Dynamic Quantile Models of Rational Behavior∗

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Abstract

This paper develops a dynamic model of rational behavior under uncertainty, in which
the agent maximizes the stream of the future $\tau$-quantile utilities, for $\tau \in (0, 1)$. That is, the
agent has a quantile utility preference instead of the standard expected utility. Quantile
preferences have useful advantages, such as robustness and ability to capture heterogeneity.
Although quantiles do not have some of the useful properties of expectations, such as
linearity and the law of iterated expectations, we show that the quantile preferences
are dynamically consistent. We also show that the corresponding dynamic problem yields
a value function, via a fixed-point argument, and establish its concavity and differentiability.
The principle of optimality also holds for this dynamic model. Additionally, we
derive the corresponding Euler equation. Empirically, we show that one can employ ex-
isting quantile regression methods for estimating and testing the economic model directly
from the stochastic Euler equation. Thus, the parameters of the model can be estimated
using known econometric techniques and interpreted as structural objects. In addition, the
methods provide microeconomic foundations for quantile regression estimation. To illus-
trate the developments, we construct an asset-pricing model and estimate the elasticity of
intertemporal substitution and discount factor parameters across the quantiles. The results
provide evidence of heterogeneity in these parameters.

Keywords: Quantile utility, dynamic programing, quantile regression, asset pricing.

JEL: C22, C61, E20, G12

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1 Introduction

Modeling dynamic economic behavior has been a concern in economics for a long time (see, e.g., Samuelson (1958), Baumol (1959), Koopmans (1960), Brock and Mirman (1972)). Dynamic models provide a flexible tool for economic analysis by considering the possibilities of change in the economic variables. These models are critical for learning about economic behavior and effects, incentives, and to design policy analysis. We contribute to this literature by developing a new dynamic model for an individual, who, when selecting among uncertain alternatives, chooses the highest one given the $\tau$-quantile of the utility distribution for $\tau \in (0, 1)$, instead of the standard expected utility. This quantile preference model is tractable, simple to interpret, and substantially broadens the scope of economic applications, because it is robust to fat tails and allows to account for heterogeneity through the quantiles.$^1$

Quantile preferences were first studied by Manski (1988) and axiomatized by Chambers (2009) and Rostek (2010). Manski (1988) develops the decision-theoretic attributes of quantile maximization and examines risk preferences of quantile maximizers. In the context of preferences over distributions, Chambers (2009) shows that monotonicity, ordinal covariance, and continuity characterize quantile preferences. Rostek (2010) axiomatizes the quantile preference in Savage (1954)’s framework, using a ‘typical’ consequence (scenario). Thus, quantile preferences are a useful alternative to the expected utility, and a plausible complement to the study of rational behavior under uncertainty.$^2$

This paper initiates the use of quantile preferences in a dynamic economic setting by providing a comprehensive analysis of a dynamic rational quantile model. As a first step in the development, we introduce dynamic programming for intertemporal decisions whereby the economic agent maximizes the present discounted value of the stream of future $\tau$-quantile utilities by choosing a decision variable in an feasible set. Our first main result establishes dynamic consistency of the quantile preferences, in the sense commonly adopted in decision theory. Second, we show that the optimization problem leads to a contraction, which therefore has a unique fixed-point. This fixed point is the value function of the problem and satisfies the Bellman equation. Third, we prove that the value function is concave and differentiable, thus establishing the quantile analog of the envelope theorem. Fourth, we show that the principle of optimality holds. Fifth, using these results, we derive the corresponding Euler equation for the infinite horizon problem.

$^1$Rostek (2010) discusses the advantages of the quantile preferences, such as robustness and ability to capture heterogeneity by offering a family of preferences indexed by quantiles.

$^2$Quantile preferences can be associated with Choquet expected utility (see, e.g., Chambers (2007); Bassett, Koenker, and Kordas (2004)). The method of Value-at-Risk, which is widespread in finance, also is an instance of quantiles (see, e.g., Engle and Manganelli (2004)).
We note that the theoretical developments and derivations in this paper are of independent interest. The main results for the dynamic quantile model – dynamic consistency, value function, principle of optimality, and Euler equation – are parallel to those of the expected utility model. However, because quantiles do not share all of the convenient properties of expectations, such as linearity and the law of iterated expectations, the generalizations of the results from expected utility to quantile preference are not straightforward.

The derivation of the Euler equation is an important feature of this paper because it allows to connect economic theory with empirical applications. We show that the Euler equation has a conditional quantile representation and relates to quantile regression econometric methods, and hence, our methods provide microeconomic foundations for quantile regression. The Euler equation, which must be satisfied in equilibrium, implies a set of population orthogonality conditions that depend, in a nonlinear way, on variables observed by an econometrician and on unknown parameters characterizing the preferences. Thus, empirically, one can employ existing general econometric methods, such as instrumental variables for nonlinear quantile regression, for estimating and testing the parameters of the model. In this fashion, these parameters can be interpreted as structural objects, and practical inference is simple to implement. In addition, varying the quantiles \( \tau \) enables one to empirically estimate a set of parameters of interest as a function of the quantiles, and hence learn about the potential underlying parameter heterogeneity among the different \( \tau \)-quantiles.\(^3\)

Finally, we briefly illustrate the methods with a dynamic asset-pricing model, which is central to contemporary economics and finance, and has been extensively used.\(^4\) We use a variation of Lucas (1978)’s model where the economic agent decides on how much to consume and save by maximizing a quantile utility function subject to a linear budget constraint. We solve the dynamic problem and obtain the Euler equation. Following a large body of literature, we specify an isoelastic utility function and estimate the implied discount factor and elasticity of intertemporal substitution (EIS) parameters at different levels of risk attitude (quantiles). The empirical results document evidence of heterogeneity in both parameters across quantiles. On the one hand, the discount factor is relatively large for lower quantiles and smaller for upper quantiles, on the other hand, the EIS coefficient is relatively smaller for the lower quantiles and larger for the upper quantiles.

More broadly, this paper contributes to the literature by robustifying economic and policy design, and capturing potential heterogeneity by varying the quantiles \( \tau \). The proposed methods could be applied to any dynamic economic problem, substituting the standard maximization of expectation by maximization of the quantile objective function. Since dynamic economic models are now routinely used in many fields, such as macroeconomics, finance, in-

\(^3\)We note that the theoretical methods do not impose restrictions across quantiles, and thus the parameter estimates might (or might not) vary across quantiles.

ternational economics, public economics, industrial organization and labor economics, among others, the proposed methods expand the scope of economic analysis and empirical applications, providing an interesting alternative to the expected utility models.

The remaining of the paper is organized as follows. Section 2 presents definitions and basic properties of quantiles. Section 3 describes the dynamic economic model and presents the main theoretical results. Section 4 discusses the estimation and inference. Section 5 illustrates the empirical usefulness of the new approach by applying it to the asset pricing model. Finally, Section 6 concludes. We relegate all proofs to the Appendix.

1.1 Review of the Literature

This paper has a broad scope and relates to a number of streams of literature in economic theory and econometrics.

First, the paper relates to the extensive literature on dynamic nonlinear rational expectations models. Many models of dynamic maximization that use expected utility have been proposed and discussed. These models have been workhorses in several economic fields. We refer the reader to more comprehensive works, such as Stokey, Lucas, and Prescott (1989) and Ljungqvist and Sargent (2012). Another related segment of the literature studies recursive utilities. We refer the reader to Epstein and Zin (1989), Marinacci and Montrucchio (2010), Bommier, Kochov, and Le Grand (2017), and Remark 3.11 below for further discussions. We extend this literature by replacing expected utility with quantile utility.

Second, this paper is related to a few works on economic models using the quantile preferences, such as Manski (1988), Chambers (2009), Bhattacharya (2009), Rostek (2010) and Giovannetti (2013). We contribute to this line of work by taking the quantile maximization to a general dynamic optimization model and deriving its properties.

Third, the paper relates to an extensive literature on estimating Euler equations. Since the contributions of Hall (1978), Lucas (1978), Hansen and Singleton (1982), and Dunn and Singleton (1986) it has become standard in economics to estimate Euler equations based on conditional expectation models. There are large bodies of literature in micro and macroeconomics on this subject. We refer the reader to Attanasio and Low (2004) and Hall (2005), and the references therein, for a brief overview. The methods in this paper derive a Euler equation that has a conditional quantile function representation and estimate it using existing econometric methods.

Finally, this paper relates to the quantile regression (QR) literature, for which there is a large body of work in econometrics.\footnote{This paper is also related to an econometrics literature on identification, estimation, and inference of general (non-smooth) conditional moment restriction models. We refer the reader to, among others, Newey and McFadden (1994), Chen, Linton, and van Keilegom (2003), Chen and Pouzo (2009), Chen, Chernozhukov, Lee, and Newey (2014), and Chen and Liao (2015).} Koenker and Bassett (1978) developed QR methods for estimation of conditional quantile functions. These models have provided a valuable tool in
economics and statistics to capture heterogeneous effects, and for robust inference when the presence of outliers is an issue (see, e.g., Koenker (2005)). QR has been largely used in program evaluation studies (Chernozhukov and Hansen (2005) and Firpo (2007)), identification of nonseparable models (Chesher (2003) and Imbens and Newey (2009)), nonparametric identification and estimation of nonadditive random functions (Matzkin (2003)), and testing models with multiple equilibria (Echenique and Komunjer (2009)). This paper contributes to the effort of providing microeconomic foundations for QR by developing a dynamic optimization decision model that generates a conditional quantile restriction (Euler equation).

2 Preliminaries

This section introduces basic concepts considered in the paper. Subsection 2.1 defines quantiles and establishes well-known basic results that are useful later. Subsection 2.2 introduces the one-period quantile preferences that will substitute the standard expected utility preferences in our analysis. Subsection 2.3 briefly defines the notion of risk associated with the quantile preferences.

2.1 Quantiles

Let $X$ be a random variable, with c.d.f. $F(\alpha) \equiv \Pr[X \leq \alpha]$. The quantile function $Q : [0, 1] \to \mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\}$ is the generalized inverse of $F$, that is,

$$Q(\tau) \equiv \begin{cases} \inf \{ \alpha \in \mathbb{R} : F(\alpha) \geq \tau \}, & \text{if } \tau \in (0, 1] \\ \sup \{ \alpha \in \mathbb{R} : F(\alpha) = 0 \}, & \text{if } \tau = 0. \end{cases}$$

(1)

The definition is special for $\tau = 0$ so that the quantile assumes a value in the support of $X$. It is clear that if $F$ is invertible, that is, if $F$ is strictly increasing, its generalized inverse coincide with the inverse, that is, $Q(\tau) = F^{-1}(\tau)$. Usually, it will be important to highlight the random variable to which the quantile refers. In this case we will denote $Q(\tau)$ by $Q_{\tau}[X]$. For convenience, throughout the paper we will focus on $\tau \in (0, 1)$, unless explicitly stated.

In Lemma 7.1 in the appendix, we develop some useful properties of quantiles, such as the fact that it is left-continuous and $F(Q(\tau)) \geq \tau$. Another well-known and useful property of quantiles is “invariance” with respect to monotonic transformations, that is, if $g : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function, then

$$Q_{\tau}[g(X)] = g(Q_{\tau}[X]).$$

(2)

Indeed, $\inf \{ \alpha \in \mathbb{R} : F(\alpha) \geq 0 \} = -\infty$, no matter what is the distribution.
For $\tau \in (0, 1]$, the conditional quantile of $X$ with respect to $Z$ is defined as:

$$Q_{\tau}[X|Z] = \inf\{\alpha \in \mathbb{R}: \Pr([X \leq \alpha]|Z = z) \geq \tau\}. \quad (3)$$

Lemma 7.2, in the appendix, generalizes (2) to conditional quantiles. More precisely, Lemma 7.2 proves that if $g : \Theta \times Z \rightarrow \mathbb{R}$ is non-decreasing and left-continuous in $Z \in Z$, then,

$$Q_{\tau}[g(\theta, \cdot)|Z = z] = g(\theta, Q_{\tau}[X|Z = z]). \quad (4)$$

This property is repeatedly used in the rest of the paper.

### 2.2 Quantile Preference

Expected utility is the widely used preference in economics and econometrics. To contextualize the difference between the expected utility and the quantile preferences, let $\mathcal{R}$ denote the set of random variables (lotteries). We say that the functional $\varphi : \mathcal{R} \rightarrow \mathbb{R}$ represents the preference $\succ$ if for all $X, Y \in \mathcal{R}$ we have:

$$X \succ Y \iff \varphi(X) \geq \varphi(Y). \quad (5)$$

In von-Neumann-Morgenstern’s expected utility, $\varphi(X) = E[u(X)]$. To be more specific, von-Neumann-Morgenstern theorem states that $\succ$ satisfies completeness, transitivity, continuity and independence if and only if there exists an utility function $u$ such that\footnote{See Kreps (1988) for more details.}

$$X \succ Y \iff E[u(X)] \geq E[u(Y)]. \quad (6)$$

This paper departs from this standard preference by adopting quantile preferences, where the functional $\varphi$ in (5) is given by a quantile function, that is, $\varphi(X) = Q_{\tau}[u(X)]$, so that:

$$X \succ Y \iff Q_{\tau}[u(X)] \geq Q_{\tau}[u(Y)]. \quad (7)$$

Manski (1988) was the first to study this preference, which was recently axiomatized by Chambers (2009) and Rostek (2010). Rostek (2010) axiomatized the quantile preferences in the context of Savage (1954)’s subjective framework. Rostek (2010) modifies Savage’s axioms to show that they are equivalent to the existence of a $\tau \in (0, 1)$, probability measure and utility function such that the functional $\varphi$ in equation (5) is a quantile function.\footnote{If $\tau \in \{0, 1\}$, the statement is more complex; see her paper for details.} In contrast, Chambers (2009) departs from a framework where the utility function and the probability distributions are in some sense already fixed. He shows that the preference satisfies monotonicity,

\footnote{Rostek (2010) also shows that the quantiles preferences are probabilistic sophisticated for $\tau \in (0, 1)$, by using a variation of the original concept of probabilistic sophistication introduced by Machina and Schmeidler (1992).}
ordinal covariance, and continuity if and only if (7) holds, that is, the preference is a quantile preference; see his paper for more details.10

2.3 Risk in the Quantile Model

Another interesting property of the quantile preference is the relationship of the risk attitude with respect to the $\tau$, identified by Manski (1988) and Rostek (2010). They show that the quantile model admits a notion of comparative risk attitude.

Rostek (2010) argues that quantile maximizers are concerned with downside risk, which can be defined as following.

**Definition 2.1.** 1. The distribution $Q \in P_o(X)$ crosses distribution $P \in P_o(X)$ from below if there exists $x \in X$, such that (i) $Q(y) \leq P(y)$ for all $y$, such that $y < x$ and (ii) $Q(y) \geq P(y)$ for all $y$, such that $y > x$.

2. Consider the class of all pairs of distributions with the single-crossing property, $\mathcal{SC} = \{(P, Q) \in P_o(X) \times P_o(X) : Q \text{ crosses } P \text{ from below}\}$. For any pair $(P, Q)$ in $\mathcal{SC}$, there exists an outcome $x$ such that $P(y < z) \geq Q(y < z)$ for all $z < x$, and $P(y > z) \geq Q(y > z)$ for all $z > x$; we will say that $P$ involves more downside risk than $Q$ with respect to $x$.

Intuitively, this comparative notion allows ranking the attractiveness of distributions by comparing the likelihood of losses with respect to outcome $x$. Say that individual $A$ is more risk-averse than individual $B$ if, for all pairs of distributions $(P, Q) \in \mathcal{SC}$, whenever $B$ weakly prefers a distribution which involves less downside risk, so does $A$. The following result establishes the connection between the risk attitude and quantiles; see Rostek (2010, section 6.1) for discussion.

**Theorem 2.2** (Rostek, 2010). In the Quantile Maximization model, $\tau < \tau'$ if and only if a $\tau$-maximizer is weakly more averse toward downside risk than a $\tau'$-maximizer.

Theorem 2.2 shows that $\succeq^{\tau'}$ is more risk-averse than $\succeq^\tau$ if and only if $\tau' < \tau$. Thus, a decision maker that maximizes a lower quantile is more “risk-averse” (in the sense used by Rostek) than one who maximizes a higher quantile. In other words, the risk-attitude can be related to the quantile rather than to the concavity of the utility function. To understand this, fix $u(x) = x^\rho$ and remember that (2) implies $\phi^\rho_\tau(X) = Q_\tau[(X)^\rho] = (Q_\tau[X])^\rho$. Thus, $\phi^\rho_\tau$ and $\phi^{\rho'}_\tau$ represent the same preference, for any $\rho, \rho' > 0$, that is,

$$X \succeq Y \iff \phi^\rho_\tau(X) \succeq \phi^\rho_\tau(Y) \iff Q_\tau[X] \succeq Q_\tau[Y] \iff \phi^{\rho'}_\tau(X) \succeq \phi^{\rho'}_\tau(Y).$$

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10Since the upper semicontinuity property is a technical condition and first-order stochastic dominance is a mild property, also satisfied by expected utility, the really important property is invariance with respect to monotonic transformations. We have stated this property in equation (2). Thus, this property could be considered the essence of the quantile preference considered here.
Therefore, we can represent this preference just by \( \varphi_\tau(X) \equiv Q_\tau[X] \). However, if \( \varphi^\rho(X) \equiv E[(X)^\rho] \) and \( 0 < \rho < \rho' < 1 \), then \( \varphi_\rho \) represents a more risk-averse preference (in the standard sense) than \( \varphi_{\rho'} \). In fact, for each given random variable \( X \), it is possible to establish a map between the coefficient of relative risk aversion \( \rho \) and the quantile \( \tau \) that captures a similar attitude towards risk.

