Algorithms for cautious reasoning in games*

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Abstract

We provide comparable algorithms for the Dekel-Fudenberg procedure, iterated admissibility, proper rationalizability and full permissibility by means of the concepts of preference restrictions and likelihood orderings. We apply the algorithms for comparing iterated admissibility, proper rationalizability and full permissibility, and provide a sufficient condition under which iterated admissibility does not rule out properly rationalizable strategies. Finally, we use the algorithms to examine an economically relevant strategic situation, namely a bilateral commitment bargaining game.

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1 Introduction

In non-cooperative game theory, a player is *cautious* if he takes into account all opponents' strategies, also strategies that seem very unlikely to be chosen by the opponents. Cautious reasoning of a player can be modeled by a *lexicographic belief* (Blume et al., 1991a). This notion allows a player i to deem some strategy s_j of an opponent j infinitely more likely than some other strategy s'_j, while still taking s'_j into account. What outcomes of a strategic game are consistent with common belief of the event that all players are rational and cautious?

Various concepts in the literature provide different answers to this question. Still, there is a common idea underlying each of these concepts, namely that player i should deem a strategy s_j of opponent j infinitely more likely than strategy s'_j whenever player i considers s_j a "better choice" for opponent j than s'_j. The question then remains what we should mean by a "better choice".

As an illustration, consider the following economic example (which is the *Spy game* of Perea, 2012, p. 262, but with another motivating story). An entrant (firm 1) and an incumbent (firm 2) must decide which type of good to bring to the market: x, y or z. The entrant expects a revenue of 3 as long as it produces a good different from the incumbent, and a revenue of 2 if it produces the same good. Its production costs for each of the goods is 2. The incumbent expects, for every production choice, a revenue of 3. The only exception is when the goods x and z are both brought on the market. Since these goods are complementary, the incumbent expects a revenue of 6 in this case. The incumbent has produced good x in the past, which would therefore have the lowest costs (normalized to 0). Producing goods y and z would cost the incumbent 1 and 2, respectively, since good y is more similar to x than z is. The profits for both firms can be found in Figure 0, where the choice of firm 1 is indicated in upper case, to differentiate from the choice of firm 2 in lower case.

Note that for firm 2, production choice y can never be rational as x strictly dominates y, whereas x and z can be rational for some belief about firm 1's choice. One could therefore argue that x and z are better choices for firm 2 than y, and hence firm 1 should deem x and z infinitely more likely than y. But then, if firm 1 takes all possible choices by firm 2 into account, its unique rational choice would be to implement production plan Y. The line of argument we have followed here corresponds to the procedure of *iterated admissibility* which iteratively eliminates all weakly dominated strategies, as it corresponds to the epistemic foundation provided

	x	y	z
X	0, 3	1, 2	1, 4
Y	1, 3	0, 2	1, 1
Z	1, 6	1, 2	0, 1

Figure 0: Illustrating iterated admissiblity and proper rationalizability.

for this procedure by Brandenburger et al. (2008)

Iterated admissibility is not the only plausible procedure for cautious reasoning, however. Consider again the example above. If firm 2 would indeed believe that firm 1 makes production choice Y, which is what iterated admissibility requires, then choice y would actually be better for firm 2 than choice z. So given that firm 1 believes that firm 2 believes that firm 1 will do the choice that iterated admissibility requires, one could argue that firm 1 should deem y infinitely *more* likely than z, and not infinitely *less* likely, as iterated admissibility imposes. Hence, by applying the procedure of iterated admissibility one may along the way impose conditions on lexicographic beliefs which need not be convincing given the prescriptions that this procedure ends up providing.

The concept of *proper rationalizability* (Schuhmacher, 1999; Asheim, 2001) takes a different viewpoint. The key condition is that a player i should deem a strategy s_j of opponent j infinitely more likely than strategy s_j' whenever he believes that opponent j, after completing his reasoning process, prefers s_j to s_j'. If the beliefs of player i satisfies this condition, we say (following Blume et al., 1991b, Definition 4) that player i *respects the preferences* of opponent j.

To see what difference this approach makes, let us return to the example. It is clear that for firm 2, choice x is better than choice y, whereas we cannot say at this stage of the reasoning processs that z is better than y. Proper rationalizability therefore only requires that firm 1 deems x infinitely more likely than y, but does not require that it deems z infinitely more likely than y. If firm 1 indeed holds such a belief, then it prefers Y to X, and hence firm 2 should deem Y infinitely more likely than X. But then, firm 2 will prefer x to y, and y to z. Hence, firm 1 should deem x infinitely more likely than y, and y infinitely more likely than z. As a consequence, firm 1 should choose production plan Z, and not Y as iterated admissibility requires.

Both concepts, *iterated admissibility* and *proper rationalizability,* are reasonable concepts with their own intuitive appeal, but may lead to completely different choices as we have seen. It therefore seems worthwile to investigate their differences and similarities in some more detail, and this is exactly what this paper seeks to accomplish.

A number of contributions, starting with Brandenburger (1992) and Börgers (1994), have shown that the *Dekel-Fudenberg procedure* (Dekel and Fudenberg, 1990), where one round of elimination of *weakly* dominated strategies is followed by iterated elimination of *strictly* dominated strategies, provides a robust answer to the question we posed initially, in the sense that the eliminated strategies are definitely incompatible with common belief of the event that all players are rational and cautious. Hence, surviving the Dekel-Fudenberg procedure, and thus being *permissible* in the terminology of Brandenburger (1992), is a necessary condition. However, the concepts of *iterated admissibility* and *proper rationalizability* might rule out more strategies. The same applies for the concept of *full permissibility* (Asheim and Dufwenberg, 2003a), which is another procedure for cautious reasoning in games.

Permissibility, iterated admissibility and full permissibility are all defined in terms of algorithms. While epistemic foundations for the former and latter were provided quickly (Brandenburger, 1992; Börgers, 1994; Asheim and Dufwenberg, 2003a), half a century elapsed between the introduction of iterated admissibility in the 1950s and the establishment of an epistemic foundation for this procedure by Brandenburger et al. (2008).

The case of proper rationalizability is different. This concept was defined by Schuhmacher (1999) and Asheim (2001) by means of epistemic conditions. Schuhmacher defines, for every $\varepsilon > 0$, the ε-proper trembling condition, which states that if a player prefers one pure strategy over another, then the probability he assigns to the latter strategy should be at most ε times the probability he assigns to the former. Proper rationalizability is obtained by imposing common belief of the ε-proper trembling condition, and then letting ε tend to zero. Schuhmacher (1999) provides an algorithm, *iteratively proper trembling,* which generates for a given $\varepsilon > 0$ the set of mixed strategy profiles that can be chosen under common belief of the ε-proper trembling condition. However, this procedure does not yield the set of properly rationalizable strategies directly, as we must still let ε go to zero, and see which strategies survive in the limit. Only later has Perea (2011) provided an algorithm that *directly* computes the set of properly rationalizable strategies.

The purpose of the present paper is to present algorithms for permissibility, iterated admissibility and full permissibility that build on the key concepts introduced by Perea (2011), thereby making such established procedures comparable to the new algorithm for proper rationalizability. Section 2 introduces these key concepts: *preference restrictions* and *likelihood orderings*. Above we have already illustrated in the game of Figure 0 how the working of iterated admissibility and proper rationalizability can be described in terms of likelihood orderings, indicating which strategies are deemed infinitely more likely than others. Section 3 introduces algorithms that iterately eliminates likelihood orderings, for the concepts of permissibility, iterated admissibility and full permissibiity. These algorithms are thus comparable with the one for proper rationalizability. Section 4 then puts these algorithms to use. In particular, we offer examples further illuminating the differences between iterated admissibility, proper rationalizability and full permissibility. Moreover, we provide a sufficient condition under which iterated admissibility does not rule out properly rationalizable strategies. Finally, we use the algorithms to examine an economically relevant strategic situation, namely a bilateral commitment bargaining game which has been analyzed by Ellingsen and Miettinen (2008). Section 5 offers concluding remarks, while an appendix contains all proofs.

2 Preference Restrictions and Likelihood Orderings

Consider a finite *strategic game* $G = (S_i, u_i)_{i \in I}$ where I is a finite set of players and where, for $i \in I$, the finite set S_i denotes player i's set of strategies and $u_i : \prod_{j \in I} S_j \to \mathbb{R}$ denotes player i's utility function. Write $S_{-i} := \prod_{j \neq i} S_j$. As usual, we extend u_i to subjective probability distributions $\lambda_i \in \Delta(S_{-i})$ over the opponent's strategy profiles, writing $u_i(s_i, \lambda_i)$ for the resulting subjective expected utility.

Say that strategy $s_i \in S_i$ is *strictly dominated* by a mixed strategy $\mu_i \in \Delta(S_i)$ on a subset of opponents' strategy profiles $S'_{-i} \subseteq S_{-i}$ if $u_i(s_i, s_{-i}) < u_i(\mu_i, s_{-i})$ for every $s_{-i} \in S'_{-i}$. Similarly, say that s_i is *weakly dominated* by μ_i on S'_{-i} if $u_i(s_i, s_{-i}) \leq u_i(\mu_i, s_{-i})$ for every $s_{-i} \in S'_{-i}$, with strict inequality for some $s'_{-i} \in S'_{-i}$.

