + z^j \right) - y^j - z^j \geq U \left( b \frac{\sum_{i=1}^{N} y^i}{\sum_{i=1, i \neq j}^{N} b^i + b} + z^j \right) - y^j - z^j \]
for all \( b \geq 0 \) and \( j = 1, \ldots, N \). However, the right-hand side of the above inequality is strictly increasing in \( b \). Thus, we need
\[ U \left( b^j \frac{\sum_{i=1}^{N} y^i}{\sum_{i=1}^{N} b^i} + z^j \right) \geq U \left( \sum_{i=1}^{N} y^i + z^j \right), \]
which can only be satisfied if $\sum_{i=1}^{N} b_i = b_j$ for $j \in \{1, \ldots, N\}$, that is, if $b_j \equiv 0$. Hence, if agents are free to bid as much as they want in the centralized market, the only possible Nash equilibria in the stage game are the ones where no trade takes place in the centralized market.

**No Noise.** Let $\tilde{\sigma}^*$ be the strategy profile where all agents behave according to the following automaton. The set of states is $W = \{C, D_b, D_g, A\}$ and the initial state is $C$. The decision rules $f_1$ and $f_2$ are given by

\[ f_1(C) = f_1(D_b) = f_1(D_g) = \text{yes}, \quad f_1(A) = \text{no}, \]
\[ f_2(C) = (0, x, x), \quad f_2(D_b) = (0, x, 0), \quad f_2(D_g) = (z, 0, \pi), \quad \text{and} \quad f_2(A) = (x, 0, 0); \]

we let $x = x^*$ and $z = 0$ if $U(x)$ is strictly concave and $x = z = \pi$ if $U(x) = \alpha x$. In turn, the transition rules $\tau_1$ and $\tau_2$ are such that

\[
\tau_1(C, a_1, a'_1) = \begin{cases} 
C & \text{if } (a_1, a'_1) \in \{(\text{yes}, \text{yes}), (\text{no}, \text{no})\} \\
D_b & \text{if } (a_1, a'_1) = (\text{no}, \text{yes}) \\
D_g & \text{if } (a_1, a'_1) = (\text{yes}, \text{no})
\end{cases}
\]

\[
\tau_1(D_g, a_1, a'_1) \equiv D_g, \quad \tau_1(D_b, a_1, a'_1) \equiv D_b, \quad \tau_1(A, a_1, a'_1) \equiv A,
\]

\[
\tau_2(w, a_2, p) = \begin{cases} 
C & \text{if } w \in \{C, D_b, D_g\} \text{ and } p = 1 \\
A & \text{if } w \in \{C, D_b, D_g\} \text{ and } p \neq 1
\end{cases}
\]

Note that no agent can ever be in states $D_b$ or $D_g$ in the decentralized market and that the price in the centralized market is $p = 1$ if one agent is in state $D_b$, another agent is in state $D_g$, and the remaining $N - 2$ agents are in state $C$. The strategy profile $\tilde{\sigma}^*$ implements the first–best.

Now consider the belief system $\hat{\mu}^*$ where: (i) an agent in state $C$ believes that all other agents are in state $C$; (ii) an agent in state $A$ believes that all other agents are in state $A$; (iii) an agent in state $D_g$ ($D_b$) believes that there is one agent in state $D_b$ ($D_g$) and the remaining agents are in state $C$. Clearly, $(\tilde{\sigma}^*, \hat{\mu}^*)$ is a consistent assessment. We work with this assessment in what follows.
Let \( \hat{V}^*_{DM} \) and \( \hat{V}^*_{CM} \) be the lifetime payoffs to an agent in state \( C \) before he enters the decentralized and centralized markets, respectively. Then
\[
\hat{V}^*_{DM} = \frac{1}{1 - \delta} \left\{ \frac{1}{2} [u(q^*) - c(q^*)] + U(x) - x \right\} \quad \text{and} \quad \hat{V}^*_{CM} = U(x) - x + \delta \hat{V}^*_{DM}.
\]
Now let \( \hat{V}^*_{A} \) be the lifetime payoff to an agent in state \( A \). Then, \( \hat{V}^*_{A} = (1 - \delta)^{-1} [U(x) - x] \).