3 Economic Model and Theoretical Results

This section describes a dynamic economic model and develops a dynamic program theory for quantile preferences. We try to follow closely Stokey, Lucas, and Prescott (1989, chapter 9). We begin in subsection 3.1 by extending the quantile preference to a dynamic environment, suitable for our analysis. Subsection 3.2 states and discusses the assumptions used for establishing the main results. Subsection 3.3 establishes the existence of recursive functions, necessary to complete the definition of the preferences. Subsection 3.4 shows that the preference is dynamically consistent. In subsection 3.5 we establish the existence of the value function and its differentiability. Subsection 3.6 states and proves, in our context, the Bellman’s Principle of Optimality, which allows to pass from plans to single period decisions and vice-versa, thus establishing that the value function corresponds to the original dynamic problem in a precise sense. Subsection 3.7 derives the Euler equation associated to this dynamic problem, which describes the agents behavior and is useful for the econometric part of the paper. Finally, subsection 3.8 illustrates the theory with an example of the one-sector growth model.

The main results in this section are generalizations to the quantile preferences’ case of the corresponding ones in Stokey, Lucas, and Prescott (1989), which focus on expected utility. First, they increase the scope of potential applications of economic models substantially by using quantile utility. Second, the generalizations are of independent interest. The demonstrations are not routine since quantiles do not possess several of the convenient properties of expectations, such as linearity and the law of iterated expectations.

3.1 Dynamic Environment and Dynamic Quantile Preference

Section 2.2 introduced and discussed the quantile preferences with respect to single period uncertainty. We adopt this preference in a dynamic environment. In such an environment, the state variable whose quantile the decision-maker/consumer is interested is given by a stream of future consumption. To describe this more formally, we introduce now a dynamic setting that will be used in the rest of the paper.

3.1.1 States and Shocks

Let \( \mathcal{X} \subset \mathbb{R}^p \) denote the state space, and \( \mathcal{Z} \subset \mathbb{R}^k \) the range of the shocks (random variables) in the model. Let \( x_t \in \mathcal{X} \) and \( z_t \in \mathcal{Z} \) denote, respectively, the state and the shock in period \( t \),
both of which are known by the decision maker at the beginning of period \( t \). We may omit the time indexes for simplicity, when it is convenient. Let \( \mathcal{Z}^t = \mathcal{Z} \times \cdots \times \mathcal{Z} \) (\( t \)-times, for \( t \in \mathbb{N} \)), \( \mathcal{Z}^\infty = \mathcal{Z} \times \mathcal{Z} \times \cdots \) and \( \mathbb{N}^0 \equiv \mathbb{N} \cup \{0\} \). Given \( z \in \mathcal{Z}^\infty \), \( z = (z_1, z_2, \ldots) \), we denote \( (z_t, z_{t+1}, \ldots) \) by \( t z \) and \( (z_t, z_{t+1}, \ldots, z_{t'}) \) by \( t z' \). A similar notation can be used for \( x \in \mathcal{X}^\infty \).

The random shocks will follow a time-invariant (stationary) Markov process. More precisely, a probability density function (p.d.f.) \( f : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+ \) establishes the dependence between \( Z_t \) and \( Z_{t+1} \), such that the process is invariant with respect to \( t \). For simplicity of notation, we will frequently represent \( Z_t \) and \( Z_{t+1} \) by \( Z \) and \( Z' \), respectively. We will assume that \( f \) and \( Z \) satisfy standard assumptions, as explicitly stated below in section 3.2.

For any topological space \( S \), we will denote by \( \sigma(S) \) the Borel \( \sigma \)-algebra. For each \( z \in \mathcal{Z} \) and \( A \in \sigma(\mathcal{Z}) \), define

\[
K(z, A) = \int_A f(z'|z)dz',
\]  

(8)

where \( f(z'|z) = \frac{f(z,z')}{\int_{\mathcal{Z}} f(z,z')dz} \). Thus, \( K \) is a probabilistic kernel, that is, (i) \( z \mapsto K(z, A) \) is measurable for every \( A \in \sigma(\mathcal{Z}) \); and (ii) \( A \mapsto K(z, A) \) is probability measure for every \( z \in \mathcal{Z} \). In other words, \( K \) represents a conditional probability, and we may emphasize this fact by writing \( K(A|z) \) instead of \( K(z, A) \). We will also abuse notation by denoting \( K(z, \{\tilde{z} : \tilde{z} \in z'\}) \) simply by \( K(z'|z) \).

### 3.1.2 Plans

At the beginning of period \( t \), the decision maker knows the current state \( x_t \) and learns the shock \( z_t \) and decides (according to preferences defined below) the future state \( x_{t+1} \in \Gamma(x_t, z_t) \subset \mathcal{X} \), where \( \Gamma(x, z) \) is the constraint set.\(^ {11} \) From this, we can define plans as follows:

**Definition 3.1.** A plan \( \pi \) is a profile \( \pi = (\pi_t)_{t \in \mathbb{N}} \) where, for each \( t \in \mathbb{N} \), \( \pi_t \) is a measurable function from \( \mathcal{X} \times \mathcal{Z}^t \) to \( \mathcal{X} \).\(^ {12} \)

The interpretation of the above definition is that a plan \( \pi_t(x_t, z^t) \) represents the choice that the individual makes at time \( t \) upon observing the current state \( x_t \) and the sequence of previous shocks \( z^t \). The following notation will simplify statements below.

**Definition 3.2.** Given a plan \( \pi = (\pi_t)_{t \in \mathbb{N}} \in \Pi \), \( x \in \mathcal{X} \) and realization \( z^\infty = (z_1, \ldots) \in \mathcal{Z}^\infty \), the sequence associated to \( (x, z^\infty) \) is the sequence \( (x^\pi_t)_{t \in \mathbb{N}^0} \in \mathcal{X}^\infty \) defined recursively by \( x^\pi_1 = x \) and \( x^\pi_t = \pi_{t-1}(x^\pi_{t-1}, z^{t-1}) \), for \( t \geq 2 \). Similarly, given \( \pi \in \Pi \), \( x, z^t \in \mathcal{X} \times \mathcal{Z}^t \), the \( t \)-sequence associated to \( (x, z^t) \) is \( (x^\pi_t)_{t=1}^T \in \mathcal{X}^t \) defined recursively as above.

\(^ {11} \)This model is very close to the one discussed in Stokey, Lucas, and Prescott (1989, Chapter 9). There are different, slightly more complicated dynamic models where the state is not chosen by the decision maker, but defined by the shock. The arguments in the current model can be extended to those models when preferences are expected utility, as Stokey, Lucas, and Prescott (1989, Chapter 9) discuss. In our setup, this extension may be more involved.

\(^ {12} \)In the expressions below, \( \pi_0(z^0) \) should be understood as just \( \pi_0 \in \mathcal{X} \).
We may write \( x_t^\pi(\cdot), x_t^\pi(x, z^1) \) or \( x_t^\pi(x, z^{∞}) \) to emphasize that \( x_t^\pi \) depends on the initial state \( x \) and on the sequence of shocks \( z^∞ \), up to time \( t \).

**Definition 3.3.** A plan \( \pi \) is feasible from \( (x, z) \in \mathcal{X} \times Z \) if \( \pi_t(x_t^\pi, z^t) \in \Gamma(x_t^\pi, z_t) \) for every \( t \in \mathbb{N} \) and \( z^∞ \in Z^∞ \) such that \( x_1^\pi = x \) and \( z_1 = z \).

We denote by \( \Pi(x, z) \) the set of feasible plans from \( (x, z) \in \mathcal{X} \times Z \). Let \( \Pi \) denote the set of all feasible plans from some point, that is, \( \Pi = \cup_{(x, z) \in \mathcal{X} \times Z} \Pi(x, z) \).

### 3.1.3 Preferences

Let \( \Omega_t \) represent all the information revealed up to time \( t \).\(^{13} \) We assume that in time \( t \) with revealed information \( \Omega_t \), the consumer/decision-maker has a preference \( \succeq_{t, \Omega_t} \) over plans \( \pi, \pi' \in \Pi(x, z) \), which is represented by a function \( V_t : \Pi \times \mathcal{X} \times Z^t \to \mathbb{R} \), that is,

\[
\pi' \succeq_{t, \Omega_t} \pi \iff V_t(\pi', x, z^1) \geq V_t(\pi, x, z^1). \tag{9}
\]

A special case of this model corresponds to the standard case of expected utility, that is,

\[
V_t(\pi, x, z^1) = E \left[ \sum_{s > t} \beta^{s-t} u(x_s^\pi, x_{s+1}^\pi, Z_s) \bigg| Z^t = z^1 \right], \tag{10}
\]

where \( u : \mathcal{X} \times \mathcal{X} \times Z \to \mathbb{R} \) is the current-period utility function. That is, \( u(x, y, z) \) denotes the instantaneous utility obtained in the current period when \( x \in \mathcal{X} \) denotes the current state, \( y \in \mathcal{X} \), the choice in the current state, and \( z \in Z \), the current shock.

The first attempt to define the dynamic quantile preference would be to substitute the expectation operator \( E \) by the quantile operator \( Q_\tau \) in (10). Although this seems the natural way to adapt the standard definition, this would lead to dynamically inconsistent preferences. The reason is that the analog of the “law of iterated expectations” does not hold for quantiles.\(^{14} \) Therefore, to define a dynamically consistent preference, we need to require this property in the construction of the preferences. The standard way to do this is to use a recursive equation. Indeed, note that that the functions \( V_t \) defined by (10) satisfy the following recursive equation:

\[
V_t(\pi, x, z^t) = u(x_t^\pi, x_{t+1}^\pi, z_t) + \beta E \left[ V_{t+1}(\pi, x, (z^t, Z_{t+1})) \bigg| Z^t = z^1 \right]. \tag{11}
\]

This basic recursive property guarantees dynamic consistency. Departing from it, we consider the following recursive equation, where the expectation operator \( E \) is substituted by the quantile operator \( Q_\tau \), that is, we impose:

\[
V_t(\pi, x, z^t) = u(x_t^\pi, x_{t+1}^\pi, z_t) + \beta Q_{\tau} \left[ V_{t+1}(\pi, x, (z^t, Z_{t+1})) \bigg| Z^t = z^1 \right]. \tag{12}
\]

\(^{13} \)With the knowledge of a fixed \( \pi, \Omega_t \) reduces to the initial state \( x_1 \) and the sequence of shocks \( z^1 \). More generally, we could take the sequence of states and shocks \( (x^1, z^1) \).

\(^{14} \)This fact is formally stated and proved in Proposition 3.7 below.
In section 3.3 below, we explicitly define a sequence of functions \( V_t \) that satisfy (12) and will specify the preferences (9). But before our formal definition, it is useful to build intuition on how the recursive equation (12) leads to an expression in quantiles that would be different from the expected utility case, developed from (11).

To see this, let us adopt \( t = 1 \) and substitute the expression of \( V_{t+1} = V_2 \) by the expression in (11) and use superscript \( E \) to denote the expected utility case, we obtain:

\[
V^E_1(\pi, x, z^1) = u(x_1^\pi, x_2^\pi, z_1) + \beta E \left[ u(x_2^\pi, x_3^\pi, z_2) + \beta^2 V^E_2(\pi, x, z^1) \bigg| Z_2 = z_2 \bigg] Z_1 = z \right].
\]

Above, we could eliminate the expectation with respect to \( Z_2 = z_2 \) using of the Law of Iterated Expectations. Since the same simplification is not possible in the quantile case, we will avoid it here. Moreover, we will put all the terms inside the expectations. That is, we can write:

\[
V^E_1(\pi, x, z^1) = E \left[ E \left[ \sum_{t=1}^{3} \beta^{t-1} u(x_t^\pi, x_{t+1}^\pi, z_t) + \beta^3 V^E_3(\pi, x, z^1) \bigg| Z_3 = z_3 \bigg] Z_2 = z_2 \bigg] Z_1 = z \right]
\]

\[
= \ldots E \left[ \sum_{t=1}^{n} \beta^{t-1} u(x_t^\pi, x_{t+1}^\pi, z_t) + \beta^n V^E_n(\pi, x, z^1) \bigg| Z_n = z_n \bigg] \ldots Z_1 = z ,
\]

where there are \( n \) expectation operators \( E \) and corresponding conditionals \( Z_t = z_t \) in the last line above. Following the same developments from (12), we obtain:

\[
V^{Q_\tau}_1(\pi, x, z^1) = u(x_1^\pi, x_2^\pi, z_1) + \beta Q_\tau \left[ u(x_2^\pi, x_3^\pi, z_2) + \beta^2 V^{Q_\tau}_2(\pi, x, z^1) \bigg| Z_2 = z_2 \bigg] Z_1 = z \right]
\]

\[
= Q_\tau \left[ Q_\tau \left[ \sum_{t=1}^{3} \beta^{t-1} u(x_t^\pi, x_{t+1}^\pi, z_t) + \beta^3 V^{Q_\tau}_3(\pi, x, z^1) \bigg| Z_3 = z_3 \bigg] Z_2 = z_2 \bigg] Z_1 = z \right]
\]

\[
= \ldots Q_\tau \left[ \sum_{t=1}^{n} \beta^{t-1} u(x_t^\pi, x_{t+1}^\pi, z_t) + \beta^n V^{Q_\tau}_n(\pi, x, z^1) \bigg| Z_n = z_n \bigg] \ldots Z_1 = z , \quad (13)
\]

where the operator \( Q_\tau[\cdot] \) and corresponding conditionals \( Z_t = z_t \) appear \( n \) times in the last line above. In order to simplify the above equation, we will introduce the following notation:

\[
Q^{n}_\tau[\cdot] = Q_\tau \left[ \ldots Q_\tau \left[ \cdot \bigg| Z_n = z_n \bigg] \ldots \right] \bigg| Z_1 = z \right] .
\]

where the operator \( Q_\tau \) and corresponding conditionals appear \( n \) times. Therefore, by using
the notation defined by (14), we are able to rewrite (13) as

\[ V_1^{Q_\tau}(\pi, x, z^t) = Q_\tau^n \left[ \sum_{t=1}^{n} \beta^{t-1} u(x_t^\pi, x_{t+1}^\pi, z_t) + \beta^n V_1^{Q_\tau}(\pi, x, z^t) \right]. \]

Our next step is to take the limit as \( n \) goes to \( \infty \). The formalization of such limit will be made in section 3.3 below, but one can now intuitively understand the following:

\[ V_1^{Q_\tau}(\pi, x, z^t) = Q_\tau^\infty \left[ \sum_{t=1}^{\infty} \beta^{t-1} u(x_t^\pi, x_{t+1}^\pi, z_t) \right] \]

(15)

as a notation for an (infinite) sequence of applications of \( Q_\tau^n[\cdot|Z^t = z^t] \).

Note that if we introduce an analogous notation of (14), that is \( E^\infty \) for a(n infinite) sequence of conditional expectations, because of the law of iterated expectations, we obtain

\[ V_1^E(\pi, x, z^t) = E^\infty \left[ \sum_{t=1}^{\infty} \beta^{t-1} u(x_t^\pi, x_{t+1}^\pi, z_t) \right] = E \left[ \sum_{t=1}^{\infty} \beta^{t-1} u(x_t^\pi, x_{t+1}^\pi, z_t) \mid Z_1 = z_1 \right], \]

which is the expression in (10). Therefore, our expression (15) is the corresponding generalization of (10): the difference, that is, the fact that we can substitute \( E^\infty \) by \( E \) but not \( Q_\tau^\infty \) by \( Q_\tau \), is explained by whether or not the law of iterated expectations hold, as Proposition 3.7 below shows.

### 3.2 Assumptions

Now we state the assumptions used for establishing the main results. We organize the assumptions in two groups. The first group collects basic assumptions, which will be assumed throughout the paper, even if they are not explicitly stated. The second group of assumptions will be used only to obtain special, desirable properties of the value function.

**Assumption 1 (Basic).** The following properties are maintained throughout the paper:

(i) \( Z \subseteq \mathbb{R}^k \) is convex;

(ii) \( f : Z \times Z \rightarrow \mathbb{R}_+ \) is continuous, symmetric and \( f(z, z') > 0 \), for all \((z, z') \in Z \times Z\);\(^{15}\)

(iii) \( X \subseteq \mathbb{R}^p \) is convex;

(iv) \( u : X \times X \times Z \rightarrow \mathbb{R} \) is continuous and bounded;

(v) The correspondence \( \Gamma : X \times Z \rightharpoonup X \) is continuous, with nonempty, compact and convex values.