Each player i's preferences over his own strategies are determined by u_i and a *lexicographic probability system* (LPS) (Blume et al., 1991a) with full support on S_{-i}. An LPS consists of a finite sequence of subjective probability distributions, $\lambda_i = (\lambda_i^1, \ldots, \lambda_i^K)$, where for each $k \in \{1, \ldots, K\}$, $\lambda_i^k \in \Delta(S_{-i})$. Player i prefers $a_i \in S_i$ to $s_i \in S_i$ if there exists $k \in \{1, \ldots, K\}$ such that (i) $u_i(a_i, \lambda_i^k) > u_i(s_i, \lambda_i^k)$ and (ii) $u_i(a_i, \lambda_i^\ell) = u_i(s_i, \lambda_i^\ell)$ for all $\ell \in \{1, \ldots, k-1\}$. The LPS $\lambda_i = (\lambda_i^1, \ldots, \lambda_i^K)$ has *full support* on S_{-i} if, for all $s_{-i} \in S_{-i}$, there exists $k \in \{1, \ldots, K\}$ such that

$\lambda_i^k(s_{-i}) > 0$. Player i deems s'_{-i} *infinitely more likely than* s''_{-i} (written $s'_{-i} \gg_i s''_{-i}$) if there exists $k \in \{1, \ldots, K\}$ such that (i) $\lambda_i^k(s'_{-i}) > 0$ and (ii) $\lambda_i^\ell(s''_{-i}) = 0$ for all $\ell \in \{1, \ldots, k\}$. For each $j \neq i$, player i deems s'_j *infinitely more likely than* s''_j (written $s'_j \gg_i s''_j$) if there exists some $s'_{-i} \in \{s'_j\} \times \prod_{j' \neq i,j} S_{j'}$ such that $s'_{-i} \gg_i s''_{-i}$ for all $s''_{-i} \in \{s''_j\} \times \prod_{j' \neq i,j} S_{j'}$. It follows that $\gg_i$ is an asymmetric and transitive binary relation both on S_{-i} and on S_j for each $j \neq i$.

The following two definitions, which are taken from Perea (2011), provide the key concepts for our algorithms.

Definition 1 (Preference restriction) A preference restriction for player i on S_i is a pair (s_i, A_i), where $s_i \in S_i$ and A_i is a nonempty subset of S_i.

The interpretation of a preference restriction (s_i, A_i) is that player i prefers some strategy in A_i to s_i. Let R_i denote a set of preference restrictions for i, let $\mathcal{R}_i^*$ denote the collection of all sets of preference restrictions for i, and let $\mathcal{R}_{-i}^* := \prod_{j \neq i} \mathcal{R}_i^*$ denote the collection of all vectors of sets of preference restrictions for i's opponents.

For any set R_i of preference restrictions, define the *choice set* $C_i(R_i)$ as follows:

$$C_i(R_i) := \{s_i \in S_i \mid \nexists A_i \subseteq S_i \text{ with } (s_i, A_i) \in R_i\}.$$

It follows that $C_i(R'_i) \cap C_i(R''_i) = C_i(R'_i \cup R''_i)$ for every R'_i, $R''_i \in \mathcal{R}_i^*$. In particular, $C_i(R'_i) \supseteq C_i(R''_i)$ whenever $R'_i \subseteq R''_i$. Let $C_{-i}(R_{-i}) := \prod_{j \neq i} C_j(R_j)$ denote the Cartesian product of the choice sets of i's opponents, given their sets of preference restrictions.

Definition 2 (Likelihood ordering) A *likelihood ordering* for player i on S_{-i} is an ordered partition $L_i = (L_i^1, L_i^2, \ldots, L_i^K)$ of S_{-i}.

A likelihood ordering $L_i = (L_i^1, L_i^2, \ldots, L_i^K)$ on S_{-i} determines the infinitely-more-likely relation of player i: $s_{-i} \gg_i s'_{-i}$ if and only if $s_{-i} \in L_i^k$ and $s'_{-i} \in L_i^{k'}$ with $k < k'$. Let $\mathcal{L}_i^*$ denote the set of all likelihood orderings on S_{-i}, and let $\tilde{\mathcal{L}}_i^*$ ($\subseteq \mathcal{L}_i^*$) denote the set of all likelihood orderings on S_{-i} which are either trivial (so that $K = 1$ and $L_i = (L_i^1) = (S_{-i})$) or partition S_{-i} into a non-empty proper subset A_{-i} and its complement (so that $K = 2$ and $L_i = (L_i^1, L_i^2) = (A_{-i}, S_{-i} \backslash A_{-i})$).

For any likelihood ordering L_i on S_{-i}, let $R_i(L_i)$ denote the set of preference restrictions *derived* from L_i in the following manner:

$$\begin{aligned} R_i(L_i) := \{(s_i, A_i) \in S_i \times 2^{S_i} \mid\ & \exists k \in \{1, \ldots, K\} \text{ and } \mu_i \in \Delta(A_i) \\ & \text{such that } s_i \text{ is weakly dominated by } \mu_i \text{ on } L_i^1 \cup \cdots \cup L_i^k\}. \end{aligned}$$

Let $R_{-i}(L_{-i}) := (R_j(L_j))_{j \neq i}$ denote the vectors of sets of preference restrictions for i's opponents given their vector $(L_j)_{j \neq i}$ of likelihood orderings.

For any non-empty set $\mathcal{L}_i$ of likelihood orderings on S_{-i}, write

$$R_i(\mathcal{L}_i) := \bigcap\nolimits_{L_i \in \mathcal{L}_i} R_i(L_i) \, .$$

It follows that $R_i(\mathcal{L}'_i) \cap R_i(\mathcal{L}''_i) = R_i(\mathcal{L}'_i \cup \mathcal{L}''_i)$ for every $\mathcal{L}'_i$, $\mathcal{L}''_i \subseteq \mathcal{L}^*_i$. In particular, $R_i(\mathcal{L}'_i) \supseteq R_i(\mathcal{L}''_i)$ whenever $\mathcal{L}'_i \subseteq \mathcal{L}''_i$. Let $R_{-i}(\mathcal{L}_{-i}) := (R_j(\mathcal{L}_j))_{j \neq i}$ denote the vectors of sets of preference restrictions for i's opponents given their vector $\mathcal{L}_{-i} := (\mathcal{L}_j)_{j \neq i}$ of non-empty sets of likelihood orderings. Let $\mathcal{L}_{-i} \neq \emptyset$ signify that $\mathcal{L}_j \neq \emptyset$ for all $j \neq i$ and let $\mathcal{L}'_{-i} \subseteq \mathcal{L}''_{-i}$ signify that $\mathcal{L}'_j \subseteq \mathcal{L}''_j$ for all $j \neq i$.

Likelihood-orderings can be related to the ordinary *belief* operator as well as the *assumption* operator, as proposed by Brandenburger et al. (2008) (and discussed by Asheim and Søvik, 2005, Section 6).

Definition 3 (Believing an event) For a given subset $A_{-i} \subseteq S_{-i}$ of opponent strategy vectors, the likelihood ordering L_i *believes* A_{-i} if, for every $s_{-i} \in S_{-i} \backslash A_{-i}$, $a_{-i} \gg_i s_{-i}$ for some $a_{-i} \in A_{-i}$.

Hence, L_i believes A_{-i} if and only if $L_i^1 \subseteq A_{-i}$.

Definition 4 (Assuming an event) For a given subset $A_{-i} \subseteq S_{-i}$ of opponent strategy vectors, the likelihood ordering L_i *assumes* A_{-i} if $A_{-i} \neq \emptyset$ and, for every $s_{-i} \in S_{-i} \backslash A_{-i}$, $a_{-i} \gg_i s_{-i}$ for every $a_{-i} \in A_{-i}$.

Hence, L_i assumes A_{-i} if and only if there exists $k \in \{1, \ldots, K\}$ such that $L_i^1 \cup \cdots \cup L_i^k = A_{-i}$. So, if the likelihood ordering L_i assumes an event A_{-i}, then it also believes the event A_{-i} (since clearly $L_i^1 \subseteq A_{-i}$), but not vice versa.

Likelihood-orderings can also be related to *respect of preferences* as introduced by Blume et al. (1991b).

Definition 5 (Respecting preferences) For a given vector $R_{-i} \in \mathcal{R}^*_{-i}$ of sets of preference restrictions, the likelihood ordering L_i *respects* R_{-i} if, for all players $j \neq i$ and every preference restriction $(s_j, A_j) \in R_j$, $a_j \gg_i s_j$ for some $a_j \in A_j$.

It follows that a likelihood ordering L_i believes the rationality of i's opponents if it respects their preferences, but not vice versa. This can be stated formally as follows.