Finally, let \( \hat{V}^*_{D_b} \) and \( \hat{V}^*_{D_g} \) be the lifetime payoffs to an agent in states \( D_b \) and \( D_g \), respectively, before he enters the centralized market. Then,
\[
\hat{V}^*_{D_b} = -\bar{\pi} + \delta \hat{V}^*_{DM} \quad \text{and} \quad \hat{V}^*_{D_g} = U(z + \bar{\pi}) - z + \delta \hat{V}^*_{DM}.
\]

**Proposition 3.** There exists \( \bar{\delta} \in (0, 1) \) independent of \( N \) such that \((\bar{\sigma}^*, \bar{\mu}^*)\) is a sequential equilibrium for all \( \delta \geq \bar{\delta} \).

**Proof:** Trivially, there are no profitable one-shot deviations in state \( A \). Let us consider incentives in state \( C \) then. An agent in the decentralized market has no profitable one-shot deviation since \( \bar{\pi} \geq c(q^*) \). Consider now an agent in the centralized market and suppose he chooses \((Z, Y, B) \neq (0, x, x)\). His flow payoff is \( U(B/p + Z) - (Z + Y) \), where
\[
p = \frac{(N - 1)x + B}{(N - 1)x + Y}.
\]
The same argument in the proof of Proposition 1 shows that the agent has no profitable one-shot deviation if \( B = Y \) (so that \( p = 1 \)). Suppose then that \( B \neq Y \), which implies that his continuation payoff is \( \hat{V}^*_{A} \). Since \( B/p \) is increasing in \( B \), the agent has no profitable one-shot deviation if
\[
U(x) - x + \frac{\delta [u(q^*) - c(q^*)]}{2(1 - \delta)} \geq U \left( \frac{B(N - 1)x + Y}{(N - 1)x + B} + Z \right) - (Z + Y).
\]
Because the right-hand side of the above equation is bounded above by \( U(\bar{\pi} + \bar{\pi}) \), it is immediate to see that there exists \( \delta' \in (0, 1) \) independent of the population size such that the agent has no profitable one-shot deviation if \( \delta \geq \delta' \).

We now check incentives for an agent in state \( D_b \) in the centralized market. Note that any deviation in state \( D_b \) induces a price \( p > 1 \). Indeed, since \( B \geq 0 \) and \( Y \leq \bar{\pi} \), the price is
equal to 1 only if the agent chooses \( Z = 0 \), \( Y = \pi \), and \( B = 0 \). Thus, a one–shot deviation \((Z, Y, B) \neq (0, \pi, 0)\) is not profitable if

\[
-\pi + \delta V_{\text{DM}}^* \geq U \left( B \frac{(N-2)x+Y}{(N-2)x+\pi+B} + Z \right) - (Y + Z) + \delta V_{A}^*,
\]

which is equivalent to

\[
-\pi + \frac{\delta}{2(1-\delta)} [u(q^*) - c(q^*)] \geq U \left( B \frac{(N-2)x+Y}{(N-2)x+\pi+B} + Z \right) - (Y + Z).
\]

Since the right–hand side of the above inequality is increasing in \( B \), a one–shot deviation is not profitable if

\[
-\pi + \frac{\delta}{2(1-\delta)} [u(q^*) - c(q^*)] \geq U \left( B \frac{(N-2)x+Y}{(N-2)x+\pi+B} + Z \right) - (Y + Z).
\]

Now observe that the right–hand side of the last inequality is bounded above by \( U(B + \pi) \), from which it is immediate to see that there exists \( \delta'' \in (0, 1) \) independent of the population size such that the agent has no profitable one–shot deviation if \( \delta \geq \delta'' \).

To finish, we check incentives for an agent in state \( D_g \) in the centralized market. Suppose the agent’s action is \((Z, Y, B)\) such that

\[
p = \left( \frac{(N-2)x+B}{(N-1)x+Y} \right) = 1.
\]

This implies that \( B = Y + x \). Hence, the payoff from a one–shot deviation is

\[
U(B/p + Z) - (Y + Z) + \delta \hat{V}_{\text{DM}}^* = U(Y + Z + x) - (Y + Z) + \delta \hat{V}_{\text{DM}}^*,
\]

which is maximized when \( Y + Z = 0 \) \((Y + Z = \pi)\) if \( U \) is strictly concave \((U(x) = \alpha x)\).