\(^{15}\)Symmetry guarantees stationarity since \( \Pr([Z_1 \in A]) = \int_Z \int_A f(z_1, z_2)dz_1dz_2 = \int_A \int_Z f(z_1, z_2)dz_1dz_2 = \Pr([Z_2 \in A]). \)
Note that in the above assumption, property (i) allows an unbounded multidimensional Markov process, requiring only that the support is convex. Property (ii) imposes continuity of \( f \), the pdf that establishes the dependence between \( Z_t \) and \( Z_{t+1} \) and requires it to be strictly positive in the support of the Markov process, \( Z \). The state space \( \mathcal{X} \) is not required to be compact, but only convex by property (iii). Property (iv) is the standard continuity assumption. Property (v) and the continuity of \( u \) required in property (iv) guarantee that an optimal choice always exist.

For some results we will also require differentiability, concavity and monotonicity assumptions.

**Assumption 2** (Differentiability, Concavity and Monotonicity). The following holds:

(i) \( Z \subseteq \mathbb{R} \) is an interval;

(ii) If \( h: Z \to \mathbb{R} \) is weakly increasing and \( z \leq z' \), then:

\[
\int_Z h(\alpha) f(\alpha|z) d\alpha \leq \int_Z h(\alpha) f(\alpha|z') d\alpha;
\]  

(iii) \( u: \mathcal{X} \times \mathcal{X} \times Z \to \mathbb{R} \) is \( \mathcal{C}^1 \), strictly concave in the first two variables and strictly increasing in the last variable;

(iv) For every \( x \in \mathcal{X} \) and \( z \leq z' \), \( \Gamma(x,z) \subseteq \Gamma(x,z') \);

(v) For all \( z \in Z \) and all \( x, x' \in \mathcal{X} \), \( y \in \Gamma(x,z) \) and \( y' \in \Gamma(x',z) \) imply

\[
\theta y + (1-\theta)y' \in \Gamma[\theta x + (1-\theta)x', z], \text{ for all } \theta \in [0,1].
\]

To work with monotonicity, we restrict the dimension of the Markov process to \( k = 1 \) in Assumption 2(i). Assumptions 2(ii) – 2(v) are standard conditions on dynamic models (see, e.g., Assumptions 9.8 – 9.15 in Stokey, Lucas, and Prescott (1989)). Assumption 2(ii) implies that whenever \( z \leq z' \),

\[
K(w|z') = \int_{\{\alpha \in Z : \alpha \leq w\}} f(\alpha|z') d\alpha \leq \int_{\{\alpha \in Z : \alpha \leq w\}} f(\alpha|z) d\alpha = K(w|z),
\]  

for all \( w \).\(^{16}\) In other words, \( K(\cdot|z') \) first-order stochastically dominates \( K(\cdot|z) \). Assumption 2(iii) allows us to establish the continuity and differentiability of the value function. Assumption 2(iv) only requires the monotonicity of the choice set. Assumption 2(v) implies that \( \Gamma(x,z) \) is a convex set for each \( (x,z) \in \mathcal{X} \times Z \), and that there are no increasing returns.

It should be noted that monotonicity also is important for econometric reasons. Indeed, Matzkin (2003, Lemma 1, p. 1345) shows that two econometric models are observationally

\[\to obtain (17), it is enough to use h(z) = -1_{\{\alpha \in Z : \alpha \leq w\}}(z) in (47).\]
equivalent if and only if there are strictly increasing functions mapping one to another. Thus, in a sense, the quantile implied by a model is the essence of what can be identified by an econometrician.

### 3.3 The Sequence of Recursive Functions

In this section, we define the sequence of functions $V_t$ that satisfy (12) and specify the preferences (9). For this, we need to define a transformation. Let $L$ be the set of real-valued functions from $X \times Z$ to $\mathbb{R}$ and let $C \subseteq L$ denote the set of bounded continuous functions from $X \times Z$ to $\mathbb{R}$, endowed with the sup norm. It is well known that $C$ is a Banach space. Let us fix $\pi \in \Pi$ and $\tau \in (0,1)$, and define the transformation $T^\pi : C \to L$ (the dependency on $\tau$ is omitted) by the following:

$$T^\pi(V)(x,z) = u(x_1^\pi, x_2^\pi, z_1) + \beta Q_{\tau}[V(x_2^\pi, Z_2)]Z_1 = z,$$

where $(x_1^\pi, z_1) = (x, z)$ and $x_2^\pi = \pi(x, z)$. We show that the image of $T^\pi$ is indeed in $C$ continuous and that $T^\pi$ is a contraction and, therefore, has a unique fixed point:

**Theorem 3.4.** $T^\pi(V)$ is continuous and bounded on $X \times Z$, that is, $T(C) \subseteq C$. Moreover, $T^\pi$ is a contraction and has a unique fixed point, denoted $V^\pi \in C$.

Now we can define $V_t$ as follows:

$$V_t(\pi, x, z^t) = V^\pi(x_1^t, z_t),$$

where $(x_1^t)_{t=1}^\infty$ is the associated t-sequence to $(x, z^t)$ (see definition 3.2). From the fact that $V^\pi$ is the unique fixed point of $T^\pi$, it is clear that (12) holds. This completes the definition of the preferences (9).

It is possible to write $V^\pi$ in a more explicit form. For this, let us define

$$V^\pi(x, z) = u(x_1^\pi, x_2^\pi, z_1) + Q_{\tau} \left[ \beta u(x_2^\pi, x_3^\pi, z_2) + Q_{\tau} \left[ \beta^2 u(x_3^\pi, x_4^\pi, z_3) + \ldots \right] \right]$$

$$= Q_{\tau} \left[ \ldots \left[ \beta^n u(x_{n+1}^\pi, x_n^\pi, z_n) \right] \ldots \right] \left[ Z_1 = z \right],$$

where the expression $Q_{\tau}^n[\cdot]$ in the last line is just a short notation for the conditional quantiles applied successively, as shown in the previous line; see (14). With this definition, we obtain:
Proposition 3.5. $V^n(x, z) = \lim_{n \to \infty} V^n(x, z)$.

Thus, the existence of the limit $\lim_{n \to \infty} V^n(x, z)$ allows us to define the notation $Q^\infty_{\tau}[,]$, that is,

\[
V^n(x, z) = Q^\infty_{\tau}\left[\sum_{t=0}^{\infty} \beta^t u(x^\tau_{t+1}, x^\tau_{t+2}, z_t)\right] = u(x^\tau_1, x^\tau_2, z_1) + Q_{\tau}\left[\beta u(x^\tau_2, x^\tau_3, z_2) + \beta^2 u(x^\tau_3, x^\tau_4, z_3) + \ldots\right]
\]

\[
\ldots + Q_{\tau}\left[\beta^n u(x^\tau_{n+1}, x^\tau_{n+2}, z_{n+1}) + \ldots\right] + \ldots + Z_2 = z_2\mid Z_1 = z
\]

We turn now to verify that this preference is dynamically consistent.

3.4 Dynamic Consistency

Our objective is to develop a dynamic theory for quantile preferences. Thus, the dynamic consistency of such preferences is of uttermost importance. In this section we formally define dynamic consistency and show that it is satisfied by the above defined preferences. The following definition comes from Maccheroni, Marinacci, and Rustichini (2006); see also Epstein and Schneider (2003).

Definition 3.6 (Dynamic Consistency). The system of preferences $\succ_{t, \Omega_t}$ is dynamically consistent if for every $t$ and $\Omega_t$ and for all plans $\pi$ and $\pi'$, $\pi_{t'}(\cdot) = \pi'_{t'}(\cdot)$ for all $t' \leq t$ and $\pi' \succ_{t+1, \Omega_{t+1}} x$ for all $\Omega'_{t+1}, x$, implies $\pi' \succ_{t, \Omega_t} x$.

In principle, there is no reason to expect that quantile preferences would be dynamically consistent. For instance, the law of iterated expectations, which is important to the dynamic consistency of expected utility, does not have an analogous for quantile preferences, as the following result shows.

Proposition 3.7. Let $\Sigma_1 \supset \Sigma_0$ be two $\sigma$-algebras on $\Omega$, $\tau \in (0, 1)$, and consider random variables $X: \Omega \to \mathbb{R}$ and $Y: \Omega \to \mathbb{R}$. Then, in general,

\[
Q_{\tau}[Q_{\tau}[X|\Sigma_1]|\Sigma_0] \neq Q_{\tau}[X|\Sigma_0].
\]

and it is possible that

\[
Q_{\tau}[X|\Sigma_1](\omega) \geq Q_{\tau}[Y|\Sigma_1](\omega), \forall \omega \in \Omega, \text{ but } Q_{\tau}[X|\Sigma_0](\omega) < Q_{\tau}[Y|\Sigma_0](\omega), \forall \omega \in \Omega.
\]

Note that (22) suggests a potential negation of dynamic consistency for quantile preferences in general. Indeed, this failure would be fatal for dynamic consistency if we had chosen the
preference to seek the maximization of $Q_\tau \left[ \sum_{t=0}^{\infty} \beta^t u(x_t^\tau, x_{t+1}^\tau, z_t) \right]$, because changing from one period to the other could imply a reversion of choices, which is exactly what (22) illustrates. However, because we have adopted as preference $Q_\tau^{\infty} \left[ \sum_{t=0}^{\infty} \beta^t u(x_t^\tau, x_{t+1}^\tau, z_t) \right]$, which involves an infinite sequence of nested conditional quantiles, as explained in section 3.1.3, where the notation $Q_\tau^{\infty} [\cdot]$ is also introduced. This is exactly what allows to obtain dynamic consistency. Indeed, in our framework, quantile preferences are dynamically consistent and amenable to the use of the standard techniques of dynamic programming, as the following result establishes.

**Theorem 3.8.** The quantile preferences defined by (9) are dynamically consistent.

This result is important, because it implies that no money-pump can be used against a decision maker with quantile preferences. Many preferences that departure from the expected utility framework do not satisfy dynamic consistency.

The reason why this result holds is that we imposed the recursive structure (12). This implies that in each period, the decision will be made taking in account a fixed succession of conditional quantiles. Thus, there is no reverse in choices. Details are given in the appendix.

### 3.5 The Value Function

In this section we establish the existence of the value function associated to the dynamic programming problem for the quantile utility and some of its properties. This is accomplished through a contraction fixed point theorem.

The first step is to define the contraction operator; this is similar to what we have defined in Section 3.3. For $\tau \in (0, 1)$, define the transformation $M^\tau : C \to C$ as

$$M^\tau(v)(x, z) = \sup_{y \in \Gamma(x, z)} u(x, y, z) + \beta Q_\tau[v(y, w)|z]. \quad (23)$$

Note that this is similar to the usual dynamic program problem, in which the expectation operator $E[\cdot]$ is in place of $Q_\tau[\cdot]$. The main objective is to show that the above transformation has a fixed point, which is the value function of the dynamic programming problem. The following result establishes the existence of the contraction $M^\tau$ under the basic assumptions assumed throughout this paper.

**Theorem 3.9.** $M^\tau$ is a contraction and has a unique fixed point $v^\tau \in C$.

The unique fixed point of the problem will be the value function of the problem. Notice that the proof of this theorem is not just a routine application of the similar theorems from the expected utility case. In particular, the continuity of the function $(x, z) \mapsto Q_\tau[v(x, w)|z]$ is not immediate as in the standard case. Since $v$ is not assumed to be strictly increasing in the second argument, it can be constant at some level. Constant values may potentially lead to
discontinuities in the c.d.f or quantile functions; see illustration in section 7.1 in the appendix. Thus, some careful arguments are needed for establishing this continuity.

The next step is to establish the differentiability and monotonicity of the value function.

**Theorem 3.10.** If Assumption 2 holds, then \( v^\tau : \mathcal{X} \times \mathcal{Z} \to \mathbb{R} \) is differentiable in \( x \), strictly increasing in \( z \), and strictly concave in an interior point \( x \). Moreover,

\[
\frac{\partial v^\tau}{\partial x_i}(x, z) = \frac{\partial u}{\partial x_i}(x, y^*, z),
\]

where \( y^* \in \Gamma(x, z) \) is the unique maximizer of (23), assumed to be interior to \( \Gamma(x, z) \).

This theorem is the most important result in the paper since it delivers interesting and important properties of the value function. It establishes that the value function that one obtains from quantile functions possesses, essentially, the same basic properties of the value function of the corresponding expected utility problem. The second part of Theorem 3.10 is very important for the characterization of the problem. It is the extension of the standard envelope theorem for the quantile utility case. Notice that since the quantile function does not have some of the convenient properties of the expectation, we assumed that \( z \) were unidimensional (see Assumption 2) in order to establish the conclusions of Theorem 3.10. However, this unidimensionality requirement does not seem overly restrictive in most practical applications. For example, it allows us to tackle the standard asset pricing model, as section 5 shows.

**Remark 3.11.** The result in Theorem 3.9 is related to that in Marinacci and Montrucchio (2010). They establish the existence and uniqueness of the value function in a more general setup. Nevertheless, we are able to provide sharper characterizations of the fixed point \( v^\tau \). In particular, Theorem 3.9 establishes that \( v^\tau \) is continuous and Theorem 3.10 that it is differentiable, concave, and increasing.

### 3.6 The Principle of Optimality

This section establishes that the principle of optimality holds in our model (Theorem 3.15 below). That is, we show that optimizing period after period, as in the recursive problem (23), yields the same result as choosing the best plan for the whole horizon of the problem. In order to do that, we have to complete three tasks. First, we define the problem of choosing plans. Next, we revisit the recursive problem to establish a result that will be useful in the sequel. Finally, we show that choosing plans for the entire horizon and solving the problem step by step as in the recursive problem, lead to the same values.

For this, let us begin by establishing that the set of feasible plans departing from \((x, z) \in \mathcal{X} \times \mathcal{Z}\) at time \( t \) is nonempty. More formally, let us define:

\[
\Pi_t(x, z) \equiv \{ \pi \in \Pi(x, z) : \exists (x, z^t) \in \mathcal{X} \times \mathcal{Z}^t, \text{ with } z_t = z, \text{ such that } x_t^\pi(x, z^t) = x \}.
\]
Thus, $\Pi_1(x, z)$ is just $\Pi(x, z)$. We have the following result regarding the set of feasible plans:

**Lemma 3.12.** For any $x \in \mathcal{X}$ and $t \in \mathbb{N}$, $\Pi_t(x, z) \neq \emptyset$.

This result allows us to define a supremum function as:

$$v^*_t(x, z) \equiv \sup_{\pi \in \Pi_t(x, z)} V_t(\pi, x, z). \quad (24)$$

We first observe that $t$ plays no role in the above equation (24), that is, we prove the following:

**Lemma 3.13.** For any $t \in \mathbb{N}$ and $(x, z) \in \mathcal{X} \times \mathcal{Z}$, $v^*_t(x, z) = v^*_1(x, z)$.

Thus, we are able to drop the subscript $t$ from (24) and write $v^*(x, z)$ instead of $v^*_t(x, z)$.

The next step is to relate $v^*$ to $v^\tau$, the solution of the functional equation studied in the previous section, which was proved to exist in Theorem 3.9 and satisfies the Bellman equation:

$$v^\tau(x, z) = \sup_{y \in \Gamma(x, z)} \{ u(x, y, z) + \beta Q_\tau[v^\tau(y, w)|z] \}. \quad (25)$$

In the rest of this section we will denote $v^\tau$ simply by $v$.

To achieve this aim, we first establish important results relating $v$ in equation (25) to the policy function that solves the original problem. In particular, the next result allows us to define the policy function:

**Lemma 3.14.** If $v$ is a bounded continuous function satisfying (25), then for each $(x, z) \in \mathcal{X} \times \mathcal{Z}$, the correspondence $\Upsilon: \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{X}$ defined by

$$\Upsilon(x, z) \equiv \{ y \in \Gamma(x, z) : v(x, z) = u(x, y, z) + \beta Q_\tau[v(y, w)|z] \} \quad (26)$$

is nonempty, upper semi-continuous and has a measurable selection.

Let $\psi: \mathcal{X} \times \mathcal{Z} \to \mathcal{X}$ be a measurable selection of $\Upsilon$. The policy function $\psi$ generates the plan $\pi^\psi$ defined by $\pi^\psi_t(z^t) = \psi(\pi_{t-1}(z^{t-1}), z_t)$ for all $z^t \in \mathcal{Z}^t$, $t \in \mathbb{N}$.

The next result provides sufficient conditions for a solution $v$ to the functional equation to the be supremum function, and for the plans generated by the associated policy function $\psi$ to attain the supremum.

**Theorem 3.15.** Let $v: \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ be bounded and satisfy the functional equation (25) and $\psi$ be defined as above. Then, $v = v^*$ and the plan $\pi^\psi$ attains the supremum in (24).