Lemma 1 *If the likelihood ordering L_i respects the vector $R_{-i} \in \mathcal{R}^*_{-i}$ of sets of preference restrictions, then it also believes the event $C_{-i}(R_{-i})$.*

Let $\mathcal{L}^b_i(R_{-i})$ denote the set of likelihood orderings for i that believe the rationality of i's opponents when the preferences of i's opponents satisfy the vector R_{-i} of sets of preference restrictions:

$$\mathcal{L}^b_i(R_{-i}) := \{L_i \in \mathcal{L}^*_i \mid L_i \text{ believes } C_{-i}(R_{-i})\} .$$

Let $\mathcal{L}^a_i(R_{-i})$ denote the set of likelihood orderings for i that assume the rationality of i's opponents when the preferences of i's opponents satisfy the vector R_{-i} of sets of preference restrictions:

$$\mathcal{L}^a_i(R_{-i}) := \{L_i \in \mathcal{L}^*_i \mid L_i \text{ assumes } C_{-i}(R_{-i})\} .$$

Finally, let $\mathcal{L}^r_i(R_{-i})$ denote the set of likelihood orderings for i that respect the vector R_{-i} of opponent sets of preference restrictions:

$$\mathcal{L}^r_i(R_{-i}) := \{L_i \in \mathcal{L}^*_i \mid L_i \text{ respects } R_{-i}\} .$$

We have seen that assumption implies belief, but not vice versa. Moreover, from Lemma 1 we know that respect of preferences implies belief of rationality, but not versa. Hence, we conclude that

$$\mathcal{L}^b_i(R_{-i}) \supseteq \mathcal{L}^a_i(R_{-i}) \cup \mathcal{L}^r_i(R_{-i})$$

for every $R_{-i} \in \mathcal{R}^*_{-i}$ with $C_{-i}(R_{-i}) \neq \emptyset$. Since the belief operator satisfies conjunction and monotonicity, the properties of the choice correspondence $C_i(\cdot)$ imply

$$\mathcal{L}^b_i(R'_{-i}) \cap \mathcal{L}^b_i(R''_{-i}) = \mathcal{L}^b_i(R'_{-i} \cup R''_{-i})$$

for every R'_{-i}, $R''_{-i} \in \mathcal{R}^*_{-i}$. However, since the assumption operator satisfies conjunction but not monotonicity, it holds for every R'_{-i}, $R''_{-i} \in \mathcal{R}^*_{-i}$ that

$$\mathcal{L}^a_i(R'_{-i}) \cap \mathcal{L}^a_i(R''_{-i}) \subseteq \mathcal{L}^a_i(R'_{-i} \cup R''_{-i}) ,$$

while the inverse inclusion need not hold. In particular, $\mathcal{L}^a_i(R'_{-i}) \cap \mathcal{L}^a_i(R''_{-i}) \neq \emptyset$ only if $C_{-i}(R'_{-i}) \subseteq C_{-i}(R''_{-i})$ or $C_{-i}(R''_{-i}) \subseteq C_{-i}(R'-i)$. Finally, Definition 5 implies

$$\mathcal{L}^r_i(R'_{-i}) \cap \mathcal{L}^r_i(R''_{-i}) = \mathcal{L}^r_i(R'_{-i} \cup R''_{-i})$$

for every R'_{-i}, $R''_{-i} \in \mathcal{R}^*_{-i}$. In particular, $\mathcal{L}^b_i(R'_{-i}) \supseteq \mathcal{L}^b_i(R''_{-i})$ and $\mathcal{L}^r_i(R'_{-i}) \supseteq \mathcal{L}^r_i(R''_{-i})$ whenever $R'_{-i} \subseteq R''_{-i}$. This conclusion need not hold for $\mathcal{L}^a_i(\cdot)$ since a likelihood ordering L_i may assume A'_{-i} but not A''_{-i} even though $A'_{-i} \subset A''_{-i}$. Hence, we may have $\mathcal{L}^a_i(R'_{-i}) \nsubseteq \mathcal{L}^a_i(R''_{-i})$ and $\mathcal{L}^a_i(R'_{-i}) \nsupseteq \mathcal{L}^a_i(R''_{-i})$ even though $R'_{-i} \subset R''_{-i}$.

3 Algorithms

In this section we provide comparable algorithms for permissibility (the Dekel-Fudenberg procedure), iterated admissibility, proper rationalizability, and full permissibility. To define these concepts, we need to introduce the following operators:

$$a_i(S'_{-i}) := \{s_i \in S_i \mid s_i \text{ is not weakly dominated by any } \mu_i \in \Delta(S_i) \text{ on } S'_{-i}\}\,,$$
$$b_i(S'_{-i}) := \{s_i \in S_i \mid s_i \text{ is not strictly dominated by any } \mu_i \in \Delta(S_i) \text{ on } S'_{-i}\}\,,$$

where S'_{-i} is a non-empty subset of S_{-i}. Note that $\emptyset \neq a_i(S'_{-i}) \subseteq b_i(S'_{-i}) \subseteq S_i$ for any non-empty subset S'_{-i} of S_{-i}.

3.1 An algorithm for permissibility

We first consider the *Dekel-Fudenberg procedure* (Dekel and Fudenberg, 1990), which is the procedure where one round of maximal elimination of weakly dominated strategies is followed by iterated maximal elimination of strictly dominated strategies. Following Brandenburger (1992), strategies surviving the Dekel-Fudenberg procedure are referred to as *permissible*. The formal definition is as follows.

Definition 6 (Permissibility) Consider the sequence defined by, for all players i, $S_i^0 = S_i$ and, for every $n \geq 1$, $S_i^n = b_i\big(S_{-i}^{n-1}\big) \cap a_i(S_{-i})$. A strategy s_i for player i is *permissible* if $s_i \in \bigcap_{n=1}^{\infty} S_i^n$.

Since $a_i(S_{-i}) \subseteq b_i(S_{-i})$ this corresponds to the Dekel-Fudenberg procedure: Elimination of weakly dominated strategies in the first round, followed by elimination of strictly dominated strategies in later rounds.

Consider the following algorithm, which iteratedly decreases the set of likelihood orderings for all players:

Ini For all players i, let $\mathcal{L}_i^0 = \mathcal{L}_i^*$.

Per For every $n \geq 1$ and all players i, let $\mathcal{L}_i^n = \mathcal{L}_i^b(R_{-i}(\mathcal{L}_{-i}^{n-1}))$.

From the properties of $\mathcal{L}_i^b(\cdot)$ and $R_i(\cdot)$, it follows that **Ini** and **Per** determine, for each player, a non-increasing sequence of sets of likelihood orderings (where non-increasing are defined w.r.t. set inclusion). As a consequence, the sequence $C_i(R_i(\mathcal{L}_i^n))$ of choice sets is non-increasing. Since the set of likelihood orderings is finite, the algorithm converges after a finite number of rounds.

For all players i, let $\mathcal{L}_i^\infty := \bigcap_{n=1}^\infty \mathcal{L}_i^n$ be the limiting set of likelihood orderings produced by the algorithm defined by **Ini** and **Per**.

Proposition 1 *Let G be a finite strategic game. Then, for all players i, a strategy s_i is permissible if and only if $s_i \in C_i(R_i(\mathcal{L}_i^\infty))$.*

Proof. See the appendix. ■

As we note in the proof, the same result is obtained if the algorithm is initiated with $\mathcal{L}_i^0 = \tilde{\mathcal{L}}_i^*$, including only likelihood orderings that are either trivial or partition S_{-i} into a non-empty proper subset and its complement. The reason is that the belief operator is concerned only with the top level element of the likelihood ordering.

3.2 An algorithm for iterated admissibility

Iterated admissibility is the procedure of iterated maximal elimination of weakly dominated strategies, which can formally be defined as follows.

Definition 7 (Iterated admissibility) Consider the sequence defined by, for all players i, $S_i^0 = S_i$ and, for every $n \geq 1$, $S_i^n = a_i\big(S_{-i}^{n-1}\big) \cap S_i^{n-1}$. A strategy s_i for player i *survives iterated admissibility* if $s_i \in \bigcap_{n=1}^\infty S_i^n$.

Consider the following algorithm, which iteratedly decreases the set of likelihood orderings for all players:

Ini For all players i, let $\mathcal{L}_i^0 = \mathcal{L}_i^*$.

IA For every $n \geq 1$ and all players i, let

$$\mathcal{L}_i^n = \mathcal{L}_i^a(R_{-i}(\mathcal{L}_{-i}^{n-1})) \cap \mathcal{L}_i^{n-1}\,.$$

It follows directly that **Ini** and **IA** determine, for each player, a non-increasing sequence of sets of likelihood orderings. As a consequence, the sequence $C_i(R_i(\mathcal{L}_i^n))$ of choice sets is non-increasing. Since the set of likelihood orderings is finite, the algorithm converges after a finite number of rounds.

For all players i, let $\mathcal{L}_i^\infty := \bigcap_{n=1}^\infty \mathcal{L}_i^n$ be the limiting set of likelihood orderings produced by the algorithm defined by **Ini** and **IA**.