Thus, there exists no profitable one–shot deviation in this case. Suppose now the agent’s action is \((Z, Y, B)\) such that \( p \neq 1 \). Since \( B/p \) is increasing in \( B \), a one–shot deviation is not profitable if

\[
U(z + \pi) - z + \frac{\delta}{2(1-\delta)} [u(q^*) - c(q^*)] \geq U \left( B \frac{(N-1)x+Y}{(N-2)x+B} + Z \right) - (Y + Z).
\]

Because the right–hand side of the above inequality is bounded above by \( U(2B + \pi) \), it is immediate to see that there exists \( \delta''' \in (0, 1) \) independent of the population size such that the agent has no profitable one–shot deviation if \( \delta \geq \delta''' \). We let \( \delta = \max\{\delta', \delta'', \delta'''\} \).

\[\square\]
Noise. Recall that $U(x) = \alpha x$. Consider the strategy profile $\tilde{\sigma}^{**}$ where all agents behave according to the following automaton. Once more, the set of states is $W = \{C, D_b, D_g, A\}$ and the initial state is $C$. The decision rules $f_1$ and $f_2$ are given by

$$f_1(C) = f_1(D_b) = f_1(D_g) = \text{yes}, \quad f_1(A) = \text{no},$$
$$f_2(C) = (\bar{x} - \varepsilon, \varepsilon, \varepsilon), \quad f_2(D_b) = (0, \bar{x}, 0), \quad f_2(D_g) = (\bar{x}, 0, \bar{x}), \quad \text{and} \quad f_2(A) = (\bar{x}, 0, 0),$$

where $\varepsilon$ is such that $(N - 1)\varepsilon < N$. Let $S_C = [1/\theta_{\text{max}}, 1/\theta_{\text{min}}]$. The transition rules $\tau_1$ and $\tau_2$ are such that

$$\tau_1(C, a_1, a_2') = \begin{cases} 
C & \text{if} \quad (a_1, a_1') \in \{(\text{yes, yes}), (\text{no, no})\} \\
D_b & \text{if} \quad (a_1, a_1') = (\text{no, yes}) \\
D_g & \text{if} \quad (a_1, a_1') = (\text{yes, no})
\end{cases},$$
$$\tau_1(D_g, a_1, a_1') \equiv D_g, \quad \tau_1(D_b, a_1, a_1') \equiv D_b, \quad \tau_1(A, a_1, a_1') \equiv A,$$
$$\tau_2(w, a_2, p) = \begin{cases} 
C & \text{if} \quad w \in \{C, D_b, D_g\} \text{ and } p \in S_C \\
A & \text{if} \quad w \in \{C, D_b, D_g\} \text{ and } p \notin S_C
\end{cases}, \text{ and } \tau_2(A, a_2, p) \equiv A.$$

Note that no agent is ever in states $D_b$ or $D_g$ in the decentralized market. Also notice that $p_t = 1/\theta$ if in period $t$ either all agents are in state $C$ or one agent is in state $D_b$, one agent is in state $D_g$, and the remaining agents are in state $C$. The strategy profile $\tilde{\sigma}^{**}$ implements the first-best.

Now let $\tilde{\mu}^{**}$ be the belief system of Proposition 3. Clearly, $(\tilde{\sigma}^{**}, \tilde{\mu}^{**})$ is a consistent assessment. We have the following result.

**Proposition 4.** There exists $\delta \in (0, 1)$ independent of $N$ such that $(\tilde{\sigma}^{**}, \tilde{\mu}^{**})$ is a sequential equilibrium for all $\delta \geq \delta$.

**Proof:** Let $\hat{V}_{\text{DM}}^{**}$ and $\hat{V}_{\text{CM}}^{**}$ be the lifetime payoffs to an agent in state $C$ before he enters the decentralized market and the centralized market, respectively. Then,

$$\hat{V}_{\text{DM}}^{**} = \frac{1}{1 - \delta} \left\{ \frac{1}{2} [u(q^*) - c(q^*)] + (\alpha - 1) \bar{x} \right\} \quad \text{and} \quad \hat{V}_{\text{CM}}^{**} = (\alpha - 1) \bar{x} + \delta \hat{V}_{\text{DM}}^{**}.$$
Now let $\hat{V}^{**}_{Ds}$, with $s \in \{b, g\}$, be the lifetime payoff to an agent in state $D_s$ in the centralized market. We have that

$$\hat{V}^{**}_{Db} = -\bar{x} + \delta \hat{V}^{**}_{DM} \quad \text{and} \quad \hat{V}^{**}_{Dg} = (2\alpha - 1) \bar{x} + \delta \hat{V}^{**}_{DM}.$$  

Finally, let $\hat{V}^{**}_{A} = (1 - \delta)^{-1} (\alpha - 1) \bar{x}$ be the lifetime payoff to an agent in state $A$.