We highlight that this generalization is not straightforward. When working with expected utility, one can employ the law of iterated expectations. However, unfortunately a similar rule does not hold for quantiles, as we have already observed in Proposition 3.7.
3.7 Euler Equation

The final step is to characterize the solutions of the problem through the Euler equation. Let \( v = v^\tau \) be the unique fixed point of \( M^\tau \), satisfying (25). By Theorem 3.10, \( v \) is differentiable in its first coordinate, satisfying

\[
\frac{\partial v^\tau}{\partial x_i}(x, z) = \frac{\partial u}{\partial x_i}(x, y^*, z) = u(x, y^*, z).
\]

Given that we have shown the differentiability of value function, we are able to apply the standard technique to obtain the Euler equation, as formalized in the following theorem:

**Theorem 3.16.** Let Assumption 2 hold and let \( \pi = \pi^\psi \) be an optimal plan, as in Theorem 3.15. Assume that \( x^\pi_{t+1} \in \text{int}(x^\tau_t, z_t) \), that is, the optima are interior, and \( z_t \mapsto \frac{\partial u}{\partial x_i}(x^\tau_t, x^\tau_{t+1}, z_t) \) is strictly increasing. Then, the following first order condition (called Euler equation in this setting) necessarily holds for every \( t \in \mathbb{N} \) and \( i = 1, \ldots, p \):

\[
u_{yi}(x^\tau_t, x^\tau_{t+1}, z_t) + \beta Q_{\tau} [u_{x_i}(x^\tau_{t+1}, x^\tau_{t+2}, z_{t+1}) | z_t] = 0. \tag{27}
\]

In the expression above, \( \nu_{yi} \) represents the derivative of \( u \) with respect to (some of the coordinates of) its second variable (\( y \)) and \( u_{x_i} \) represents the derivative of \( u \) with respect to (some of the coordinates of) its first variable (\( x \)).

Theorem 3.16 provides the Euler equation, that is the optimality conditions for the quantile dynamic programming problem. This result is the generalization the traditional expected utility to the quantile utility. The Euler equation in (27) is displayed as an implicit function, nevertheless for any particular application, and given utility function, one is able to solve an explicitly equation as a conditional quantile function.

The proof of Theorem relies on a result about the differentiability inside the quantile function. Indeed, for a general function \( h \), we have \( \frac{\partial}{\partial x_i} Q_{\tau} [h(x, Z)] \neq Q_{\tau} \left[ \frac{\partial h}{\partial x_i}(x, Z) \right] \). However, we are able to establish this differentiability under our assumptions. We are not aware of this result in the theory of quantiles, and given its usefulness, we state it here:

**Proposition 3.17.** Assume that \( h : \mathcal{X} \times \mathcal{Z} \to \mathbb{R} \) and \( x \mapsto Q_{\tau} [h(x, Z)] \) are differentiable and that \( h \) and \( d \) are increasing in \( z \), where \( d(z) = h(x_i', x_{-i}, z) - h(x_i, x_{-i}, z) \) for \( x_i', x_i \) satisfying \( 0 < x_i' - x_i < \epsilon \), for some small \( \epsilon > 0 \). Then,

\[
\frac{\partial Q_{\tau} [h(x, Z)]}{\partial x_i} = Q_{\tau} \left[ \frac{\partial h}{\partial x_i}(x, Z) \right].
\]

3.8 Example: One-Sector Growth Model

We provide a simple example to illustrate the quantile maximization utility model: the one sector-growth model (see, e.g., Brock and Mirman (1972)). We also compare the results with the corresponding model for the expected utility maximization.

Consider the one sector-growth model with the quantile maximization utility. Using the
notation introduced in (20), we can write the consumer problem can be written as

$$
\max_{(c_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta_t^t U(c_t),
$$

subject to the following constraints:

$$
c_t + k_{t+1} = z_t h(k_t) \quad \text{(29)}
$$

$$
0 \leq k_{t+1} \leq z_t h(k_t),
$$

where $c_t$ denotes the amount of consumption good, $k_t$ is stock of capital, $z_t$ is the shock, $U(\cdot)$ is the utility function, and $h(\cdot)$ is the technology. Note that $\beta_\tau$ is written with a subscript $\tau$, to emphasize the fact that we may have a different parameter for each $\tau \in (0, 1)$.

From the results in Section 3.5, the corresponding value function for problem (28)–(29) can be expressed as

$$
v(k, z) = \max_{y \in [0, z_t h(k_t)]} \left\{ U(zh(k) - y) + \beta_\tau Q_\tau [v(y, z')|z] \right\}.
$$

It is easy to verify that this model satisfies Assumptions 1 and 2, and hence Theorems 3.9, 3.10, and 3.15. From Theorem 3.16, the Euler equation has the following representation:

$$
-U'(z_t h(k_t) - k_{t+1}) + \beta_\tau Q_\tau [U'(z_{t+1} h(k_{t+1}) - k_{t+2})z_{t+1} h'(k_{t+1})|z_t] = 0.
$$

By noting that $c_t = z_t h(k_t) - k_{t+1}$ and rearranging one can express the above equation as

$$
Q_\tau \left[ \beta_\tau \frac{U'(c_{t+1})}{U'(c_t)} z_{t+1} h'(k_{t+1}) - 1 \right] z_t = 0. \quad \text{(30)}
$$

Now we move our attention to the standard expected utility model, which can be written as

$$
\max_{(c_t)_{t=0}^{\infty}} E \left[ \sum_{t=0}^{\infty} \beta^t U(c_t) \right],
$$

subject to the same constraints in equation (29).

This problem can be rewritten and the associated value function is:

$$
v(k, z) = \max_{y \in [0, z_t h(k_t)]} \left\{ U(zh(k) - y) + \beta E [v(y, z')|z] \right\}.
$$

Finally, the Euler equation can be written as

$$
-U'(z_t h(k_t) - k_{t+1}) + \beta E [U'(z_{t+1} h(k_{t+1}) - k_{t+2})z_{t+1} h'(k_{t+1})|z_t] = 0,
$$

20
and by rearranging the previous equation we obtain

\[ E \left[ \frac{\beta U'(c_{t+1})}{U'(c_t)} z_{t+1} h'(k_{t+1}) - 1 \middle| z_t \right] = 0. \]  

(32)

When comparing the Euler equations in (30) and (32) one can notice similarities and differences. The expressions inside the conditional quantile in (30) and the conditional expectation in (32) are practically the same, except that, for the quantile model, the parameters depend on the quantile \( \tau \). That is, for each \( \tau \), we will have (potentially) different \( \beta(\tau) \) and parameters of the utility function \( U(\cdot) \) and technology \( h(\cdot) \).

4 Estimation and Inference

In the previous section, we derived the Euler equation for the quantile utility model. In this section we briefly review procedures for estimation and inference of conditional quantile functions using instrumental variables (IV) quantile regression (QR) for nonlinear models developed in de Castro, Galvao, and Kaplan (2017).\(^{17}\) It has been standard in the literature to estimate Euler equations derived from the expected utility models. It is an important exercise to learn about the structural parameters that characterize the economic problem of interest. After parametrizing the utility function, the restrictions imply a conditional average model and the parameters are commonly estimated by the generalized method of moments (GMM) of Hansen (1982). Estimation and inference of (non-smooth) GMM have been discussed by, among others, Newey and McFadden (1994), Chen, Linton, and van Keilegom (2003), Chen and Pouzo (2009), and Chen and Liao (2015).

For a given parametrized utility function, one is able to isolate the implicit quantile function defined by equation (27), thus obtaining the following potentially nonlinear conditional quantile model:

\[ Q_\tau[m(y_t, x_t, \theta_{0\tau}) | \Omega_t] = 0, \]  

(33)

where \( \tau \in (0, 1) \) is a quantile of interest, \( m(\cdot) \) is a function known up to a finite dimensional vector of parameter of interest \( \theta_{0\tau} \), \( \Omega_t \) denotes the \( \sigma \)-field generated by \( \{z_s, s \leq t\} \) that contains the information up to time \( t \), and \( (y_t, x_t, z_t) \) are the observable variables. The quantile model in equation (33) is valid for each given quantile \( \tau \). We aim to estimate the parameters \( \theta_{0\tau} \) that describe the Euler equation for specified quantiles of interest.

The model in (33) can be represented by non-smooth conditional moment restrictions as

\[ E[\tau - 1\{m(y_t, x_t, \theta_{0\tau}) \leq 0\} | z_t] = 0, \]  

(34)

\(^{17}\)In a seminal paper Koenker and Bassett (1978) introduced QR methods, which have been employed largely in economic applications.
where $1\{\cdot\}$ is the indicator function. Since $E[1\{m(y_t, x_t, \theta_0, \tau) \leq 0\} | z_t] = F(m(y_t, x_t, \theta_0, \tau) | z_t]$, when $F(\cdot)$ is invertible, one is able to recover (33) from (34).

For a given quantile index $\tau$, we estimate the parameter vector $\theta_{0\tau}$ using the method of moments. Rewrite the conditional quantile function in (34) as the following moment condition

$$E[z_t \{1\{m(y_t, x_t, \theta_0, \tau) \leq 0\} - \tau\}] = 0, \quad (35)$$

with endogenous vector $y_t \in \mathcal{Y} \subseteq \mathbb{R}^{d_y}$, full instrument vector $z_t \in \mathcal{Z} \subseteq \mathbb{R}^{d_z}$ that contains $x_t \in \mathcal{X} \subseteq \mathbb{R}^{d_x}$ as a subset, known function $m(\cdot)$, and parameters $\theta_{0\tau} \in \mathcal{B} \subseteq \mathbb{R}^{d_\beta}$.

We now present the smoothed IVQR (SIVQR) estimator. Let the population map $M : \mathcal{B} \times \mathcal{T} \mapsto \mathbb{R}^{d_z}$ be

$$M(\theta, \tau) \equiv E[g_t^u(\theta, \tau)], \quad (36)$$

$$g_t^u(\theta, \tau) \equiv g^u(y_t, x_t, z_t, \theta, \tau) \equiv z_t \{1\{m(y_t, x_t, \theta) \leq 0\} - \tau\}, \quad (37)$$

where superscript “$u$” denotes “unsmoothed.” The population moment condition (35) is

$$M(\theta_{0\tau}, \tau) = 0. \quad (38)$$

The method of moments is constructed by replacing the population moments, the expectation $E(\cdot)$, with their corresponding sample expectation $\hat{E}(\cdot)$, i.e., the sample average. Analogous to (36), using (37), the unsmoothed sample moment map is

$$\hat{M}_t^u(\theta, \tau) \equiv \hat{E}[g^u(y, x, z, \theta, \tau)] \equiv \frac{1}{T} \sum_{t=1}^{T} g_t^u(\theta, \tau). \quad (39)$$

Even if population system (38) has a unique solution, the unsmoothed system $\hat{M}_t^u(\theta, \tau) = 0$ may have zero or multiple solutions. Although this issue can be worked around theoretically, smoothing addresses it directly. Hence, the SIVQR estimator we consider is the solution to the system of smoothed sample moments, shown in (41) below. With smoothing (no “$u$” superscript), the sample analogs of (36), (37), and (38) are

$$g_{Tt}(\theta, \tau) \equiv g_T(y_t, x_t, z_t, \theta, \tau) \equiv z_t[\tilde{I}(-m(y_t, x_t, \theta) / h_T) - \tau], \quad (40)$$

$$\hat{M}_T(\theta, \tau) \equiv \frac{1}{T} \sum_{t=1}^{T} g_{Tt}(\theta, \tau),$$

$$\hat{M}_T(\hat{\theta}_\tau, \tau) = 0, \quad (41)$$

where $h_T$ is a bandwidth (sequence), $\tilde{I}(\cdot)$ is a smoothed version of the indicator function $1\{\cdot > 0\}$, and $I(\cdot)$ may stand for “indicator-like function” or “integral of kernel.” An example function $\tilde{I}(\cdot)$ is the integral of a fourth-order polynomial kernel and has been used by Horowitz
(1998) and Whang (2006). The double subscript on \( g_{Tt} \) is a reminder that we have a triangular array setup because \( g_{Tt} \) depends on the bandwidth sequence \( h_T \) in addition to \((y_t,x_t,z_t)\), the observed random variables.

Given a random sample \( \{(y_t,x_t,z_t) : t = 1,...T\} \), for any given quantile \( \tau \), the parameters of interest, \( \theta_{0\tau} \), can be estimated by (40) and (41). The objective function depends only on the available sample information, the known function \( m(\cdot) \), and the unknown parameters. Solutions of the above problem are denoted by \( \hat{\theta}_\tau \), the SIVQR estimator. de Castro, Galvao, and Kaplan (2017) discuss and give conditions for identification of the parameters of interest, and consider estimation and inference with weakly dependent data. The parameter \( \theta_{0\tau} \) is “locally identified” if there exists a neighborhood of \( \theta_{0\tau} \) in which only \( \theta_{0\tau} \) satisfies (35). This property holds if the partial derivative matrix of the right-hand side of (35) with respect to the \( \theta \) argument is full rank.\(^{18}\)

de Castro, Galvao, and Kaplan (2017) establish the asymptotic properties, namely consistency and asymptotic normality, of the estimator given in (40) and (41).

**Theorem 4.1** (de Castro, Galvao and Kaplan, 2017). Under standard regularity conditions, as \( T \to \infty \), the estimator is consistent, i.e., \( \hat{\theta}_\tau \xrightarrow{p} \theta_{0\tau} \), and

\[
\sqrt{T} (\hat{\theta}_\tau - \theta_{0\tau}) \xrightarrow{d} N(0, G^{-1} \Sigma_{M_T} [G^T]^{-1}),
\]

where \( \Sigma_{M_T} = \lim_{T \to \infty} \text{Var} \left( T^{-1/2} \sum_{t=1}^{T} g_{Tt}(\theta_{0\tau}, \tau) \right) \), \( G = \frac{\partial}{\partial \theta} E[z_t 1\{m(y_t,x_t,z_t,\theta) \leq 0\}]_{\theta = \theta_{0\tau}} = -E[z_t d_t^\top f(0|z_t, d_t)] \), \( d_t \equiv \nabla_{\theta} m(y_t,x_t,\theta_{0\tau}) \), and \( f_{m|z,d}(\cdot|z,d) \) is the conditional pdf of \( m_t \) given \( z_t = z \) and \( d_t = d \).

Given the result in Theorem 4.1, one is able to estimate the variance-covariance matrix and conduct practical inference for the parameters of interest.

A few key observations should be noted. First, for a given random sample \( \{(y_t,x_t,z_t) : t = 1,...T\} \), we are able to apply the SIVQR methods and estimate the parameters \( \theta_{0\tau} \) across different quantiles by simply varying \( \tau \). Second, for any given example, applying the SIVQR requires specifying the function \( m(\cdot) \), the observable variables \((y_t,x_t)\), and the instruments \( z_t \).

The instruments are used to achieve a valid orthogonality condition for the Euler equation, that is, the (conditional) moment condition equals to zero. The idea is that by imposing certainty equivalence on the nonlinear rational expectations model, the instruments help to circumvent some of the difficulties in obtaining a complete characterization of the stochastic equilibrium.\(^{19}\)

Third, we can allow for conditional heteroskedasticity and can conduct statistical inference without explicitly characterizing the dependence of the conditional variances on the information

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\(^{18}\)Identification for general nonlinear semiparametric and nonparametric conditional moment restrictions models is presented in Chen, Chernozhukov, Lee, and Newey (2014).

\(^{19}\)In the literature, it is standard to estimate Euler equations for conditional average models by parametrizing the utility function and estimating the parameters of interest using instrumental variables GMM (Hansen (1982)).
set. In the context of the asset pricing models discussed in Section 5 below, for example, this feature of our estimation procedure allows the conditional variances of asset yields to fluctuate with movements of variables in the conditioning information set.

Remark 4.2. In this paper, we are interested in estimating the conditional quantile functions to learn about the underlying heterogeneity among quantiles. Nevertheless, it is possible to see the quantile \( \tau \) as a parameter to be estimated together with the parameters \( \theta_{0\tau} \). Bera, Galvao, Montes-Rojas, and Park (2016) develop an approach that delivers estimates for the coefficients together with a representative quantile. In their framework, \( \tau \) captures a measure of asymmetry of the conditional distribution of interest and is associated with the “most probable” quantile in the sense that it maximizes the entropy.

5 Application: Asset Pricing Model

This section illustrates the usefulness of the new quantile utility maximization methods through an empirical example. We apply the methodology to the standard intertemporal allocation of consumption model, which is central to contemporary economics and finance. It has been used extensively in the literature and has had remarkable success in providing empirical estimates for the study of the elasticity of intertemporal substitution (EIS) and discount-factor parameters. We refer the readers to Campbell (2003), Cochrane (2005), and Ljungqvist and Sargent (2012), and the references therein, for a comprehensive overview.


We employ a variation of Lucas (1978)’s endowment economy (see, also, Hansen and Singleton (1982)). The economic agent decides on the intertemporal consumption and savings (assets to hold) over an infinity horizon economy, subject to a linear budget constraint. The decision generates an intertemporal policy function, which is used to estimate the parameters of interest for a given utility function.