Proposition 2 *Let G be a finite strategic game. Then, for all players i, a strategy s_i survives iterated admissibility if and only if $s_i \in C_i(R_i(\mathcal{L}_i^\infty))$.*

Proof. See the appendix. ■

Proposition 2 echoes Brandenburger et al.'s (2008, Theorem 9.1) epistemic characterization of iterated admissibility (see also the observation that Stahl (1995) makes in his theorem) by pointing out that iterated admissibility corresponds to likelihood orderings where strategies eliminated in a later round are deemed infinitely more likely than strategies eliminated in an earlier round, and surviving strategies are deemed infinitely more likely than strategies eliminated in any round. Thus, when we evaluate the procedure of iterated admissibility by considering how our algorithm eliminates likelihood orderings, our evaluation is consistent with Brandenburger et al.'s (2008) epistemic characterization.

3.3 An algorithm for proper rationalizability

We then consider *proper rationalizability*, a concept defined by Schuhmacher (1999) and characterized by Asheim (2001). We refer to these references for details.

Consider the following algorithm, which iteratedly decreases the set of likelihood orderings for all players:

Ini For all players i, let $\mathcal{L}_i^0 = \mathcal{L}_i^*$.

PR For every $n \geq 1$ and all players i, let $\mathcal{L}_i^n = \mathcal{L}_i^r(R_{-i}(\mathcal{L}_{-i}^{n-1}))$.

From the properties of $\mathcal{L}_i^r(\cdot)$ and $R_i(\cdot)$, it follows that **Ini** and **PR** determine, for each player, a non-increasing sequence of sets of likelihood orderings. As a consequence, the sequence $C_i(R_i(\mathcal{L}_i^n))$ of choice sets is non-increasing. Since the set of likelihood orderings is finite, the algorithm converges after a finite number of rounds.

For all players i, let $\mathcal{L}_i^\infty := \bigcap_{n=1}^\infty \mathcal{L}_i^n$ be the limiting set of likelihood orderings produced by the algorithm defined by **Ini** and **PR**.

Proposition 3 *Let G be a finite strategic game. Then, for all players i, a strategy s_i is properly rationalizable if and only if $s_i \in C_i(R_i(\mathcal{L}_i^\infty))$.*

Proof. Perea (2011). ■

3.4 An algorithm for full permissibility

We finally consider a procedure for the concept of *fully permissible sets*, as defined by Asheim and Dufwenberg (2003a) for 2-player games. Full permissibility selects *sets* of strategies, rather than individual strategies, for both players. To formally define this concept, let Σ_i denote the collection of all non-empty subsets of strategies in S_i, for both players i. Furthermore, introduce the following operator:

$$\alpha_i(\Sigma'_j) := \{A_i \in \Sigma_i \mid \exists(\emptyset \neq)\, \Sigma''_j \subseteq \Sigma'_j \text{ s.t. } A_i = a_i\big(\cup_{A_j \in \Sigma''_j} A_j\big) \cap a_i(S_j)\}\,,$$

where $j \neq i$ and Σ'_j is a non-empty subset of Σ_j. Note that $\emptyset \neq \alpha_i(\Sigma'_j) \subseteq \Sigma_i$ for any non-empty subset Σ'_j of Σ_j.

Definition 8 (Full permissibility) Consider the sequence defined by, for both players i, $\Sigma_i^0 = \Sigma_i$ and, for every $n \geq 1$, $\Sigma_i^n = \alpha_i\big(\Sigma_j^{n-1}\big)$. A strategy set A_i for player i is *fully permissible* if $A_i \in \bigcap_{n=1}^{\infty} \Sigma_i^n$.

Consider the following algorithm, which iteratedly decreases the set of likelihood orderings for all players:

Ini For all players i, let $\mathcal{L}_i^0 = \mathcal{L}_i^*$.

FP For every $n \geq 1$ and all players i, let

$$\mathcal{L}_i^n = \{L_i \in \mathcal{L}_i^* \mid \exists(\emptyset \neq)\, \mathcal{L}_{-i} \subseteq \mathcal{L}_{-i}^{n-1} \text{ such that } L_i \text{ assumes } A_{-i} \text{ if and only if } A_{-i} \in \{\cup_{L_{-i} \in \mathcal{L}_{-i}} C_{-i}(R_{-i}(L_{-i})), S_{-i}\}\}\,.$$

It follows that **Ini** and **FP** determine, for each player, a non-increasing sequence of sets of likelihood orderings. As a consequence, the sequence $(C_i(R_i(L_i)))_{L_i \in \mathcal{L}_i^n}$ of collections of choice sets is non-increasing. Since the set of likelihood orderings is finite, the algorithm converges after a finite number of rounds.

For all players i, let $\mathcal{L}_i^\infty := \bigcap_{n=1}^{\infty} \mathcal{L}_i^n$ be the limiting set of likelihood orderings produced by the algorithm defined by **Ini** and **FP**.

Proposition 4 *Let G be a finite 2-player strategic game. Then, for both players i, A_i is a fully permissible set if and only if there exists $L_i \in \mathcal{L}_i^\infty$ such that $A_i = C_i(R_i(L_i))$.*

Proof. See the appendix. ■

We can use the algorithm defined by **Ini** and **FP** to define the concept of fully permissible sets for games with more than two players:

Definition 9 Let G be a finite strategic game. Then, for all players i, A_i is a fully permissible set if there exists $L_i \in \mathcal{L}_i^\infty$ such that $A_i = C_i(R_i(L_i))$.

As for permissibility, we can initiate the algorithm for full permissibility with $\mathcal{L}_i^0 = \tilde{\mathcal{L}}_i^*$, including only likelihood orderings that are either trivial or partition S_{-i} into a non-empty proper subset and its complement. Indeed, $\mathcal{L}_i^n \subseteq \tilde{\mathcal{L}}_i^*$ for every $n \geq 1$ and all players i also when the algortithm is initiated with $\mathcal{L}_i^0 = \mathcal{L}_i^*$.

4 Applying the algorithms

In this section we put the algorithms to work. In the first subsection four examples illustrate how the algorithms lead to sequences of sets of likelihood orderings. This sheds light on differences between iterated admissibility, proper rationalizability and full permissibility. Iterated admissibility results in a strict refinement of permissibility in all four examples, proper rationalizability strictly refines permissibility in examples 2 and 3, and full permissibility strictly refines permissibility in examples 2 and 4. However, even when two different concepts (like iterated admissibility and proper rationalizability in examples 2 and 3) give rise to the same prescription, there are interesting differences in the working of the algorithms in terms of the likelihood orderings that are eliminated along the way. In particular, example 3 illustrates how iterated admissibility and proper rationalizability promote backward induction through two different sequences of elimination, while example 4 does the same for how iterated admissibility and full permissiblity promote forward induction.

In the second subsection we build on insights conveyed by the examples and provide through Proposition 5 a sufficient condition ensuring that any properly rationalizable strategy survives iterated admissibility. In particular, since proper equilibrium always exists and any strategy being used with positive probability in a proper equilibrium is properly rationalizable, we reach the following conclusion: If a game, for which iterated admissibility leads to a unique strategy for each player, satisfies the sufficient condition of Proposition 5, then the surviving strategies are the unique properly rationalizable strategies and the corresponding strategy profile is the unique proper equilibrium.

In the third subsection we consider a contribution on commitment bargaining (Ellingsen and Miettinen, 2008) to show the usefulness and appeal of the concept of proper rationalizability in an economically relevant situation. In particular, we use the algorithm of Section 3.3 to show how proper rationalizability yields the outcomes Ellingsen and Miettinen point to in their propositions, while other concepts do not.

	L	R
U	1, 1	1, 1
M	0, 1	2, 0
D	1, 0	0, 1

Figure 1: Iterated admissibility rules out properly rationalizable strategies (G_1).

4.1 Examples

The examples are games G_1–G_4, which are illustrated by Figures 1–4. The corresponding Tables 1–4 provide the order in which likelihood orderings are eliminated by the algorithms for permissibility, iterated admissibility, proper rationalizability and full permissibility in each of these examples.[1]

[Table 1 about here.]

In game G_1 (discussed by Asheim and Dufwenberg, 2003a) the algorithm for permissibility rules out likelihood orderings for player 2 where D is at the top level, while the algorithm for proper rationalizability in addition requires that player 2 respects the preferences of player 1 by deeming D infinitely less likely than U (as U weakly dominates D and is thus preferred by player 1). Since this does not imply anything about the relative likelihood of M and D, which is what the preferences of player 2 depend on, there is no elimination of likelihood orderings for player 1. Thus, for permissibility and proper rationalizability, the algorithm converges after one round. The algorithm for full permissibility also rules out that the top level element of a surviving likelihood ordering is a singleton set containing only R or M. However, all three concepts eliminate only strategy D in this example.

In contrast, the algorithm for iterated admissibility works by eliminating all likelihood orderings for player 2 but those that assume $\{U, M\}$, thus deeming D infinitely less likely than *both* U and M in the first round. This in turn means that player 2 prefers L to R, determining $(\{L\}, \{R\})$ as the sole surviving likelihood ordering for player 1 in round 2, and that player 1 prefers U to M, determining $(\{U\}, \{M\}, \{D\})$ as the sole surviving likelihood ordering for player 2 in round 3. Thus, iterated admissibility eliminates both strategies D and M for player 1 and strategy R for player 2.