It is immediate to see that there is no profitable one–shot deviation in state $A$. We start with incentives in state $C$. An agent in the decentralized market has no profitable one–shot deviation since $\bar{x} \geq c(q^*)$. Consider now an agent in the centralized market and let $(Z, Y, B) \neq (\bar{x} - \varepsilon, \varepsilon, \varepsilon)$ be his action choice. The agent’s flow payoff gain is

$$\alpha \left( \frac{B Y + (N - 1)\varepsilon}{B + (N - 1)\varepsilon} + Z \right) - (Y + Z) - (\alpha - 1) \bar{x}$$

$$= \frac{1}{B + (N - 1)\varepsilon} \left\{ (\alpha - 1)(Y + Z - \bar{x})[B + (N - 1)\varepsilon] + \alpha(B - Y)(N - 1)\varepsilon \right\}$$

$$\leq \frac{\alpha(B - Y)(N - 1)\varepsilon}{B + (N - 1)\varepsilon},$$

since $Y + Z \leq \bar{x}$. Clearly, the agent has no profitable one–shot deviation with $B \leq Y$.

Suppose then that $B > Y$. In this case, there exists $\hat{\theta}' = \hat{\theta}'(B, Y) > \theta_{min}$ such that

$$\frac{1}{\hat{\theta}'(N - 1)\varepsilon + B} = \frac{1}{\theta_{min}}.$$  

As a result, any realization $\theta_t < \hat{\theta}'$ triggers a reversion to autarky. The probability of such an event is

$$\int_{\theta_{min}}^{\theta_{min}(N - 1)\varepsilon + B} \omega(\theta) d\theta \geq \frac{\lambda \theta_{min}(B - Y)}{Y + (N - 1)\varepsilon}.$$  

Thus, a one–shot deviation is not profitable if

$$\frac{\alpha(N - 1)\varepsilon}{B + (N - 1)\varepsilon} \leq \frac{\lambda \theta_{min}}{Y + (N - 1)\varepsilon} \frac{\delta}{(1 - \delta)} \left\{ \frac{1}{2} [u(q^*) - c(q^*)] \right\}.$$  

Since $(N - 1)\varepsilon < \bar{N}$, there exists $\delta' \in (0, 1)$ independent of the population size such that a one–shot deviation is not profitable if $\delta \geq \delta'$.

We now check incentives for an agent in state $D_b$ in the centralized market. Suppose the agent’s action is $(Z, Y, B) \neq (0, \bar{x}, 0)$. Then,