Let \( c_t \) denote the amount of consumption good that the individual consumes in period \( t \). At the beginning of period \( t \), the consumer has \( x_t \) units of the risky asset, which pays dividend \( z_t \). The price of the consumption good is normalized to one, while the price of the risky asset in period \( t \) is \( p(z_t) \). Then, the consumer decides how many units of the risky asset \( x_{t+1} \) to save for the next period, and its consumption \( c_t \), satisfying:

\[
\begin{align*}
    c_t + p(z_t) x_{t+1} & \leq \left[ z_t + p(z_t) \right] x_t, \\
    c_t, x_{t+1} & \geq 0.
\end{align*}
\]

(42)  (43)
Using the notation introduced in (20), we can write the consumer problem as the maximization of
\[
Q_t^\infty \left[ \sum_{t=0}^\infty \beta_t^x U(c_t) \right] \equiv U(c_0) + Q_t \left[ \beta_t U(c_1) + Q_t \left[ \beta_t^2 U(c_2) + Q_t \left[ \beta_t^3 U(c_3) + \cdots \right] \right] \right] \Omega_1 \Omega_0.
\]

(44)

subjected to (42) and (43), where \( \beta_t \) is (0,1) is the discount factor for the quantile \( \tau \in (0,1) \) of interest, and \( U : \mathbb{R}_+ \rightarrow \mathbb{R} \) is the utility function. Because we have a single agent, the holdings must not exceed one unit. In fact, in equilibrium, we must have \( x_t^* \in [1, \forall t, k \). Let \( \bar{x} > 1 \) and \( \mathcal{X} = [0, \bar{x}] \).

From (42), we can determine the consumption entirely from the current and future states, that is, \( c_t = z_t \cdot x_t + p(z_t) \cdot (x_t - x_{t+1}) \). Following the notation of the previous sections, we denote \( x_t \) by \( x \), \( x_{t+1} \) by \( y \), and \( z_t \) by \( z \). Then, the above restrictions are captured by the feasible correspondence \( \Gamma : \mathcal{X} \times \mathcal{Z} \rightarrow \mathcal{X} = \mathcal{X} \) defined by:
\[
\Gamma(x, z) \equiv \{ y \in \mathcal{X} : p(z) \cdot y \leq (z + p(z)) \cdot x \}.
\]

(45)

For each pricing function \( p : \mathcal{Z} \rightarrow \mathbb{R}_+ \), define the utility function as:
\[
u(x, y, z) \equiv U[z \cdot x + p(z) \cdot (x - y)].
\]

(46)

We assume the following:

**Assumption 3.**

(i) \( \mathcal{Z} \subseteq \mathbb{R} \) is a bounded interval and \( \mathcal{X} = [0, \bar{x}] \);

(ii) \( U : \mathbb{R}_+ \rightarrow \mathbb{R} \) is given by \( U(c) = \frac{1}{1-\gamma} c^{1-\gamma} \), for \( \gamma > 0 \);

(iii) \( z \) follows a Markov process with pdf \( f : \mathcal{Z} \times \mathcal{Z} \rightarrow \mathbb{R}_+ \), which is continuous, symmetric, \( f(z, w) > 0 \), for all \( (z, w) \in \mathcal{Z} \times \mathcal{Z} \) and satisfies the property: if \( h : \mathcal{Z} \rightarrow \mathbb{R} \) is weakly increasing and \( z \leq z' \), then:
\[
\int_{\mathcal{Z}} h(\alpha) f(\alpha | z) d\alpha \leq \int_{\mathcal{Z}} h(\alpha) f(\alpha | z') d\alpha;
\]

(47)

(iv) \( z \mapsto z + p(z) \) is \( C^1 \) and non-decreasing, with \( z (\ln(z + p(z)))' \geq \gamma \).

Assumptions 3(i) – (ii) are standard in economic applications. Assumption 3(i) specifies an isoelastic utility function (constant elasticity of substitution – CES). This is a standard assumption in a large body of the literature, as for example, among others, Hansen and Singleton (1982), Stock and Wright (2000), and Campbell (2003). Condition 3(ii) states that the idiosyncratic shocks follow a Markov process. Assumption 3(iii) means that a high value of the dividend today makes a high value tomorrow more likely. It implies Assumption 2(ii).
Assumption 3(iv), \( z \rightarrow z + p(z) \) is non-decreasing, is natural. It states that the price of the risky asset and its return are a non-decreasing function of the dividends. Note that it is natural to expect that the price is non-decreasing with the dividends, but Assumption 3(iv) is even weaker than this, as it allows the price to decrease with the dividend; only \( z + p(z) \) is required to be non-decreasing.

Given Assumption 3, we can verify the assumptions for establishing the quantile utility model in the asset pricing model context. Thus, we have the following:

**Lemma 5.1.** Assumption 3 implies Assumptions 1 and 2 and Theorem 3.16 holds.

Therefore, Theorems 3.9 and 3.10 imply the existence of a value function \( v \), which is strictly concave and differentiable in its first variable, satisfying

\[
v(x, z) = \max_{y \in \Gamma(x, z)} Q_\tau[g(x, y, z, \cdot)\big| z],
\]

where \( g(x, y, z, w) = u(x, y, z) + \beta v(y, w) \). Also, \( \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} \). Note that

\[
\begin{align*}
\frac{\partial u}{\partial x}(x, y, z) &= U'[z \cdot x + p(z) \cdot (x - y)] (z + p(z)); \\
\frac{\partial u}{\partial y}(x, y, z) &= U'[z \cdot x + p(z) \cdot (x - y)] (-p(z)).
\end{align*}
\]

Because, in equilibrium, the holdings are \( x_t = 1 \) for all \( t \), we can derive the Euler equation as in (27) for this particular problem to obtain:

\[
-p(z_t)U'(c_t) + \beta \tau Q_\tau[U'(c_{t+1}) (z_{t+1} + p(z_{t+1})) | \Omega_t] = 0. \tag{48}
\]

Let us define the return by: \( 1 + r_{t+1} \equiv \frac{z_{t+1} + p(z_{t+1})}{p(z_t)} \). Thus, the Euler equation in (48) simplifies to

\[
Q_\tau \left[ \beta \tau (1 + r_{t+1}) \frac{U'(c_{t+1})}{U'(c_t)} \bigg| \Omega_t \right] = 1.
\]

In addition, Assumption 3(i) implies that the ratio of marginal utilities can be written as

\[
\frac{U'(c_{t+1})}{U'(c_t)} = \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma}.
\]

Finally, from the previous two equations, the Euler equation can be rewritten as

\[
Q_\tau \left[ \beta \tau (1 + r_{t+1}) \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} - 1 \bigg| \Omega_t \right] = 0. \tag{49}
\]

\footnote{In our dataset, when regressing the returns on the dividends, we find a statistically positive correlation.}
5.1 Estimation

The Euler equation in (49) possesses a nonlinear conditional quantile function representation as in (33). Thus, one is able to estimate the parameters of interest using the quantile regression methods described in Section 4. We aim to estimate the parameters \((\gamma_\tau, \beta_\tau)\). Given a random sample \{\((c_t, r_t, w_t) : t = 1, \ldots, T\}\}, where \(c_t\) denotes the consumption, \(r_t\) is the real return on the asset, and \(w_t\) is a vector of instrumental variables, we are able to apply the SIVQR methods and, for each quantile \(\tau \in (0, 1)\), estimate the parameters. In this way, we uncover the potential underlying heterogeneity across the quantiles. The smoothing bandwidth is \(h = 10^{-4}\). The bandwidth for the estimation of the variance-covariance matrix is usually \(c = 1\). One advantage of the quantile Euler equation is that it may be log-linearized with no approximation error, differently from the standard Euler equation. Thus, we use a model as

\[
Q_{1-\tau} \left[ \ln \left( \frac{c_{t+1}}{c_t} + 1 \right) / \gamma_\tau - \gamma_\tau^{-1} \ln(\beta_\tau) - \gamma_\tau^{-1} \ln(1 + r_{t+1}) \right] = 0.
\]

From \(\ln(\beta_\tau)/\gamma_\tau\) and \(1/\gamma_\tau\) we are able to recover the parameters of interest.

It is important to discuss the interpretation of the parameters of interest \((\beta_\tau, \gamma_\tau)\). We notice that as discussed in Section 2.3, and as in Epstein and Zin (1989, 1991), it is possible to separate the risk attitude from the intertemporal substitutability in the quantile model. First, the present notion of risk preference differs in several respects from the one familiar in the expected utility literature. The quantile \(\tau\) captures the risk attitude in the model. As discussed previously, given that the notion of risk attitude is comparative and captured by varying the quantile index, we do not characterize agents as risk-averse, risk-neutral, or risk-seeking, but instead estimate the model for several different quantiles. Thus, an important point in the application is to compare estimates across quantiles, that is, different measures of risk. Second, for a given quantile \(\tau\), \(\beta_\tau\) is the usual discount factor. Finally, from the discussion in Section 2.3 and equation (44), one can notice that the parameter \(1/\gamma_\tau\) captures the standard measure of EIS implicit in the CES utility function. Thus, by employing the quantile maximization model, for each specific risk attitude \(\tau\), we are able to estimate the associated discount factor and EIS.

Additional considerations are in order when estimating the parameters in (49). First, equation (49) is an equilibrium condition. This is commonly used in the literature to derive orthogonality conditions based on instrumental variables that can be used to estimate the parameters of the utility function. Second, when bringing (49) to the data, rational expectations is an underlying assumption. This means that the conditional quantile function operator in (49) coincides with the theoretical given all information available to the consumer at time \(t\). Thus, the conditional quantile function is valid over time. Third, we abstract from the presence of the “taste-shock” (or measurement error). All of these assumptions and simplifications have been largely discussed in terms of models for estimating conditional averages (see, e.g., Attanasio and Low (2004)). Because the main objective of this paper is to provide a first view of the quantile utility maximization problem, we use these assumptions for simplicity.
5.2 Data

We use a data set that is common in the literature for modeling stock prices, as discussed in the previous section. We use monthly data from 1959:01 to 2015:11, which produces 683 observations. As is standard in the literature (see, e.g., Hansen and Singleton (1982)), two different measures of consumption were considered: nondurables, and nondurables plus services. The monthly, seasonally adjusted observations of aggregate nominal consumption (measured in billions of dollars unit) of nondurables and services were obtained from the Federal Reserve Economic Data. Real per capita consumption series were constructed by dividing each observation of these series by the corresponding observation of population, deflated by the corresponding CPI (base 1973:01).

Each measure of consumption was paired with four sets of stock returns from the Center for Research in Security Prices (CRSP) U.S. Stock database, which contains month-end prices for primary listings for the New York Stock Exchange (NYSE). We use the equally-weighted average of returns (EWR) (including and excluding dividends) on the NYSE. The nominal returns were converted to real returns by dividing by the deflator associated with the measure of consumptions. Instruments include lags of log real consumption growth, nominal interest rate, inflation, and a log dividend-price ratio for equities. We use two instruments (same excluded instruments used in Yogo (2004)): constant, and the linear projection of the real interest rate onto a constant and nominal interest rate, inflation, and log consumption growth. All instruments are lagged twice to avoid problems with time aggregation in consumption data.

5.3 Results

Now we present the empirical results. Because the literature reports results for conditional mean models, for comparison purposes, we also include estimates of the standard conditional expectation regression IV (TSLS) version of the model.

The results for the estimates of the parameters of interest at different quantiles are reported in Table 1 and Figure 1. We present estimates using consumption of nondurables and stock return with and without dividends. The panels on the left display the estimates for EWR excluding dividends (EWRwo), and the right panel including dividends (EWRw). The results for consumption nondurables plus services are qualitatively similar. For brevity, we omit them.

First we consider the estimates of the discount factor using EWRwo. The results are presented in the left panels in Table 1 and Figure 1, and show empirical evidence of heterogeneity in the discount factor parameter. The estimates are decreasing over the quantiles. For the upper quantiles the estimates are close to 0.80. The table shows that, for low quantiles, the discount factor estimates are larger than one. This result is in line with the results in Epstein and Zin (1991). Nevertheless, for the first four deciles we are not able to reject the null hypothesis of the discount factors being statistically equal to one. Overall, Table 1 shows evidence that the discount factor is smaller for upper quantiles (higher risk aversion). In the same way,
the smaller risk aversion, the more patient. The results for the TSLS case are consistent with the literature and show a discount factor of 0.992.

Table 1: SIVQR and TSLS estimates for Discount Factor and EIS

<table>
<thead>
<tr>
<th>EWRwo</th>
<th>EWRw</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$</td>
<td>$\beta_\tau$</td>
</tr>
<tr>
<td>0.1</td>
<td>1.156*</td>
</tr>
<tr>
<td>0.2</td>
<td>1.061*</td>
</tr>
<tr>
<td>0.3</td>
<td>1.033*</td>
</tr>
<tr>
<td>0.4</td>
<td>1.006*</td>
</tr>
<tr>
<td>0.5</td>
<td>0.991*</td>
</tr>
<tr>
<td>0.6</td>
<td>0.979*</td>
</tr>
<tr>
<td>0.7</td>
<td>0.968*</td>
</tr>
<tr>
<td>0.8</td>
<td>0.802***</td>
</tr>
<tr>
<td>0.9</td>
<td>0.806***</td>
</tr>
<tr>
<td>TSLS</td>
<td>0.992*</td>
</tr>
</tbody>
</table>

This table shows coefficients returned from applying SIVQR and TSLS methods to estimate the Euler equation. *, **, and *** represent statistical significance at the 1%, 5%, and 10% levels, respectively.

Next we consider the estimates the EIS, $1/\gamma_\tau$. The left panels in Table 1 and Figure 1 present the results using EWRwo. The first interesting observation is that the results document evidence of heterogeneity in EIS across quantiles. In particular, the table shows that overall the EIS increases across quantiles, especially for $\tau \in (0.1, 0.7)$, such that EIS is relatively larger for upper quantiles. The smaller EIS, for low quantiles (less risk preferring), means less sensitivity to changes in intertemporal consumption. There is an existing literature exploring whether the EIS varies with the level of consumption (or wealth) which rejects the constant-EIS hypothesis (Blundell, Browning, and Meghir (1994); Attanasio and Browning (1995); Atkeson and Ogaki (1996)). In this paper we shed light on the discussion and allow the EIS to vary with the risk attitude, indexed by the quantile. We show that the EIS varies substantially in this case.

The right panels in Table 1 and Figure 1 display the estimates when considering EWRw. They serve as a robustness check. The results are qualitatively similar to those in the left panel and Figure 1. The coefficients of EIS present variation over the quantiles. The relative EIS increases across quantiles. The discount factor estimates also present heterogeneity, especially for upper quantiles. The discount factor is smaller for larger quantiles (more risk-taking), which suggests less patient. On the contrary, for lower quantiles, the risk aversion is large, as is the discount factor, providing evidence that more risk-averse is related to more patient.

**Remark 5.2.** It should be noted that our model does not control for income or wealth. Thus, the agents that correspond to low quantiles do not necessarily correspond to low income, but to low risk aversion. This observation is important to avoid confusion with the results in a branch
of the literature that links discount factors with income and wealth (see, e.g., Hausman (1979), and Lawrance (1991)). Moreover, there is empirical evidence that documents small discount factors estimates. This literature estimates discount factors by using a quasi-hyperbolic discount function (see, e.g., Paserman (2008), Fang and Silverman (2009), and Laibson, Maxted, Repetto, and Tobacman (2015)). In contrast to these streams of literature, this paper abstracts from a relationship between discount rates and poverty and employs a simple model to estimate the discount factor. Our objective is to illustrate the potential empirical application of the quantile utility maximization model. We leave the connection with income and wealth and related extensions for future research.

In all, the application illustrates that the new methods serve as an important alternative tool to study economic behavior, in particular, asset pricing. The methods allow one to estimate the discount factor and EIS at different levels of risk attitude (quantiles). Our empirical results document heterogeneity in both discount factor and EIS across quantiles.

6 Summary and Open Questions

This paper develops a dynamic model of rational behavior under uncertainty for an agent maximizing the quantile utility function indexed by a quantile \( \tau \in (0, 1) \). More specifically, an agent maximizes the stream of future \( \tau \)-quantile utilities, where the quantile preferences induce the quantile utility function. We show dynamic consistency of the preferences and that this dynamic problem yields a value function, using a fixed-point argument. We also obtain desirable properties of the value function. In addition, we derive the corresponding Euler equation.
Empirically, we show that one can employ existing general instrumental variables nonlinear quantile regression methods for estimating and testing the rational quantile models directly from stochastic Euler equations. An attractive feature of this method is that the parameters of the dynamic objective functions of economic agents can be interpreted as structural objects. Finally, to illustrate the methods, we construct an asset-pricing model and estimate the implied discount factor and elasticity of intertemporal substitution (EIS) parameters for different quantiles. The results suggest evidence of heterogeneity in both parameters, as both discount factor and EIS vary as a function of the quantiles.

Many issues remain to be investigated. The extension of the quantile maximization model from considering a single quantile to multiple quantiles simultaneously would be important. Extensions of the methods to general equilibrium models pose challenging new questions. In addition, aggregation of the quantile preferences is also a critical direction for future research. Applications to asset pricing and consumption models would appear to be a natural direction for further development of quantile utility maximization models.
7 Appendix

7.1 Properties of Quantiles

The following picture illustrates the c.d.f. $F$ of a random variable $X$, and its corresponding quantile function $Q(\tau) = \inf\{\alpha \in \mathbb{R} : F(\alpha) \geq \tau\}$, for $\tau > 0$.\(^{21}\) In this case, $X$ assumes the value 3 with 50% probability and is uniform in $[1, 2] \cup [4, 5]$ with 50% probability. This picture is useful to inspire some of the properties that we state below. Note, for instance, the discontinuities and the values over which the quantile is constant.