[1] For permissibility and full permissibility we restrict ourselves to likelihood orderings that are either trivial or partition the opponent's strategy set into a proper subset and its complement since—as noted in the main text—this is immaterial for the outcome.

	L	R
U	1, 1	1, 0
M	0, 1	2, 1
D	1, 0	0, 1

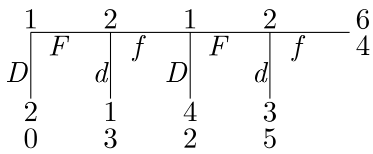
Figure 2: IA and proper rationalizability make same prescription (G_2).

The key difference in game G_1 between the algorithms for iterated admissibility and the other concepts is that the algorithm for iterated admissibility insists that both U and M be infinitely more likely than D, even though only U weakly dominates D. It follows from the structure of game G_1 that player 2 prefers L to R if player 2 believes that M is infinitely more likely than D. The algorithms for the other concepts do not reach this conclusion, and thus player 2 need not prefer L to R. Under iterated admissibility the sole surviving likelihood ordering for player 1 entails the belief that L is infinitely more likely than R, implying that player 1 prefers D to M. Nevertheless, the sole surviving likelihood ordering for player 2 entails the belief that D is infinitely *less* likely than M.

[Table 2 about here.]

Compare game G_1 to game G_2, for which both iterated admissibility and proper rationalizability prescribe only U for player 1 and only L for player 2. Also in this game, the algorithm for proper rationalizability rules out all likelihood orderings for player 2 but those where D is infinitely less likely than U (as U weakly dominates D and is thus preferred by player 1), while the algorithm for iterated admissibility goes further by eliminating all likelihood orderings but those where D is infinitely less likely than *both* U and M in the first round. However, in this example the preferences of player 2 depends on the relative likelihood of U and D and thus U being infinitely more likely than D is sufficient for player 2 preferring L to R. For both algorithms this determines $(\{L\}, \{R\})$ as the sole surviving likelihood ordering for player 1 in round 2, and implies that player 1 prefers both U and D to M.

In the algorithm for iterated admissibility this entails that player 2 assumes $\{U\}$, implying that U is infinitely more likely than both M and D. Since all likelihood orderings but those where both U and M are infinitely more likely than D have already been eliminated, $(\{U\}, \{M\}, \{D\})$ ends up as the sole surviving likelihood ordering for player 2 in round 3. However, as player 1 prefers D to M and the algorithm for proper rationalizability requires player 2 to respect the preferences



	d	fd	ff
D	2, 0	2, 0	2, 0
FD	1, 3	4, 2	4, 2
FF	1, 3	3, 5	6, 4

Figure 3: Backward induction in a four-legged centipede game (G_3).

of player 1, this algorithm yields $(\{U\}, \{D\}, \{M\})$ as the sole surviving likelihood ordering for player 2 in round 3.

A key observation for game G_2 is that U weakly dominates D, and that L weakly dominates R on both $\{U\}$ (which is the strategy used to eliminate D in the first round of iterated admissibility) and $\{U, M\}$ (which is the set of strategies for player 1 surviving the first round of iterated admissibility). The same kind of observation can be made for the centipede game, which we turn to next.

[Table 3 about here.]

Also in the four-legged centipede game illustrated in Figure 3 both iterated admissibility and proper rationalizability make the same prescription, namely the backward induction outcome (D, d). However, as for game G_2, the algorithms in terms of likelihood orderings do not coincide. In the first round, the algorithm for proper rationalizability requires that player 1 respects the preferences of player 2 by deeming *ff* infinitely less likely than *fd* (as *fd* weakly dominates *ff* and is thus preferred by player 2). The algorithm for iterated admissibility goes further by eliminating all likelihood orderings for player 1 but those that assume $\{d, fd\}$, thus deeming *ff* infinitely less likely than both *d* and *fd*. Even though the set of likelihood orderings for player 1 that assume $\{d, fd\}$ is a strict subset of those that deem *fd* infinitely more likely than *ff*, it turns out that deeming *fd* infinitely more likely than *ff* is sufficient for player 1 to prefer *FD* to *FF*. Likewise, in the second round, even though the set of likelihood orderings for player 2 that assume $\{D, FD\}$ is a strict subset of those that deem *FD* infinitely more likely than *FF*, it turns out that deeming *FD* infinitely more likely than *FF* is sufficient for player 2 to prefer *d* to *fd*.

Note that in the second round, *FD* weakly dominates *FF* on both $\{fd\}$ (which is the strategy used to eliminate *ff* in the first round of iterated admissibility) and $\{d, fd\}$ (which is the set of strategies for player 2 surviving the first round of iterated admissibility). Likewise, in the third round, *d* weakly dominates *fd* and *ff* on both $\{FD\}$ (which is the strategy used to eliminate *FF* in the second round of iterated

	L	R
U	3, 1	0, 0
M	0, 0	1, 3
D	2, 2	2, 2

Figure 4: Forward induction in the battle of the sexes with outside option (G_4).

admissibility) and $\{D, FD\}$ (which is the set of strategies for player 1 surviving the second round of iterated admissibility). Similar conclusions hold for any centipede game independent of size and illustrates how both iterated admissibility and proper rationalizability correspond to the procedure of backward induction in such games.[2]

The algorithm for permissibility works similarly in games G_2 and G_3 as in game G_1. In particular, in game G_2 it does not require player 2 to deem U infinitely more likely than D (even though U weakly dominates D and is thus preferred by player 1). Thus, this algorithm does *not* allow us to conclude that player 2 prefers L to R, and therefore does not determine $(\{L\}, \{R\})$ as the sole surviving likelihood ordering for player 1. In contrast, the algorithm for full permissibility does lead to $(\{L\}, \{R\})$ as the sole surviving likelihood ordering for player 1 in game G_2. Hence, it prescribes the outcome (U, L), thus coinciding with the algorithms for iterated admissibility and proper rationalizability in this respect. However, the algorithm for full permissibility does not directly conclude that U infinitely more likely than D. Rather, as shown in Table 2, this conclusion is reached through a process that is more involved than for the algorithms for iterated admissibility and proper rationalizability.

[Table 4 about here.]

To illustrate the algorithm for full permissibility in another game where this concept has as much cutting power as iterated admissibility, but where in contrast to game G_2 it is more restrictive than proper rationalizability, we include the battle of the sexes with outside option as game G_4. In this game, both iterated admissibility and full permissibility prescribe the forward induction outcome (U, L) (see Asheim and Dufwenberg, 2003b, p. 319). However, the process at which player 2 is lead

[2] For finite perfect information games without relevant payoff ties, proper rationalizability leads to the unique profile of backward induction *strategies* (Schuhmacher, 1999; Asheim, 2001), and iterated admissibility leads to the backward induction *outcome* (see Battigalli, 1997, pp. 52–53, for relevant references). While the algorithms of Sections 3.2 and 3.3 correspond to the backward induction *procedure* in the subclass of centipede games, this does not hold for the whole class of finite perfect information games without relevant payoff ties.

to conclude that U is infinitely more likely than M (leading to a preference for L over R) is different for the two algorithms. For iterated admissibility this follows directly from assuming $\{U, D\}$, thus deeming M infinitely less likely than *both* U and D, even though only D weakly dominates M. For full permissibility the process is more involved, as illustrated in Table 4.

The examples of Figures 1–4 show that there are no logical relationships between proper rationalizability and full permissibility, while suggesting that iterated admissibility refines proper rationalizability and full permissibility, which in turn refine permissibility. Which of these relations are general properties? This is a question which we consider in the next subsection.

4.2 The relations between the algorithms

The properties of $\mathcal{L}_i^b(\cdot)$ and $R_i(\cdot)$ combined with Lemma 1 imply *both*

$$\mathcal{L}_i^a(R_{-i}(\mathcal{L}'_{-i})) \subseteq \mathcal{L}_i^b(R_{-i}(\mathcal{L}'_{-i}))\,,$$
$$\mathcal{L}_i^r(R_{-i}(\mathcal{L}'_{-i})) \subseteq \mathcal{L}_i^b(R_{-i}(\mathcal{L}'_{-i}))\,,$$
$$\left.\begin{array}{l}\{L_i \in \mathcal{L}_i^* \mid \exists(\emptyset \neq)\,\mathcal{L}_{-i} \subseteq \mathcal{L}'_{-i} \text{ such that } L_i \text{ assumes } A_{-i} \\ \text{ if and only if } A_{-i} \in \{\cup_{L_{-i}\in\mathcal{L}_{-i}} C_{-i}(R_{-i}(L_{-i})), S_{-i}\}\}\end{array}\right. \subseteq \mathcal{L}_i^b(R_{-i}(\mathcal{L}'_{-i}))$$

for any vector $\mathcal{L}'_{-i}$ of non-empty sets of likelihood orderings for i's opponents, *and*

$$\mathcal{L}_i^b(R_{-i}(\mathcal{L}'_{-i})) \subseteq \mathcal{L}_i^b(R_{-i}(\mathcal{L}''_{-i}))$$

if $\mathcal{L}'_{-i} \subseteq \mathcal{L}''_{-i}$, signifying that $\mathcal{L}'_j \subseteq \mathcal{L}''_j$ for all $j \neq i$. Thus, if $\mathcal{L}'_j \subseteq \mathcal{L}''_j$ for all $j \neq i$, then the set of likelihood orderings determined for i by **Per** on the basis of $\mathcal{L}''_{-i}$ is always a superset of those sets determined for i by **IA**, **PR** and **FP** on the basis of $\mathcal{L}'_{-i}$. This means that Propositions 1–4 can be used to establish the (already known) result that each of the concepts iterated admissibility, proper rationalizability and full permissibility refine the concept of permissibility. The examples illustrate that these refinements might be strict.