$$p = \frac{1}{\theta_t} \frac{(N - 2)\varepsilon + \bar{x} + B}{(N - 2)\varepsilon + Y}.$$
and so the agent’s flow payoff gain is
\[
\alpha \left[ B \frac{(N - 2)\varepsilon + Y}{(N - 2)\varepsilon + \bar{\varepsilon} + B} + Z \right] - (Y + Z) + \bar{x} \\
= \alpha(Y + Z) - (Y + Z) + \bar{x} + \alpha \frac{\alpha}{(N - 2)\varepsilon + \bar{\varepsilon} + B} \{B(N - 2)\varepsilon - [(N - 2)\varepsilon + \bar{x}]Y\} \\
\leq \alpha \bar{x} + \alpha \frac{(B - Y + \bar{x})[(N - 2)\varepsilon + \bar{x}]}{(N - 2)\varepsilon + \bar{x} + B},
\]
since \(Y + Z \leq \bar{x}\). Now note that any deviation by the agent induces an increase in the price level for every realization of \(\theta\). Hence, there exists \(\hat{\theta}'' = \hat{\theta}''(B, Y) > \theta_{\text{min}}\) such that
\[
\frac{1}{\hat{\theta}''} \frac{(N - 2)\varepsilon + \bar{x} + B}{(N - 2)\varepsilon + Y} = \frac{1}{\theta_{\text{min}}}.
\]
As a result, any realization \(\theta_t < \hat{\theta}''\) triggers a reversion to autarky. The probability of such an event is
\[
\int_{\hat{\theta}''}^{\theta_{\text{min}}} \frac{(N - 2)\varepsilon + \bar{x} + B}{(N - 2)\varepsilon + Y} \omega(\theta) d\theta \geq \frac{\Delta\theta_{\text{min}}}{\theta_{\text{min}}} \frac{B - Y + \bar{x}}{(N - 2)\varepsilon + Y}.
\]
Thus, a one-shot deviation is not profitable if
\[
\frac{1}{\theta_{\text{min}}} \frac{(N - 2)\varepsilon + \bar{x}}{(N - 2)\varepsilon + \bar{x} + B} \leq \frac{\Delta\theta_{\text{min}}}{(N - 2)\varepsilon + Y} \frac{\delta}{(1 - \delta)} \left\{ \frac{1}{2} [u(q^*) - c(q^*)] \right\}.
\]
Since \((N - 1)\varepsilon < \overline{N}\), there exists \(\delta'' \in (0, 1)\) independent of the population size such that a one-shot deviation is not profitable if \(\delta \geq \delta''\).

To finish, consider the incentives for an agent in state \(D_g\) in the centralized market. Suppose the agent’s action is \((Z, Y, B) \neq (\bar{x}, 0, \bar{x})\). Then,
\[
p = \frac{1}{\hat{\theta}_t} \frac{(N - 2)\varepsilon + B}{(N - 2)\varepsilon + \bar{x} + Y},
\]
and the agent’s flow payoff gain is
\[
\alpha \left[ B \frac{(N - 2)\varepsilon + Y}{(N - 2)\varepsilon + \bar{x} + B} + Z \right] - (Y + Z) - (2\alpha - 1) \\
= (\alpha - 1)(Z + Y - \bar{x}) + \alpha \left[ B \frac{(N - 2)\varepsilon + \bar{x} + Y}{(N - 2)\varepsilon + B} - \bar{x} - Y \right] \\
= (\alpha - 1)(Z + Y - \bar{x}) + (N - 2)\varepsilon \frac{B - Y}{(N - 2)\varepsilon + B}.
\]
Thus, a profitable deviation must involve $B > Y + \pi$. Consider such a deviation. Then, there exists $\tilde{\theta}'' = \tilde{\theta}''(B, Y) > \theta_{\min}$ such that

$$\frac{1}{\tilde{\theta}''(N-2)\varepsilon + B} = \frac{1}{\theta_{\min}}.$$  

As a result any realization $\theta_t < \tilde{\theta}''$ triggers a reversion to autarky. The probability of such a reversion is

$$\int_{\theta_{\min}}^{\tilde{\theta}''(N-2)\varepsilon + B} \omega(\theta) d\theta \geq \Lambda \theta_{\min} \frac{B - \pi - Y}{(N-2)\varepsilon + \pi + Y}.$$  

Thus, a one–shot deviation is not profitable if

$$\frac{\alpha(N-2)\varepsilon}{(N-2)\varepsilon + B} \leq \frac{\Lambda \theta_{\min} \delta}{(N-2)\varepsilon + \pi + Y} \left\{ \frac{1}{2} \left[ u(q^*) - c(q^*) \right] \right\},$$

Because $(N-1)\varepsilon < N$, there exists $\delta'' \in (0, 1)$ independent of the population size such that a one–shot deviation is not profitable if $\delta \geq \delta''$. We let $\delta = \max\{\delta', \delta'', \delta'''\}$. \hfill \Box

Propositions 3 and 4 are true without the restriction that $\pi \geq c(q^*)$. As in the proofs of Propositions 1 and 2, we assume in our candidate equilibria that the only punishment for an agent who defects in the decentralized market is his payoff loss in the subsequent round of trading in the centralized market. The condition $\pi \geq c(q^*)$ can be dropped if we modify our strategy and assume that after a deviation takes place in the decentralized market, the agent who deviates must play the action $(0, \pi, 0)$ and the agent who suffers the deviation must play the action $(\pi, 0, \pi)$ for a sufficiently large number of periods.