![Figure 5: c.d.f. and quantile function of a random variable.](image)

The following lemma is an auxiliary result that will be helpful for the derivations below.

**Lemma 7.1.** The following statements are true:

(i) $Q$ is increasing, that is, $\tau \leq \hat{\tau} \implies Q(\tau) \leq Q(\hat{\tau})$.

(ii) $\lim_{\tau \downarrow \hat{\tau}} Q(\tau) \geq Q(\hat{\tau})$.

(iii) $Q$ is left-continuous, that is, $\lim_{\tau \uparrow \hat{\tau}} Q(\tau) = Q(\hat{\tau})$.

(iv) $\Pr\{\{z : z < Q(\tau)\}\} \leq \tau \leq \Pr\{\{z : z \leq Q(\tau)\}\} = F(Q(\tau))$.

(v) If $g : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function, then $Q_{\tau}[g(X)] = g(Q_{\tau}[X])$.

(vi) If $g, h : \mathbb{R} \to \mathbb{R}$ are such that $g(\alpha) \leq h(\alpha), \forall \alpha$, then $Q_{\tau}[g(Z)] \leq Q_{\tau}[h(Z)]$.

(vii) $F$ is continuous if and only if $Q$ is strictly increasing.

(viii) $F$ is strictly increasing if and only if $Q$ is continuous.

**Proof.** (i) Let us first assume $\tau > 0$. If $\tau \leq \hat{\tau}$, then $\{\alpha \in \mathbb{R} : F_Z(\alpha) \geq \tau\} \ni \{\alpha \in \mathbb{R} : F_Z(\alpha) \geq \hat{\tau}\}$. This implies $Q_Z(\tau) \leq Q_Z(\hat{\tau})$. Next, if $\sup\{\alpha \in \mathbb{R} : F_Z(\alpha) = 0\} = -\infty$, there is nothing else to prove. If $\sup\{\alpha \in \mathbb{R} : F_Z(\alpha) = 0\} = x \in \mathbb{R}$, then $F_Z(x - \epsilon) = 0$ for any $\epsilon > 0$. Let $\hat{\tau} > 0$. Then, $y \in \{\alpha \in \mathbb{R} : F_Z(\alpha) = \hat{\tau}\} \implies y > x - \epsilon$, which in turn implies $Q_Z(\hat{\tau}) > x - \epsilon$. Since $\epsilon > 0$ is arbitrary, this implies $Q_Z(\hat{\tau}) > x = Q_Z(0)$, which concludes the proof.

(ii) From (i), $\lim_{\tau \downarrow \hat{\tau}} Q_Z(\tau) \geq \inf_{\tau \uparrow \hat{\tau}} Q_Z(\tau) \geq Q_Z(\hat{\tau})$. Figure 5 illustrates (for example for $\hat{\tau} = 0.25$) that the inequality can be strict.

(iii) From (i), we know that $\lim_{\tau \downarrow \hat{\tau}} Q_Z(\tau) \leq Q_Z(\hat{\tau})$. For the other inequality, assume that $\lim_{\tau \uparrow \hat{\tau}} Q_Z(\tau) > 2\epsilon < Q_Z(\hat{\tau}) < \infty$, for some $\epsilon > 0$. This means that for each $k \in \mathbb{N}$, we can find $\alpha_k \in \{\alpha : F_Z(\alpha) \geq \hat{\tau} - \frac{1}{k}\}$ such

---

\(^{21}\)For $\tau = 0$, $Q(0) = \sup\{\alpha \in \mathbb{R} : F(\alpha) = 0\}$ is just the lower limit of the support of the variable.
that \( Q_z(\tau - \frac{1}{k}) \leq \alpha^k \leq Q_z(\tau - \frac{1}{k}) + \epsilon < Q_z(\tau) - \epsilon \). We may assume that \( \{\alpha^k\} \) is an increasing sequence bounded by \( Q_\phi(\tau) \) and thus converges to some \( \bar{\alpha} \in \mathbb{R} \). Then, \( \lim_{k \to \infty} Q_z(\tau) = \bar{\alpha} \leq Q_\phi(\tau) - \epsilon < Q_\phi(\tau) \).

Since \( F_z(\alpha^k) \geq \tau - \frac{1}{k} \) and \( F_z \) is upper semi-continuous, \( F_z(\bar{\alpha}) \geq \tau \), which implies that \( \alpha \geq Q_z(\tau) \), a contradiction. Now, assume that \( Q_z(\tau) = \infty \). Since \( \lim_{\alpha \to \infty} F_z(\alpha) = 1 \), the set \( \{\alpha \in \mathbb{R} : F_z(\alpha) \geq \tau\} \) is non-empty for all \( \tau < 1 \), that is, \( Q_z(\tau) = \infty \) for all \( \tau < 1 \). Thus, \( \bar{\tau} = 1 \). If \( \lim_{k \to \infty} Q_z(\tau) = x \in \mathbb{R} \), then \( F_z(x + 1) \geq 1 - \epsilon \) for all \( \epsilon > 0 \), which implies that \( F_z(x + 1) = 1 \) and \( Q_z(1) \leq x + 1 \), a contradiction.

(iv) As above, if \( Q_z(\tau) = \infty \), then \( \tau = 1 \), which implies \( 1 = \Pr(\{w : z < \infty\}) = \Pr(\{w : z \leq \epsilon\}) \) and there is nothing to prove. Let \( \alpha = Q_z(\tau) < \infty \). If \( \alpha^k \downarrow \bar{\alpha} \) is such that \( F_z(\alpha^k) \geq \tau \), then \( F_z(\alpha^k) \geq \tau \), by the well-known upper-semicontinuity of \( F_z \). That is, \( \tau \leq F_z(Q_z(\tau)) \). For the other inequality, let \( \alpha^k \uparrow \bar{\alpha} = Q_z(\tau) \). Since \( \alpha^k < \bar{\alpha} \), then \( \Pr[\mathcal{Z} < \alpha^k] \leq \tau \), by the definition of \( \bar{\alpha} \). Thus, \( \Pr[\mathcal{Z} < \alpha^k] \leq \Pr[\mathcal{Z} < \alpha^k] < \tau \) and \( \Pr[\mathcal{Z} < \bar{\alpha}] \leq \sup_k \Pr[\mathcal{Z} < \alpha^k] < \tau \).

(v) The proof is direct as follows:

\[
Q_\tau(g(Z)) = \inf\{\alpha \in \mathbb{R} : \Pr[g(Z) \leq \alpha] \geq \tau\} = \inf\{\alpha \in \mathbb{R} : \Pr[Z \in g^{-1}(\alpha)] \geq \tau\} = \inf\{\alpha \in \mathbb{R} : g^{-1}(\alpha) \in \beta, \Pr[Z \in \beta] \geq \tau\} = \inf\{g(\beta) : \Pr[Z \in \beta] \geq \tau\} = g(\inf\{\beta : \Pr[Z \in \beta] \geq \tau\}) = g(Q_\tau(Z)).
\]

(vi) Since \( g \leq h \), then for any \( \alpha \), \( \{z : g(z) \leq \alpha\} \supseteq \{z : h(z) \leq \alpha\} \), which implies \( F_{g(Z)}(\alpha) = \Pr[g(Z) \leq \alpha] \geq \Pr[h(Z) \leq \alpha] = F_{h(Z)}(\alpha) \). If \( \tau > 0 \), \( \{\alpha \in \mathbb{R} : \Pr[g(Z) \leq \alpha] \geq \tau\} \supseteq \{\alpha \in \mathbb{R} : \Pr[h(Z) \leq \alpha] \geq \tau\} \). Taking infima, we obtain \( Q_{g(Z)}(\tau) \leq Q_{h(Z)}(\tau) \). On the other hand, \( \{\alpha \in \mathbb{R} : F_{h(Z)}(\alpha) = 0\} \subseteq \{\alpha \in \mathbb{R} : F_{g(Z)}(\alpha) = 0\} \) and taking the supremum in both sides we obtain the same conclusion.

(vii) Assume that \( F_z \) is discontinuous at \( x_0 \), that is, \( \lim_{y \to x_0} F_z(y) = y_0 < y_1 = F_z(x_0) \). If \( y_0 < y_2 < y_3 < y_1 \), then \( Q_z(y_2) = \inf\{\alpha : F_z(\alpha) \geq y_2\} = \inf\{\alpha : F_z(\alpha) \geq y_3\} = Q_z(y_3) \), that is, \( Q_z \) is not strictly increasing. Conversely, assume that \( Q_z \) is not strictly increasing, that is, there exists \( y_2 < y_3 \) such that \( Q_z(y_2) = Q_z(y_3) = x \). By definition, this means that \( F_z(x - \epsilon) < y_2 < y_3 < F_z(x + \epsilon) \), for all \( \epsilon > 0 \). But this implies that \( F_z \) is not continuous at \( x \).

(viii) Suppose that \( F_z \) is not strictly increasing, that is, there exists \( x_1 < x_2 \) such that \( F_z(x_1) = F_z(x_2) = y \). Then, \( Q_z(y - \epsilon) = \inf\{\alpha : F_z(\alpha) \geq y - \epsilon\} \leq x_1 < x_2 \leq \inf\{\alpha : F_z(\alpha) \geq y + \epsilon\} = Q_z(y + \epsilon) \). Thus, \( Q_z \) cannot be continuous at \( y \). Conversely, assume that \( Q_z \) is not continuous at \( y_0 \). Since \( Q_z \) is increasing by (i) and left-continuous by (iii), this means that \( Q_z(y_0) = x_0 < x_1 = \lim_{y \to y_0} Q_z(y) \). If \( x_0 < x_2 < x_1 \), then \( F_z(x_2) < y_0 \), otherwise \( \lim_{y \to y_0} Q_z(y) < x_2 \). By (iv), we have \( y_0 < F_z(Q_z(y_0)) = F_z(x_0) < F_z(x_2) \leq y_0 \), that is, \( F_z \) is not strictly increasing between \( x_0 \) and \( x_2 \).

Let \( \Theta \) be a set (of parameters) and \( g : \Theta \times Z \times \mathbb{R} \to \mathbb{R} \) be a measurable function. We denote by \( Q_\tau[g(\theta, \cdot)Z] \) the quantile function associated with \( g \), that is:

\[
Q_\tau[g(\theta, \cdot)Z] = \inf\{\alpha \in \mathbb{R} : \Pr([g(\theta, W) \leq \alpha]Z = z) \geq \tau\}.
\]

The following Lemma generalizes equation (2) to conditional quantiles.
Lemma 7.2. Let \( g: \Theta \times \mathcal{Z} \rightarrow \mathbb{R} \) be non-decreasing and left-continuous in \( \mathcal{Z} \). Then,

\[
Q_\tau[g(\theta, \cdot)|z] = g(\theta, Q_\tau[w|z]).
\]

(51)

It is useful to illustrate the above result with an example. Let us define the function \( g_{ab}: [1, 5] \rightarrow \mathbb{R} \)
by:

\[
g_{ab}(x) = \begin{cases} 
7, & \text{if } x < a \\
b, & \text{if } x = a \\
10, & \text{if } x > a
\end{cases}
\]

The function \( g_{ab} \) thus defined is non-decreasing if \( b \in [7, 10] \) and it is left-continuous if \( b = 7 \).

Consider the r.v. \( X \) whose c.d.f. \( F \) and quantile function \( Q \) are shown in Figure 5 above. Let \( F_{ab} \)
and \( Q_{ab} \) denote respectively the c.d.f. and quantile functions associated to \( g_{ab}(\mathcal{Z}) \). Figure 6 shows
\( Q_\tau[g_{ab}(w)|z] \) and \( Q_{\tau}(w|z) \) for \( a \in [1, 5] \) and \( b \in [7, 10] \). The point of discontinuity is a function
of \( a \) ( \( h(a) \in [0, 1] \)).

![Figure 6a: \( g_{ab}(Q_\tau[w|z]) \).](image1)

![Figure 6b: \( Q_\tau[g_{ab}(w)|z] \).](image2)

Proof of Lemma 7.2: For a contradiction, let us first assume that

\[
Q_\tau[g(\theta, \cdot)|z] > g(\theta, Q_\tau[w|z]) \equiv \hat{\alpha}.
\]

This means that \( \hat{\alpha} \notin \{\alpha \in \mathbb{R}: \Pr(\{w: g(\theta, w) \leq \alpha\}|z) \geq \tau\} \), that is,

\[
\Pr(\{w: g(\theta, w) \leq \hat{\alpha}\}|z) < \tau.
\]

Since \( \hat{\alpha} = g(\theta, Q_\tau[w|z]) \) and \( g \) is non-decreasing in \( w \), \( \{w: w \leq Q_\tau[w|z]\} \subset \{w: g(\theta, w) \leq \hat{\alpha}\} \). Thus,
\( \Pr(\{w: w \leq Q_\tau[w|z]\}) < \tau \), but this contradicts Lemma 7.1(iv).

Conversely, assume that

\[
Q_\tau[g(\theta, \cdot)|z] < g(\theta, Q_\tau[w|z]).
\]

This means that there exists \( \tilde{\alpha} < g(\theta, Q_\tau[w|z]) \) such that

\[
\Pr(\{w: g(\theta, w) \leq \tilde{\alpha}\}|z) \geq \tau.
\]

Let \( \tilde{w} \) be the supremum of the set \( \{w: g(\theta, w) \leq \tilde{\alpha}\} \). Since \( g \) is non-decreasing and left-continuous,
\( g(\theta, \tilde{w}) \leq \tilde{\alpha} \). Moreover,

\[
\Pr(\{w: w \leq \tilde{w}\}|z) = \Pr(\{w: g(\theta, w) \leq \tilde{\alpha}\}|z) \geq \tau.
\]

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Thus, \( \tilde{w} \in \{ \alpha \in \mathbb{R} : \Pr(\{ w : w \leq \alpha \}) > \tau \} \), which implies that \( \tilde{w} \geq Q_\tau[w|z] \). Thus, \( \tilde{\alpha} \geq g(\theta, \tilde{w}) \geq g(\theta, Q_\tau[w|z]) > \tilde{\alpha} \), which is a contradiction. 

The following Corollary to the above Lemma will be useful.

**Corollary 7.3.** Let \( T \in \mathbb{N} \cup \{ \infty \} \), \( h : \Theta \times Z^T \times Z \rightarrow \mathbb{R} \), \( g : \Lambda \times Z^T \times Z \rightarrow \mathbb{R} \) be non-decreasing and left-continuous in \( Z \). Then,

\[
Q_\tau[h(\theta, z^T, Q_\tau[g(\lambda, z^T, z_{t+1})]|z_t)] = Q_\tau[h(\theta, z^T, Q_\tau[g(\lambda, z^T, Q_\tau[\tau_{z+1}])]|z)] .
\]

**Proof.** Let \( X \) denote the random variable \( Q_\tau[g(\lambda, z^T, z_{t+1})]|z_t) \) and similarly, let \( Y \) denote \( g(\lambda, z^T, Q_\tau[\tau_{z+1}]) \). Then, by Lemma 7.2, \( X = Y \). Therefore, \( h(\theta, z^T, X) = h(\theta, z^T, Y) \) and the result follows. 

The following result will be useful below.

**Proposition 7.4.** Given the random variables \( X \) and \( Y \), assume that there exists random variable \( Z \) and continuous and increasing functions \( h \) and \( g \) such that \( X = h(Z) \) and \( Y = g(Z) \). Then \( Q_\tau[X + Y] = Q_\tau[X] + Q_\tau[Y] \).

**Proof.** Let \( Z, h \) and \( g \) be as in the definition. Define \( \tilde{h}(Z) = h(Z) + g(Z) \). This function is clearly continuous and increasing. Therefore,

\[
Q_\tau[X + Y] = Q_\tau[h(Z)] = h(Q_\tau[Z]) = h(Q_\tau[Z]) + g(Q_\tau[Z]) = Q_\tau[h(Z)] + Q_\tau[g(Z)] = Q_\tau[X] + Q_\tau[Y].
\]

by applying Lemma 7.2 twice. 

### 7.2 Proofs of Section 3

**Proof of Theorem 3.4:** This is essentially the same proof of Theorem 3.9, presented in detail below. Thus, we omit it. 