The other conjecture suggested by examples of Figures 1–4, namely that iterated admissibility refines proper rationalizability and full permissibility, is not true. Asheim and Dufwenberg (2003a, p. 216) show that there is no logical relationship between iterated admissibility and full permissibility: in their game G_4 (illustrated in Asheim and Dufwenberg, 2003a, Figure 4) strategy b survives iterated admissibility but does not appear in any fully permissible set, while strategy f appears in a fully permissible set but does not survive iterated admissibility. Likewise, our example in the introduction, illustrated in Figure 0 (see also Perea, 2012, p. 262), shows that there is no logical relationship between iterated admissibility and proper rationalizability: in the game of Figure 0 proper rationalizability uniquely selects strategy Z, whereas iterated admissibility uniquely selects strategy Y.

As we have seen in games G_2 and G_3, there are examples where proper rationalizability has at least as much cutting power as iterated admissibility. In the following proposition we generalize insights gained through these examples to provide a sufficient condition under which iterated admissibilty does not rule out properly rationalizable strategies. Hence, under these conditions, the restrictions on lexicographic beliefs that the procedure of iterated admissibility imposes along the way are convincing also given the prescriptions that this procedure ends up providing.

Proposition 5 *Consider a finite 2-player strategic game G where the procedure of iterated admissibility leads to the sequence* $\langle S_1^n, S_2^n \rangle_{n=0}^{\infty}$ *of surviving strategy sets. Suppose that there exists a sequence* $\langle A_1^n, A_2^n \rangle_{n=0}^{\infty}$ *of strategy sets satisfying, for both players* i, $A_i^0 = S_i$ *and for each* $n \in \mathbb{N}$,

- $A_i^n \subseteq S_i^n$,
- *if* $S_i^n \neq S_i^{n-1}$, *then, for every* $s_i \in S_i \backslash S_i^n$, s_i *is weakly dominated by every* $a_i \in A_i^n$ *on either (*A_j^{n-1} *and* S_j^{n-1}*) or* S_j,
- *if* $S_i^n = S_i^{n-1}$, *then* $A_i^n = A_i^{n-1}$.

Then, for both players i, *if* s_i *is properly rationalizable, then* $s_i \in \bigcap_{n=1}^{\infty} S_i^n$.

Proof. See the appendix. ■

Both G_2 of Figure 2 and G_3 of Figure 3 can be used to illustrate Proposition 5. In G_2, the procedure of iterated admissibility yields the following sequence of strategy sets: $S_1^1 = S_1^2 = \{U, M\}$ and $S_1^n = \{U\}$ for $n \geq 3$, and $S_2^1 = \{L, R\}$ and $S_2^n = \{L\}$ for $n \geq 2$. Choose $A_1^n = \{U\}$ for $n \geq 1$, and $A_2^1 = \{L, R\}$ and $A_2^n = \{L\}$ for $n \geq 2$. It is straightforward to check that the conditions of Proposition 5 are satisfied; in particular, L weakly dominates R on both $A_1^1 = \{U\}$ and $S_1^1 = \{U, M\}$, and U weakly dominates M on $A_2^2 = S_2^2 = \{L\}$, and weakly dominates D on S_2.

In G_3, the procedure of iterated admissibility yields the following sequence of strategy sets: $S_1^1 = \{D, FD, FF\}$, $S_1^2 = S_1^3 = \{D, FD\}$ and $S_1^n = \{D\}$ for $n \geq 4$, and $S_2^1 = S_2^2 = \{d, fd\}$ and $S_2^n = \{d\}$ for $n \geq 3$. Choose $A_1^1 = \{D, FD, FF\}$, $A_1^2 = A_1^3 = \{FD\}$ and $A_1^n = \{D\}$ for $n \geq 4$, and $A_2^1 = A_2^2 = \{fd\}$ and $A_2^n = \{d\}$ for $n \geq 3$. Again, it is straightforward to check that the conditions of Proposition 5 are satisfied; in particular, *FD* weakly dominates *FF* on both $A_2^1 = \{fd\}$ and $S_2^1 = \{d, fd\}$, d weakly dominates both *fd* and *ff* on both $A_1^2 = \{FD\}$ and $S_1^2 = \{D, FD\}$, and D weakly dominates both *FD* and *FF* on $A_2^3 = S_2^3 = \{d\}$.

4.3 Commitment bargaining

The algorithms of Section 3 can be applied for the purpose of analyzing economically significant models, independently of whether the sufficient condition of Proposition 5 is satisfied. In particular, they can be used for comparing iterated admissibility to properly rationalizable strategies in specific strategic situations. In this subsection we consider a model of bilateral commitment bargaining due to Ellingsen and Miettinen (2008, Section I).

Ellingsen and Miettinen (2008) reexamine the problem of observable commitments in bargaining, first studied by Schelling (1956) and later formalized by Crawford (1982). Ellingsen and Miettinen (2008) extends Crawford's (1982) analysis by considering variants of iterated admissibility and refinements of Nash equilibrium. Here we show how some of the outcomes that Ellingsen and Miettinen (2008) suggest, in particular through their Lemma 2 and Proposition 2, can be obtained by using proper rationalizability instead of iterated admissibility. There is actually a mistake in their Lemma 2, but we will come back to this later.

In order to turn their strategic situation where two players bargain over real numbered fractions of a surplus of size 1 into a *finite* one-stage game with simultaneous moves, we introduce a smallest money unit g. We measure all variables in terms of numbers of the smallest money unit, and assume that k units of the smallest money unit equals the total surplus (i.e., $k \cdot g = 1$). Hence, players 1 and 2 bargain over a surplus of size k.

Each player i chooses, simultaneously with the other, either to commit to some demand $s_i \in \{0, 1, \ldots, k\}$ or to wait and remain uncommitted. Let w denote the waiting strategy. Hence the strategy set of each player i is $S_i = \{0, 1, \ldots, k\} \cup \{w\}$. If both players choose w, then each player i receives $\beta_i > 1$, where $\beta_1 + \beta_2 = k$.

In the case with certain commitments and no commitment costs (Ellingsen and Miettinen, 2008, Section I) the payoffs are as follows: If only one player i makes a commitment s_i, then i receives s_i and the other player receives $k - s_i$. If both players make commitments, then each player i receives $x_i(s_i, s_j) \in \{s_i, s_i + 1, \ldots, k - s_j\}$, with $x_1(s_1, s_2) + x_2(s_2, s_1) \leq k$, if $s_1 + s_2 \leq k$ and nothing otherwise.

The payoff function $u_i(s_i, s_j)$ of each player i can be summarized as follows:

$$u_i(s_i, s_j) = \begin{cases} x_i(s_i, s_j) & \text{if } s_i + s_j \leq k\,, \\ 0 & \text{if } s_i + s_j > k\,, \\ s_i & \text{if } s_i \neq w \text{ and } s_j = w\,, \\ k - s_j & \text{if } s_i = w \text{ and } s_j \neq w\,, \\ \beta_i & \text{if } s_i = w = s_j\,. \end{cases}$$

Ellingsen and Miettinen (2008) show through the proof of their Lemma 2 that, for each player i, iterated admissibility leads to the elimination of $0, 1, \ldots, \beta_i$ in the first round, and $\beta_i + 1, \beta_i + 2, \ldots, k - 1$ in the second round, leaving k and w as the surviving strategies. Actually, with only k and w as the surviving strategies, w is eliminated in the third round, since choosing k yields player i a payoff of 0 if the opponent also chooses k and k if the opponent chooses w, while choosing w yields player i a payoff of 0 if the opponent chooses k and β_i ($< k$) if the opponent also chooses w. Hence, the correct statement of Ellingsen and Miettinen's (2008) Lemma 2 is that only k is iteratively weakly undominated.

Ellingsen and Miettinen (2008) use Lemma 2 in their subsequent Proposition 2 to focus on Nash equilibria involving only the strategies k and w (including asymmetric equilibria where one commits to the entire surplus and the other waits), as opposed to the plethora of unrefined Nash equilibria that this game gives rise to (cf. Crawford, 1982). Their Proposition 2 states that only the two asymmetric equilibria along with the symmetric equilibrium where both claim the entire surplus are consistent with two rounds of elimination of weakly dominated strategies. This statement is correct, but it begs the question: *why stop with two rounds of weak elimination?* As the following proposition shows, proper rationalizability provides a reason for considering only the strategies k and w.