**Proof of Proposition 3.5:** Let \( L \) be a bound for \( V^{\tau} \). Using repeated times the recursive property (12), we can write

\[
V^{\tau}(x, z) = u(x_1, x_2, z_1) + Q_\tau[\beta u(x_2, x_3, z_2) + Q_\tau[\beta^2 u(x_3, ...) + \ldots
\]

\[
\ldots + Q_\tau[\beta^n u(x_{n+1}, x_{n+2}, z_n)] + \beta^{n+1} V^{\tau}(x_{n+1}, Z_n) | Z_n = z_n ] \ldots | Z_1 = z \]

\[
\leq u(x_1, x_2, z_1) + Q_\tau[\beta u(x_2, x_3, z_2) + Q_\tau[\beta^2 u(x_3, ...) + \ldots
\]

\[
\ldots + Q_\tau[\beta^n u(x_{n+1}, x_{n+2}, z_n) + \beta^{n+1} L] | Z_n = z_n ] \ldots | Z_1 = z \]

\[
= V^{\tau}(x, z) + \beta^{n+1} L,
\]

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Therefore, we have used the property of quantiles that $Q_{\tau}[X + \alpha] = \alpha + Q_{\tau}[X]$ for $\alpha \in \mathbb{R}$; see Lemma 7.2. Repeating the same argument with the lower bound $-L$, we can write:

$$V^n(x, z) - \beta^{n+1}L \leq V^n(x, z) - V^n(x, z).$$

This concludes the proof.

**Proof of Proposition 3.7:** Let $\Omega = \{1, 2, 3, 4\}$ and $P(\{\omega\}) = 1/4$ for all $\omega \in \Omega$. Define $\Sigma_0 = \{\emptyset, \Omega\}$ and $\Sigma_1 = \{\emptyset, E_1, E_2, \Omega\}$, where $E_1 = \{1, 2\}$ and $E_2 = \{3, 4\}$. Let $X(\omega) = \omega$. Then for $\tau \in (0.5, 0.75)$,

$$Q_{\tau}[X|\Sigma_1]_{\omega} = \begin{cases} 2, & \text{if } \omega \in E_1 \\ 4, & \text{if } \omega \in E_2 \end{cases}$$

Therefore, $Q_{\tau}[Q_{\tau}[X|\Sigma_1]|\Sigma_0] = 4$ but $Q_{\tau}[X|\Sigma_0] = Q_{\tau}[X] = 3$, which establishes (21).

To see (22), consider $\Omega = [0, 4]$, $\Sigma_0 = \{\emptyset, \Omega\}$ and let $\Sigma_1$ be generated by the partition $\{E_1, E_2\}$, where $E_1 = [1, 2]$ and $E_2 = [2, 4]$. Consider $P$ as the uniform distribution on $\Omega$. Let $X$ and $Y$ be two random variables with c.d.f. given respectively by $F_X(x) = \frac{1}{4} \left[ x - \frac{1}{4} \sin(\pi x) \right]$ and $F_Y(x) = \frac{1}{4} \left[ x + \frac{1}{4} \sin(\pi x) \right]$. The graphs of these two c.d.f.s are shown in Figure 7 below. Let $\tau \in (0.5, 0.75)$.

![Figure 7: Graph of X and Y, with respective quantiles.](image)

In the graph above, we plot the quantiles for $\tau = \frac{5}{8} \in (0.5, 0.75)$. We can easily see that $Q_{\tau}[X|\Sigma_1]_{(\omega)} \geq Q_{\tau}[Y|\Sigma_1]_{(\omega)}$, $\forall \omega \in \Omega$, but $Q_{\tau}[X] = Q_{\tau}[X|\Sigma_0] < Q_{\tau}[Y|\Sigma_0] = Q_{\tau}[Y]$, that is, (22) holds.

**Proof of Theorem 3.8:** Assume that plans $\pi$ and $\pi'$ are such that $\pi_{t'}(\cdot) = \pi'_{t'}(\cdot)$ for all $t' \leq t$ and $\pi'_{t+1, \Omega_{t+1, x}} \pi$ for all $\Omega_{t+1, x}$. From (9), this means that

$$V_{t+1}(\pi', x, z^{t+1}) \geq V_{t+1}(\pi, x, z^{t+1}), \forall (x, z^t) \in \mathcal{A} \times \mathcal{Z}^{t+1}. \quad (52)$$
Therefore,
\[
V_t(\pi', x, z^t) = u(x^{\pi'}_{t}, x_{t+1}^{\pi'}, z_t) + \beta Q_x \left[ V_{t+1}(\pi', x, (Z_t, z_{t+1})) \right]_{Z_t = z^t} \\
\geq u(x^{\pi'}_{t}, x_{t+1}^{\pi'}, z_t) + \beta Q_x \left[ V_{t+1}(\pi, x, (Z_t, z_{t+1})) \right]_{Z_t = z^t} \\
= u(x^{\pi'}_{t}, x_{t+1}^{\pi'}, z_t) + \beta Q_x \left[ V_{t+1}(\pi, x, (Z_t, z_{t+1})) \right]_{Z_t = z^t} \\
= V_t(\pi, x, z^t),
\]
where the first and last equalities come from the recursive equation (12), the first inequality comes from (52) and Lemma 7.1(vi), while the equality in the third line comes from the fact that the plans agree on all times up to \( t \), that is, \( x^{\pi'}_{t} = x^\pi_t \) and \( x^{\pi'}_{t+1} = \pi'_t(x^\pi_t, z^t) = \pi_t(x^\pi_t, z^t) = x^\pi_{t+1} \). This establishes the claim.

**Proof of Theorem 3.9:** We organize the proof in a series of Lemmas.

**Lemma 7.5.** If \( v \in \mathcal{C} \), the map \( (y, z) \mapsto Q_y[v(y, w)|z] \) is continuous.

**Proof.** Consider a sequence \( (y^n, z^n) \to (y^*, z^*) \). Since \( v \) and \( f \) are continuous, \( v(y^n, w) \to v(y^*, w) \) and
\[
m^n(\alpha) \equiv \Pr(\{w : v(y^n, w) \leq \alpha\}|z^n) \to \Pr(\{w : v(y^*, w) \leq \alpha\}|z^*) \equiv m^*(\alpha). \tag{53}
\]
Let \( \alpha^n \equiv \inf\{ \alpha \in \mathbb{R} : m^n(\alpha) \geq \tau \} = Q_y[v(y^n, \cdot)|z^n] \) and \( \alpha^* \equiv \inf\{ \alpha \in \mathbb{R} : m^*(\alpha) \geq \tau \} = Q_y[v(y^*, \cdot)|z^*] \). We want to show that \( \alpha^n \to \alpha^* \).

In general, \( m^n(\cdot) \) and \( m^*(\cdot) \) may fail to be continuous, but they are right-continuous and (weakly) increasing by Lemma 7.1. Moreover, \( m^* \) and \( m^n \) are strictly increasing in the range of \( v \). More precisely, for each \( y \), define \( \mathcal{R}(y) \equiv \{ \alpha \in \mathbb{R} : \exists w \text{ such that } v(y, w) = \alpha \} \). We claim that if \( \alpha < \alpha', \alpha, \alpha' \in \mathcal{R}(y) \), then \( m^*(\alpha') > m^*(\alpha) \), and similarly for \( m^n \).\footnote{Note that \( m^n \) and \( m^* \) are the corresponding c.d.f. functions for \( v \). Thus, proving that those functions are strictly increasing in the range of \( v \) leads to continuity of the quantile with respect to \( \tau \), by (an adaptation of) Lemma 7.1(viii). But this is not what we need: we want continuity in \( (y, z) \). We prefer to offer here a direct and detailed argument, although long.}

Indeed, assume that \( \exists w, w' \text{ such that } v(y, w) = \alpha \text{ and } v(y, w') = \alpha' \). The set \( P = \{ \alpha w + (1 - \alpha)w' : \alpha \in [0, 1] \} \) is contained in \( \mathcal{Z} \) because this is convex. Thus, \( \{v(y, p) : p \in P\} \) is connected, that is, a nonempty interval. We conclude that, since \( v \) is continuous, the set \( \{w \in \mathcal{Z} : \alpha < v(y, w) < \alpha' \} \) is a nonempty and open interval. (This implies, in particular, that \( R(y) \) is an interval.) Since \( f(\cdot|z) \) is strictly positive in \( \mathcal{Z} \), we conclude that
\[
m^*(\alpha') - m^*(\alpha) \geq \Pr(\{w \in \mathcal{Z} : \alpha < v(y, w) < \alpha'\}|z) > 0,
\]
which establishes the claim. By Lemma 7.1(iv), we have
\[
m^n(\alpha^n) \geq \tau \text{ and } m^*(\alpha^*) \geq \tau. \tag{54}
\]
We will show that \( \alpha^n \to \alpha^* \) by first establishing \( \lim \inf_n \alpha^n \geq \alpha^* \) and then \( \alpha^* \geq \lim \sup_n \alpha^n \).

Suppose that \( \underline{\alpha} \equiv \lim \inf_n \alpha^n < \alpha^* \). This means that there exists \( \varepsilon > 0 \) and for each \( j, n_j > j \) such that \( \alpha^{n_j} < \underline{\alpha} + \varepsilon < \alpha^* \). By the definition of \( \alpha^* \), \( \underline{\alpha} < \alpha^* \) implies \( m^*(\underline{\alpha}) < \tau \). However, by (54), \( m^n(\alpha^{n_j}) \geq \tau \), which implies \( m^n(\underline{\alpha}) \geq \tau \) and \( m^*(\underline{\alpha}) \geq \tau \), by (53). This contradiction establishes that \( \lim \inf_n \alpha^n \geq \alpha^* \).
If \( \tilde{\alpha} \equiv \limsup_n \alpha^n > \alpha^* \), there exists \( \varepsilon > 0 \) and for each \( j, n_j > j \) such that
\[
\tilde{\alpha} + \varepsilon > \alpha^{n_j} > \tilde{\alpha} - \varepsilon > \tilde{\alpha} - 2\varepsilon > \alpha^* + \varepsilon. \tag{55}
\]
Recall that \( \alpha^n = \inf\{\alpha \in \mathbb{R} : m^n(\alpha) \geq \tau\} \). Therefore, \( \alpha^{n_j} > \tilde{\alpha} - \varepsilon \) implies \( m^{n_j}(\tilde{\alpha} - \varepsilon) < \tau \). Thus, \( m^{n_j}(\alpha^* + \varepsilon) < m^{n_j}(\tilde{\alpha} - \varepsilon) < \tau \). This implies that
\[
m^*(\alpha^*) \leq m^*(\tilde{\alpha} - 2\varepsilon) \leq m^*(\tilde{\alpha} - \varepsilon) = \lim_{n \to \infty} m^{n_j}(\tilde{\alpha} - \varepsilon) \leq \tau \leq m^*(\alpha^*). 
\]
Therefore, \( m^* \) is constant between \( \alpha^* \) and \( \tilde{\alpha} - 2\varepsilon \). This will be a contradiction if we show that \( \alpha^*, \tilde{\alpha} - 2\varepsilon \in R(y^*) \).

Since \( m^*(\alpha^*) = \Pr\{w : v(y^*, w) \leq \alpha^*\} > 0 \), \( \{w : v(y^*, w) \leq \alpha^*\} \neq \emptyset \) and there exists some \( \alpha \in R(y^*) \cap (-\infty, \alpha^*]. \) On the other hand, if \( \{w : \tilde{\alpha} - 2\varepsilon \leq v(y^*, w) \leq \tilde{\alpha} + 2\varepsilon\} = \emptyset \), then for sufficiently high \( j \), \( \{w : \tilde{\alpha} - \varepsilon \leq v(y^{n_j}, w) \leq \tilde{\alpha} + \varepsilon\} = \emptyset \). In this case, \( m^{n_j}(\tilde{\alpha} - \varepsilon) = m^{n_j}(\tilde{\alpha} + \varepsilon) \equiv \tau^{n_j} \). But this would imply either \( \alpha^{n_j} \leq \tilde{\alpha} - \varepsilon \), if \( \tau^{n_j} \geq \tau \) or \( \alpha^{n_j} \geq \tilde{\alpha} + \varepsilon \), if \( \tau^{n_j} < \tau \). In either case, we have a contradiction with \( \alpha^{n_j} \in (\tilde{\alpha} - \varepsilon, \tilde{\alpha} + \varepsilon) \) as required in (55). This contradiction shows that there exists \( \alpha' \in R(y^*) \cap [\tilde{\alpha} - 2\varepsilon, \tilde{\alpha} + 2\varepsilon] \). Since \( \alpha^*, \alpha' \in R(y^*) \), we have \([\alpha^*, \tilde{\alpha} - 2\varepsilon] \subset [\alpha, \alpha'] \subset R(y^*) \). This concludes the proof. \( \square \)

**Lemma 7.6.** For each \( v \in C \) the supremum in (23) is attained and \( M^T(v) \in C \). Moreover, the optimal correspondence \( \Upsilon : X \times Z \rightrightarrows X \) defined by
\[
\Upsilon(x, z) \equiv \arg \max_{y \in E(x, z)} Q_x[u(x, y, z) + \beta v^T(y, w)|z] \tag{56}
\]
is nonempty and upper semi-continuous.

**Proof.** Let
\[
g(x, y, z, w) = u(x, y, z) + \beta v(y, w). \tag{57}
\]
By Lemma 7.2, \( Q_x[g(x, y, z, \cdot)|z] = u(x, y, z) + \beta Q_x[v(y, \cdot)|z] \). By Lemma 7.5, \( Q_x[g(x, y, z, \cdot)|z] \) is continuous in \( (x, y, z) \). From Berge's Maximum Theorem, the maximum is attained, the value function \( M^T(v) \) is continuous and \( \Upsilon \) is nonempty and upper semi-continuous. \( M^T(v) \) is bounded because \( u \) and \( v \), hence \( g \), are bounded. Therefore, \( M^T(v) \in C \). \( \square \)

We conclude the proof of Theorem 3.9 by showing that \( M^T \) satisfies Blackwell's sufficient conditions for a contraction.

**Lemma 7.7.** \( M^T \) satisfies the following conditions:

(a) For any \( v, v' \in C \), \( v \leq v' \) implies \( M^T(v) \leq M^T(v') \).

(b) For any \( a \geq 0 \) and \( x \in X \), \( M(v + a)(x) \leq M(v)(x) + \beta a \), with \( \beta \in (0, 1) \).

Then, \( \|M(v) - M(v')\| \leq \beta \|v - v'\| \), that is, \( M \) is a contraction with modulus \( \beta \). Therefore, \( M^T \) has a unique fixed-point \( v^* \in C \).

**Proof.** To see (a), let \( v, v' \in C \), \( v \leq v' \) and define \( g \) as in (57) and analogously for \( g' \), that is, \( g'(x, y, z, w) = u(x, y, z) + \beta v'(y, w) \). It is clear that \( g \leq g' \). Then, by Lemma 7.1(vi), \( Q_x[g(\cdot)|z] \leq Q_x[g'(\cdot)|z] \), which implies (a).
To verify (b), we use the monotonicity property (Lemma 7.2):

\[ Q_x[u(x, y, z) + \beta(v(x, z) + \alpha)]z = Q_x[u(x, y, z) + \beta v(x, z)z] + \beta \alpha. \]

Thus, \( M^\tau(v + \alpha) = M^\tau(v) + \beta \alpha \), that is, (b) is satisfied with equality. \( \square \)

**Proof of Theorem 3.10:** Let assumption 2 hold. It is convenient to introduce the following notation. Let \( C' \subset C \) be the set of the functions \( v : X \times Z \to \mathbb{R} \) which are concave in its first argument. It is easy to see that \( C' \) is a closed subset of \( C \). Let \( C'' \subset C' \) be the set of strictly concave functions. If we show that \( M^\tau(C') \subset C'' \), then the fixed-point of \( M^\tau \) will be strictly concave in \( x \). (See, for instance, Stokey, Lucas, and Prescott (1989, Corollary 1, p. 52).)

**Lemma 7.8.** Let assumption 2 hold. \( M^\tau(C') \subset C'' \). Therefore, \( v^* \in C'' \). Moreover, the optimal correspondence \( \Gamma : X \times Z \Rightarrow X \) defined by (56) is single-valued. Therefore, we can denote it by a function \( y^*(x, z) \).

**Proof.** Let \( \alpha \in (0, 1) \), and consider \( x_0, x_1 \in X \), \( x_0 \neq x_1 \). For \( i = 0, 1 \), let \( y_i \in \Gamma(x_i, z) \) attain the maximum, that is,

\[ M^\tau(v) (x_i, z) = u(x_i, y_i, z) + \beta Q_x[v(y_i, w)z] = Q_x[g(x_i, y_i, z, w)z]. \]

Let \( x_{\alpha} = \alpha x_0 + (1 - \alpha) x_1 \) and \( y_{\alpha} = \alpha y_0 + (1 - \alpha) y_1 \). First, let us observe that

\[ g(x_{\alpha}, y_{\alpha}, z, w) = u(x_{\alpha}, y_{\alpha}, z) + \beta v(y_{\alpha}, w) \]
\[ > \alpha u(x_0, y_0, z) + (1 - \alpha) u(x_1, y_1, z) + \beta v(y_\alpha, w) \]
\[ \geq \alpha u(x_0, y_0, z) + (1 - \alpha) u(x_1, y_1, z) + \beta [\alpha v(y_0, w) + (1 - \alpha) v(y_1, w)] \]
\[ = \alpha g(x_0, y_0, z, w) + (1 - \alpha) g(x_1, y_1, z, w), \]

where the first inequality comes from the strict concavity of \( u \) and the second, from the concavity of \( v \). That is, \( g \) is strictly quasiconcave, which establishes that \( \Gamma(x, z) \) is single-valued. Therefore,

\[ Q_x[g(x_{\alpha}, y_{\alpha}, z, w)z] > Q_x[\alpha g(x_0, y_0, z, w) + (1 - \alpha) g(x_1, y_1, z, w)w]. \]

Note that the variables \( X = g(x_0, y_0, z, w) \) and \( Y = g(x_1, y_1, z, w) \) satisfy the assumption of Proposition 7.4 since \( v \) is nondecreasing in \( w \) (holding \( z \) fixed). Therefore,

\[ Q_x[g(x_{\alpha}, y_{\alpha}, z, w)z] > \alpha Q_x [g(x_0, y_0, z, w)z] + (1 - \alpha) Q_x [g(x_1, y_1, z, w)z] \]
\[ = \alpha M^\tau(v)(x_0, z) + (1 - \alpha) M^\tau(v)(x_1, z). \] (58)

Therefore,

\[ M^\tau(v)(x_{\alpha}, z) > Q_x[g(x_{\alpha}, y_{\alpha}, z, w)z] \]
\[ > \alpha M^\tau(v)(x_0, z) + (1 - \alpha) M^\tau(v)(x_1, z), \]

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Thus, as we wanted to show.

**Lemma 7.9.** Let assumption 2 hold. If \( h : Z \to \mathbb{R} \) is weakly increasing and \( z \leq z' \), then \( Q_z[h(w)|z] \leq Q_z[h(w)|z'] \).

**Proof.** From Assumption 2(ii), if \( h : Z \to \mathbb{R} \) is weakly increasing and \( z \leq z' \):

\[
\int_Z h(\alpha) [-1_{(\alpha \in Z; \alpha \leq w)}] f(\alpha|z) \, d\alpha \leq \int_Z h(\alpha) [-1_{(\alpha \in Z; \alpha \leq w)}] f(\alpha|z') \, d\alpha.
\]

Thus,

\[
\int_{(\alpha \in Z; \alpha \leq w)} h(\alpha) f(\alpha|z) \, d\alpha \geq \int_{(\alpha \in Z; \alpha \leq w)} h(\alpha) f(\alpha|z') \, d\alpha. \tag{59}
\]

If we define \( H(w|z) = \Pr ([h(W) \leq h(w)]|Z = z) \), then (59) can be written as:

\[
H(w|z) \geq H(w|z').
\]

Observe that \( Q_z[h(w)|z] = \inf\{ \alpha \in \mathbb{R} : H(\alpha|z) \geq \tau \} \) and, whenever \( z \leq z' \), \( H(w|z') \leq H(w|z) \), for all \( w \). Therefore, if \( z \leq z' \), then

\[
\{ \alpha \in \mathbb{R} : H(\alpha|z) \geq \tau \} \supset \{ \alpha \in \mathbb{R} : H(\alpha|z') \geq \tau \},
\]

which implies that

\[
Q_z[h(w)|z] = \inf\{ \alpha \in \mathbb{R} : H(\alpha|z) \geq \tau \} \leq \inf\{ \alpha \in \mathbb{R} : H(\alpha|z') \geq \tau \} = Q_z[h(w)|z'],
\]

as we wanted to show.

**Lemma 7.10.** Let assumption 2 hold. If \( v \in C \) is increasing in \( z \) then \( M^\tau(v) \) is strictly increasing in \( z \).

**Proof.** Let \( z_1, z_2 \in Z \), with \( z_1 < z_2 \). For \( i = 1, 2 \), let \( y_i \in \Gamma(x, z_i) \) realize the maximum, that is,

\[
M^\tau(v) (x, z_i) = u(x, y_i, z_i) + \beta Q_z[v(y_i, w)|z_i].
\]

Since \( u \) is strictly increasing in \( z \), we have:

\[
M^\tau(v) (x, z_1) = u(x, y_1, z_1) + \beta Q_z[v(y_1, w)|z_1] < u(x, y_1, z_2) + \beta Q_z[v(y_1, w)|z_1].
\]

From Lemma 7.9, we have \( Q_z[v(y_1, w)|z_1] \leq Q_z[v(y_1, w)|z_2] \), which gives:

\[
M^\tau(v) (x, z_1) < u(x, y_1, z_2) + \beta Q_z[v(y_1, w)|z_2].
\]

From Assumption 2, \( \Gamma(x, z) \subseteq \Gamma(x, z') \), that is, \( y_1 \in \Gamma(x, z_2) \). Optimality thus implies that:

\[
u(x, y_1, z_2) + \beta Q_z[v(y_1, w)|z_2] \leq u(x, y_2, z_2) + \beta Q_z[v(y_2, w)|z_2] = M^\tau(v) (x, z_2).
\]

Therefore, \( M^\tau(v) (x, z_1) < M^\tau(v) (x, z_2) \), which shows strict increasingness in \( z \).

We conclude the proof of Theorem 3.10 by showing differentiability of \( v \), which follows from an
easy adaptation of Benveniste and Scheinkman (1979)'s argument. For completeness and reader's convenience, we reproduce it here. Given \((x, z)\), let \(y^*(x, z) \in \Gamma(x, z)\) be unique maximum as established in Lemma 7.8. Thus, for all \((x, z)\), we have:

\[ v(x, z) = u(x, y^*(x, z), z) + \beta Q_x[\nu(y^*(x, z), w)|z]. \]

Fix \(x_0\) in the interior of \(X\) and define:

\[ \tilde{w}(x, z) = u(x, y^*(x_0, z), z) + \beta Q_x[\nu(y^*(x_0, z), w)|z]. \]

From the optimality, for a neighborhood of \(x_0\), we have \(\tilde{w}(x, z) \leq v(x, z)\), with equality at \(x = x_0\), which implies \(\tilde{w}(x, z) - \tilde{w}(x_0, z) \leq v(x, z) - v(x_0, z)\). Note that \(\tilde{w}\) is concave and differentiable in \(x\) because \(u\) is. Thus, any subgradient \(p\) of \(v\) at \(x_0\) must satisfy

\[ p \cdot (x - x_0) \geq v(x, z) - v(x_0, z) \geq \tilde{w}(x, z) - \tilde{w}(x_0, z). \]

Thus, \(p\) is also a subgradient of \(\tilde{w}\). But since \(\tilde{w}\) is differentiable, \(p\) is unique. Therefore, \(v\) is a concave function with a unique subgradient. Therefore, it is differentiable in \(x\) (cf. Rockafellar (1970, Theorem 25.1, p. 242)) and its derivative with respect to \(x\) is the same as that of \(\tilde{w}\), that is,

\[ \frac{\partial v^*}{\partial x_i}(x, z) = \frac{\partial \tilde{w}}{\partial x_i}(x, z) = \frac{\partial u}{\partial x_i}(x, y^*(x, z), z), \]

as we wanted to show. \(\square\)

**Proof of Lemma 3.12:** By Stokey, Lucas, and Prescott (1989, Theorem 7.6), \(\Gamma\) has a measurable selection. Therefore, the argument in Stokey, Lucas, and Prescott (1989, Lemma 9.1) establishes the result. \(\square\)

We need the following notation in the next proof. Let \(\Pi \in \mathbb{N} \cup \{\infty\}\) and \(S : Z^T \to Z^{T-1}\) be the shift operator, that is, given \(z = (z_1, z_2, \ldots, z_T) \in Z^T\), \(S(z) = (z_2, \ldots, z_T) \in Z^{T-1}\). Abusing notation, let \(S : \Pi \to \Pi\) also denote the shift operator for plans, that is, given \(\pi \in \Pi\), \(\pi^t = S(\pi) \in \Pi\) is defined as follows: for each given \(z^\infty \in Z^\infty\), \(\pi_t(x, S(z^{t+1})) = \pi_{t+1}(x, z^{t+1})\). Let \(S_t : \Pi \to \Pi\) be the composition of \(S\) with itself \(t\) times.

**Proof of Lemma 3.13:** Let \(t \geq 2\) (otherwise there is nothing to prove). Since \(\Pi_t(x, z) \subset \Pi_1(x, z)\) by definition, we have \(v^*_t(x, z) \leq v_t(x, z)\). Suppose, for an absurd, that there exists \(\pi \in \Pi(x, z)\) such that

\[ V_1(\pi, x, z) > v^*_1(x, z). \quad (60) \]

Let \(\tilde{\pi}\) and \((\tilde{x}, \tilde{z}).\) be such that \(S_{t-1}(\tilde{\pi}) = \pi, x_t^\ast(\tilde{x}, \tilde{z}) = x\) and \(\tilde{z}_t = z\). Then, \(V_t(\tilde{z}, \tilde{x}, \tilde{z}) = V_1(\pi, x, z)\). Since \(v^*_t(x, z) \geq V_t(\tilde{z}, \tilde{x}, \tilde{z})\), this establishes a contradiction with (60). \(\square\)

**Proof of Lemma 3.14:** If \(v\) is bounded and satisfies (25), then it is the unique fixed-point of the contraction \(M^\ast\). Thus, the proof of Theorem 3.9 establishes, via the Maximum Theorem, the claims. \(\square\)

**Proof of Theorem 3.15:** Assume that \(v\) satisfies (25). It is sufficient to show that (i) \(v(x, z) \geq V_1(\pi, x, z)\) for any \(\pi \in \Pi(x, z)\) and \((x, z) \in \mathcal{X} \times \mathcal{Z}\); and (ii) \(v(x, z) = V_1(\pi^\ast, x, z)\). Let \(\pi \in \Pi(x, z)\). We
have:

\[
\nu(x, z) = \sup_{y \in \Gamma(x_1, z_1)} u(x_1^*, y, z_1) + \beta Q_\tau [\nu(y, z_2)] z_1 \\
\geq u(x_1^*, x_2^*, z_1) + \beta Q_\tau [\nu(x_2^*, z_2)] z_1 \\
= u(x_1^*, x_2^*, z_1) + \beta Q_\tau \left( \sup_{y \in \Gamma(x_2^*, z_2)} \left\{ u(x_2^*, y, z_2) + \beta Q_\tau [\nu(y, z_3)] z_2 \right\} \right) z_1 \\
\geq u(x_1^*, x_2^*, z_1) + Q_\tau \nu(x_2^*, x_3^*, z_2) + Q_\tau \beta^2 [\nu(x_3^*, z_3)] z_2 + \ldots
\]

where the two inequalities come from the definition of sup, and the equalities from (25) and Corollary 7.3. Repeating the same arguments, we obtain:

\[
\nu(x, z) \geq u(x_1^*, x_2^*, z_1) + Q_\tau \left[ \beta u(x_2^*, x_3^*, z_2) + Q_\tau \beta^2 [\nu(x_3^*, x_4^*, z_3)] z_3 + \ldots \right. \\
\left. + Q_\tau \beta^n u(x_{n+1}^*, x_{n+2}^*, z_n) + \beta^{n+1} [\nu(x_n^*, z_n)] z_n \right] z_1 = z_n \ldots Z_1 = z.
\]

Repeating the arguments in the proof of Proposition 3.5, we can conclude that the limit of the right hand size when \( n \to \infty \) is \( V'(x, z) = V_1(\pi, x, z) \). Thus, we have established that \( \nu(x, z) \geq V_1(\pi, x, z) \). Since \( \pi \) was arbitrary, then \( \nu(x, z) \geq V^*(x, z) \). On the other hand, for \( \pi^0 \) the inequalities above hold with equality and we obtain \( \nu(x, z) = V^*(x, z) \).

**Proof of Theorem 3.16:** Let \( g(x, y, z, w) \equiv u(x, y, z) + \beta Q_\tau [\nu^y(y, w)] z \) and \( y^*(x, z) \) be an interior solution of the problem (25). Observe that \( \nu^y \) is increasing in \( w \), differentiable in its first variable and for \( 0 < x_i - x_i < \epsilon \), for some small \( \epsilon > 0 \),

\[
\nu^y(x_i, x_{i-1}, z) - \nu^y(x_i, x_{i-1}, z) = \int_x^{x_i} \frac{\partial \nu^y}{\partial x_i} (\alpha, x_{i-1}, z) d\alpha = \int_x^{x_i} \frac{\partial u}{\partial x_i} (\alpha, x_{i-1}, z) d\alpha
\]

is increasing in \( z \) because \( \frac{\partial u}{\partial x_i} \) is. Therefore, the assumptions of Proposition 3.17 are satisfied and we conclude that \( \frac{\partial Q_\tau}{\partial x_i} [\nu^y(x, z)] = Q_\tau \left[ \frac{\partial \nu^y}{\partial x_i} (x, z) \right] \). Since \( u \) is differentiable in \( y \), so is \( g \). Since \( y^*(x, z) \) is interior, the following first order condition holds:

\[
\frac{\partial g}{\partial y_i} (x, y^*(x, z), z, Q_\tau [\nu^y(w)]) = \frac{\partial u}{\partial y_i} (x, y^*(x, z), z) + \beta Q_\tau \left[ \frac{\partial \nu^y}{\partial x_i} (y^*(x, z), w) \right] = 0.
\]

Now we apply Theorem 3.10 and its expression: \( \frac{\partial \nu^y}{\partial x_i} (x, z) = \frac{\partial u}{\partial x_i} (x, y^*(x, z), z) \), to conclude that

\[
\frac{\partial u}{\partial y_i} (x, y^*(x, z), z) + \beta Q_\tau \left[ \frac{\partial u}{\partial x_i} (y^*(x, z), y^*(x, z), w) \right] = 0. \tag{61}
\]

Now, we have just to put the notation of a sequence. For this, let \( \pi = (x_{i}) \) denote an optimal path beginning at \( (x_0, z_0) \), (61) can be rewritten, substituting \( x \) for \( x_1^*, y^*(x, z) \) for \( x_{i+1}^* \), \( y^*(y^*(x, z), w) \) for
increasing in \( z \), which allows us to conclude that
\[
\frac{\partial u}{\partial y_i} (x_i^{\pi}, x_{i+1}^{\pi}, z_t) + \beta Q_t \left[ \frac{\partial u}{\partial x_i} (x_i^{\pi}, x_{i+1}^{\pi}, z_{t+1}) \right] z_t = 0. \tag{62}
\]
which we wanted to establish.

\textbf{Proof of Proposition 3.17:} Fix \( x = (x_i, x_{i-1}) \), with the usual meaning and \( \delta > 0 \). Define \( X = \hat{h}(z) = h(x_i + \delta, x_{i-1}, z) - h(x_i, x_{i-1}, z) \) and \( Y = \hat{g}(z) = h(x_i, x_{i-1}, z) \). Since \( h \) and \( d(z) = h(x_i + \delta, x_{i-1}, z) - h(x_i, x_{i-1}, z) \) are increasing in \( z \) by assumption, the random variables \( X \) and \( Y \) satisfy the assumptions of Proposition 7.4, which allows us to conclude that
\[
Q_t[h(x_i + \delta, x_{i-1}, z)] = Q_t[X + Y] = Q_t[X] + Q_t[Y]
= Q_t[h(x_i + \delta, x_{i-1}, z) - h(x_i, x_{i-1}, z)] + Q_t[h(x_i, x_{i-1}, z)].
\]

Therefore,
\[
\frac{Q_t[h(x_i + \delta, x_{i-1}, z)] - Q_t[h(x_i, x_{i-1}, z)]}{\delta} = Q_t \left[ \frac{h(x_i + \delta, x_{i-1}, z) - h(x_i, x_{i-1}, z)}{\delta} \right].
\]
Taking the limit when \( \delta \to 0 \) on both sides above, we obtain:
\[
\frac{\partial Q_t}{\partial x_i^{\pi}} [h(x, z)] = Q_t \left[ \frac{\partial h}{\partial x_i^{\pi}} (x, z) \right],
\]
as we wanted to show.

\textbf{7.3 Proofs of Section 5}

\textbf{Proof of Lemma 5.1:} Assumption 1 (i)–(iii) and (v) are immediate. Since \( Z \) and \( X \) are bounded, and \( U \) and \( z \to z + p(z) \) are \( C^1 \), \( u \) is \( C^1 \) and bounded. Thus, Assumption 1 is satisfied. Similarly, Assumptions 2 are easily seen to be satisfied. It remains to verify the assumption of Theorem 3.16, namely that \( \frac{\partial u}{\partial x_i} (x_i^{\pi}, x_{i+1}^{\pi}, z_t) \) is strictly increasing in \( z_t \), which happens if and only if \( \log \frac{\partial u}{\partial x_i} (x_i^{\pi}, x_{i+1}^{\pi}, z_t) \) is strictly increasing in \( z_t \). Since
\[
\log \frac{\partial u}{\partial x} (x, y, z) = -\gamma \log [z \cdot x + p(z) \cdot (x - y)] + \log (z + p(z)),
\]
and \( x_t^{\pi} = x_{i+1}^{\pi} = 1 \), we need to verify only that \( -\gamma \log [z \cdot x]' + [\log (z + p(z))]' > 0 \). This is equivalent to \( \gamma < z [\log (z + p(z))]' \), which is contained in Assumption 3(iv).

\textbf{References}