Proposition 6 *Consider the finite version of Ellingsen and Miettinen's (2008, Section I) bilateral commitment bargaining game with zero commitment cost. The properly rationalizable strategies for each player are to commit to the whole surplus, i.e., to choose the strategy k, or to wait, i.e., to choose the strategy w.*

Proof. See the appendix. ■

The proof of Proposition 6 consists of two parts. The one part uses the algorithm of Section 3.3 to show that no strategy but k and w can be properly rationalizable. Since w weakly dominates $0, 1, \ldots, \beta_j$ for player j, respect of j's preferences forces player i to deem w infinitely more likely than each of $0, 1, \ldots, \beta_j$. This in turn implies that k weakly dominates $\beta_i + 1, \beta_i + 2, \ldots, k - 1$ for player i. Hence, only k and w can be best responses when players are cautious.

The other part uses the result of Asheim (2001, Proposition 2) — that any strategy being used with positive probability in a proper equilibrium is properly rationalizable — to show that k and w are properly rationalizable. In particular, the asymmetric equilibria where one player commits to the entire surplus and the other waits are proper. In addition, there is a proper equilibrium where both players choose k with probability 1.[3] In any proper equilibrium, at most one player attains positive payoff and no strategy but k and w is assigned positive probability. Thus, the concept of proper equilibrium focuses precisely on the equilibria highlighted in Ellingsen and Miettinen's (2008) Proposition 2.[4]

Ellingsen and Miettinen (2008, Section II) also consider a variant of Crawford's (1982) bilateral commitment bargaining game where commitments are uncertain. In their Proposition 4 they show that only k survives iterated admissibility if commitments are uncertain. Actually, the iterations involve one round of weak elimination, followed by two rounds of strict elimination. Hence, only k is permissible, and it follows from the algorithms of Sections 3.1 and 3.3 that only k is properly rationalizable (and thus, (k, k) is the only proper equilibrium). In their Propositions 1 and 3 they consider costly commitments. In this case, it can be shown that every strategy surviving iterated elimination of *strictly* dominated strategies is properly rationalizable. Hence, in all variants considered by Ellingsen and Miettinen (2008), proper rationalizability and proper equilibrium yield the outcomes they point to in their propositions, while other concepts do not.

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[3] This equilibrium involves likelihood orderings where $k - 1$ and w are at the second level. See the Claim of the Appendix.

[4] Even though at most one player attains positive payoff in any perfect equilibrium, there exists, for each player i and any strategy $\ell \in \{\beta_i + 1, \beta_i + 2, \ldots, k - 1\}$, a perfect equilibrium in which player i assigns positive probability to ℓ. This requires that this player also assigns sufficient positive probability to w, so that k is the unique best response for the other player. See the Claim of the Appendix. Hence, the concept of perfect equilibrium can *not* be used to rule out all equilibria but the ones highlighted in Ellingsen and Miettinen's (2008) Proposition 2.

5 Concluding remarks

In our opinion, proper rationalizability is an attractive concept which is based on appealing epistemic conditions. However, its applicability has been hampered by the lack of an algorithm leading directly to the properly rationalizable strategies. With Perea's (2011) algorithm, this roadblock has been removed. Here we have compared proper rationalizability to permissibility (i.e., the Dekel-Fudenberg procedure), iterated admissibility and full permissibility by presenting comparable algorithms for the three latter concepts. Through a bilateral commitment bargaining game due to Crawford (1982) and Ellingsen and Miettinen (2008) we have illustrated the usefulness of proper rationalizability in economic applications.

The four algorithms eliminate likelihood orderings. Likelihood orderings model cautious behavior, as they require that each player takes into all opponents strategies, also those that seem unlikely to be chosen. There might also be other interesting elimination procedures that can be captures in terms of likelihood orderings. A particularly interesting example is the *reasoning-based expected utility procedure* defined by Cubitt and Sugden (2011). This procedure determines, for each player and every iteration, a positive and a negative subset of the player's strategy set (the two subsets having a non-empty intersection) as follows:

(i) A set of allowable probability distribution is determined by assigning positive weight to every strategy in the opponent's positive set and zero weight to every strategy in the opponent's negative set.

(ii) The player's positive set consists of strategies being a best reply to every allowable probability distribution, while the player's negative set consists of strategies not being a best reply to any allowable probability distribution.

In terms of likelihood orderings this requires the top level element to include every strategy in the opponent's positive set and to exclude every strategy in the opponent's negative set. However, the resulting algorithm is different since the partitional nature of likelihood orderings induces cautious behavior: all opponent strategies, also those in the negative set are taken into account.

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A Proofs

Proof of Lemma 1. Assume that L_i respects R_{-i}. Suppose $s_j \notin C_j(R_j)$ for some $j \neq i$, implying that there exists A_j such that $(s_j, A_j) \in R_j$. Since L_i respects R_{-i}, $a_j \gg_i s_j$ for some $a_j \in A_j$. Thus, if $s_j \notin C_j(R_j)$, then there exists $a_{-i} \in S_{-i}$ such that $a_{-i} \gg_i s_{-i}$ for all $s_{-i} \in \{s_j\} \times \prod_{j' \neq i,j} S_{j'}$, implying that $L_i^1 \cap \{s_j\} \times \prod_{j' \neq i,j} S_{j'} = \emptyset$. Therefore, if $s_{-i} \in L_i^1$, then $s_{-i} \in \prod_{j \neq i} C_j(R_j) = C_{-i}(R_{-i})$, establishing the lemma. ■

In order to prove Proposition 1, we need the following lemma.

Lemma 2 *Let $s_i \in S_i$, $A_i \subseteq S_i$ and $S'_{-i} \subseteq S_{-i}$. Then, s_i is strictly dominated by some $\mu_i \in \Delta(A_i)$ on S'_{-i} if and only for every $(\emptyset \neq) S''_{-i} \subseteq S'_{-i}$ strategy s_i is weakly dominated by some $\mu'_i \in \Delta(A_i)$ on S''_{-i}.*

Proof. *Only if.* If there exists $\mu_i \in \Delta(A_i)$ such that μ_i strictly dominates s_i on S'_{-i}, then, for every $(\emptyset \neq) S''_{-i} \subseteq S'_{-i}$, $\mu_i \in \Delta(A_i)$ weakly dominates s_i on S''_{-i}.

If. Suppose there does not exist $\mu_i \in \Delta(A_i)$ such that μ_i strictly dominates s_i on S'_{-i}. Hence, by Pearce (1984, Lemma 3), there exists $\lambda_i \in \Delta(S'_{-i})$ such that $u(s_i, \lambda_i) \geq u(s'_i, \lambda_i)$ for all $s'_i \in A_i$. Then, by Pearce (1984, Lemma 4), there does not exist $\mu'_i \in \Delta(A_i)$ such that μ'_i weakly dominates s_i on $S''_{-i} := \text{supp}\lambda_i \subseteq S'_{-i}$. ■

By Lemma 2 it follows that the operator $b_i(S'_{-i})$ can be expressed as follows:

$$b_i(S'_{-i}) = \{s_i \in S_i \mid \exists (\emptyset \neq) S''_{-i} \subseteq S'_{-i} \text{ s.t. } s_i \in a_i(S''_{-i})\} \, ,$$

and the combined operator used to define permissibility (in Definition 6) becomes:

$$\begin{aligned} b_i(S'_{-i}) \cap a_i(S_{-i}) = \{s_i \in S_i \mid \exists (\emptyset \neq) S''_{-i} \subseteq S'_{-i} \\ \text{s.t. } s_i \in a_i(S''_{-i}) \cap a_i(S_{-i})\} \, . \end{aligned} \tag{A1}$$

Note the analogy to the definition of full permissibility in Definition 8.

Proof of Proposition 1. Consider, for all players i, the sequence $\langle S_i^n \rangle_{n=0}^{\infty}$ defined in Definition 6. We show, by induction on n, that $C_i(R_i(\mathcal{L}_i^n)) = S_i^{n+1}$ for all players i and every $n \geq 0$.

Part (i). For $n = 0$, we have that $\mathcal{L}_i^0 = \mathcal{L}_i^*$ and hence,

$$R_i(\mathcal{L}_i^0) = \{(s_i, A_i) \mid \exists \mu_i \in \Delta(A_i) \text{ such that } s_i \text{ is weakly dominated by } \mu_i \text{ on } S_{-i}\}.$$

Therefore, $C_i(R_i(\mathcal{L}_i^0)) = a_i(S_{-i}) = b_i(S_{-i}^0) \cap a_i(S_{-i}) = S_i^1$ for all players i, since $S_{-i}^0 = S_{-i}$ and $a_i(S_{-i}) \subseteq b_i(S_{-i})$.

Part (ii). Now, let $n \geq 1$, and assume that for all players i, $C_i(R_i(\mathcal{L}_i^{n-1})) = S_i^n$. We show that, for all players i, $C_i(R_i(\mathcal{L}_i^n)) = S_i^{n+1}$.

Fix a player i. By definition, $\mathcal{L}_i^n = \mathcal{L}_i^b(R_{-i}(\mathcal{L}_{-i}^{n-1}))$. We have that

$$\begin{aligned} \mathcal{L}_i^b(R_{-i}(\mathcal{L}_{-i}^{n-1})) &= \{L_i \in \mathcal{L}_i^* \mid L_i \text{ believes } C_{-i}(R_{-i}(\mathcal{L}_{-i}^{n-1}))\} \\ &= \{L_i \in \mathcal{L}_i^* \mid L_i \text{ believes } S_{-i}^n\} \\ &= \{L_i \in \mathcal{L}_i^* \mid L_i^1 \subseteq S_{-i}^n\}, \end{aligned}$$

by our induction assumption. But then,

$$\begin{aligned} R_i(\mathcal{L}_i^n) = \{(s_i, A_i) \mid &\text{ for every } L_i^1 \subseteq S_{-i}^n \text{ there is } \mu_i \in \Delta(A_i) \text{ such that} \\ & s_i \text{ is weakly dominated by } \mu_i \text{ on } L_i^1 \text{ or on } S_{-i}\} \end{aligned}$$

and

$$\begin{aligned} C_i(R_i(\mathcal{L}_i^n)) &= \{s_i \in S_i \mid \exists (\emptyset \neq) L_i^1 \subseteq S_{-i}^n \text{ s.t. } s_i \in a_i(L_i^1) \cap a_i(S_{-i})\} \\ &= b_i(S_{-i}^n) \cap a_i(S_{-i}) = S_i^{n+1} \end{aligned}$$

by (A1) and Definition 6, thus concluding the proof.

Note that the proof above would also apply to the case where $\mathcal{L}_i^0 = \tilde{\mathcal{L}}_i^*$, restricting to likelihood orderings that consist of one or two levels only. The reason is that the restrictions on the sets $\mathcal{L}_i^n$ of likelihood orderings only apply to the first level of the likelihood orderings, and not to further levels. ■

Proof of Proposition 2. Consider, for all players i, the sequence $\langle S_i^n \rangle_{n=0}^{\infty}$ defined in Definition 7. We show, by induction on n, that $C_i(R_i(\mathcal{L}_i^n)) = S_i^{n+1}$ for all players i and every $n \geq 0$.

Part (i). For $n = 0$, it follows by part (i) of the proof of Proposition 1, that $C_i(R_i(\mathcal{L}_i^0)) = a_i(S_{-i}) = a_i(S_{-i}^0) \cap S_i = S_i^1$ for all players i.

Part (ii). Let $n \geq 1$, and assume that, for all players i, $C_i(R_i(\mathcal{L}_i^m)) = S_i^{m+1}$ for every $m \in \{0, \ldots, n-1\}$. We show that, for all players i, $C_i(R_i(\mathcal{L}_i^n)) = S_i^{n+1}$.

Fix a player i. By definition, we have that

$$\mathcal{L}_i^n = \mathcal{L}_i^a(R_{-i}(\mathcal{L}_{-i}^0)) \cap \mathcal{L}_i^a(R_{-i}(\mathcal{L}_{-i}^1)) \cap \ldots \cap \mathcal{L}_i^a(R_{-i}(\mathcal{L}_{-i}^{n-1})).$$

By the induction assumption, we know that $C_{-i}(R_{-i}(\mathcal{L}^m_{-i})) = S_i^{m+1}$ for every $m \in \{0, \ldots, n-1\}$, and hence

$$\begin{aligned}\mathcal{L}_i^a(R_{-i}(\mathcal{L}^m_{-i})) &= \{L_i \in \mathcal{L}_i^* \mid L_i \text{ assumes } C_{-i}(R_{-i}(\mathcal{L}^m_{-i}))\} \\ &= \{L_i \in \mathcal{L}_i^* \mid L_i \text{ assumes } S_{-i}^{m+1}\} \\ &= \{L_i \in \mathcal{L}_i^* \mid \exists k \in \{1, \ldots, K\} \text{ such that } L_i^1 \cup \cdots \cup L_i^k = S_{-i}^{m+1}\}\end{aligned}$$

for every $m \in \{0, \ldots, n-1\}$. This implies that

$$\mathcal{L}_i^n = \{L_i \in \mathcal{L}_i^* \mid \forall m \in \{1, \ldots, n\}, \exists k \in \{1, \ldots, K\} \text{ such that } L_i^1 \cup \cdots \cup L_i^k = S_{-i}^m\}\,.$$

Therefore, $R_i(\mathcal{L}_i^n)$ contains exactly those preference restrictions (s_i, A_i) such that s_i is weakly dominated by some $\mu_i \in \Delta(A_i)$ on some S_{-i}^m with $m \le n$:

$$\begin{aligned}R_i(\mathcal{L}_i^n) = \{(s_i, A_i) \mid & \text{ there are } m \in \{0, \ldots, n\} \text{ and } \mu_i \in \Delta(A_i) \\ & \text{such that } s_i \text{ is weakly dominated by } \mu_i \text{ on } S_{-i}^m\}\end{aligned}$$

and

$$C_i(R_i(\mathcal{L}_i^n)) = a_i(S_{-i}^0) \cap a_i(S_{-i}^1) \cap \cdots \cap a_i(S_{-i}^n) = S_i^{n+1},$$

which completes the proof. ■

Proof of Proposition 4. Consider, for both players i, the sequence $\langle \Sigma_i^n \rangle_{n=0}^\infty$ defined in Definition 8. Consider also, for both players i, the sequence $\langle \tilde{\mathcal{L}}_i^n \rangle_{n=0}^\infty$ defined by

Ini* For both players i, let $\tilde{\mathcal{L}}_i^0 = \tilde{\mathcal{L}}_i^*$.

and **FP**. Note that $\mathcal{L}_i^1 \subseteq \tilde{\mathcal{L}}_i^0 \subseteq \mathcal{L}_i^0$, so by induction, for every $n \ge 1$, $\mathcal{L}_i^{n+1} \subseteq \tilde{\mathcal{L}}_i^n \subseteq \mathcal{L}_i^n$. Since also the algorithm defined by **Ini*** and **FP** converges after a finite number of rounds, as the set of likelihood orderings is finite, we have that $\tilde{\mathcal{L}}_i^\infty := \bigcap_{n=1}^\infty \tilde{\mathcal{L}}_i^n$ equals $\mathcal{L}_i^\infty$. Thus, it is sufficient to show that there exists $L_i \in \tilde{\mathcal{L}}_i^n$ such that $A_i = C_i(R_i(L_i))$ if and only if $A_i \in \Sigma_i^{n+1}$, for both players i and every $n \ge 0$. We show this by induction on n.

Part (i). For $n = 0$, we have that $\tilde{\mathcal{L}}_i^0 = \tilde{\mathcal{L}}_i^*$ and thus, $L_i \in \tilde{\mathcal{L}}_i^0$ if and only if $L_i = (L_i^1) = S_j$ or $L_i = (L_i^1, L_i^2) = (A_j, S_j \backslash A_j)$ for some non-empty proper subset A_j of S_j. Hence, there is $L_i \in \tilde{\mathcal{L}}_i^0$ such that $(s_i, A_i) \in R_i(L_i)$ if and only if there exist $(\emptyset \neq)\, A_j \subseteq S_j$ and $\mu_i \in \Delta(A_i)$ such that s_i is weakly dominated by μ_i on A_j or S_j. Therefore, there is $L_i \in \tilde{\mathcal{L}}_i^0$ such that $A_i \in C_i(R_i(L_i))$ if and only if $A_i = a_i(A_j) \cap a_i(S_j)$ for some $(\emptyset \neq)\, A_j \subseteq S_j$. It now follows from the definition of the operator $\alpha_i(\Sigma_j')$ that there is $L_i \in \tilde{\mathcal{L}}_i^0$ such that $A_i \in C_i(R_i(L_i))$ if and only if $A_i \in \alpha_i(\Sigma_j) = \alpha_i(\Sigma_j^0) = \Sigma_i^1$, since $\Sigma_j^0 = \Sigma_j$.

Part (ii). Now, let $n \ge 1$, and assume that for both players i, there exists $L_i \in \tilde{\mathcal{L}}_i^{n-1}$ such that $A_i = C_i(R_i(L_i))$ if and only if $A_i \in \Sigma_i^n$.

Fix a player i. By **FP**, $L_i \in \tilde{\mathcal{L}}_i^n$ is equivalent to there existing $(\emptyset \neq)\, \mathcal{L}_j \subseteq \tilde{\mathcal{L}}_j^{n-1}$ such that L_i assumes A_j if and only if $A_j \in \{\cup_{L_j \in \mathcal{L}_j} C_j(R_j(L_j)), S_j\}$. By the induction assumption this is equivalent to there existing $(\emptyset \neq)\, \Sigma_j'' \subseteq \Sigma_j^n$ such that L_i assumes A_j if and only if

$A_j \in \{\cup_{A''_j \in \Sigma''_j} A''_j, S_j\}$. Hence, $L_i \in \tilde{\mathcal{L}}_i^n$ if and only if $L_i = (L_i^1) = S_j$ for some $(\emptyset \neq)\, \Sigma''_j \subseteq \Sigma_j^n$ satisfying $\cup_{A''_j \in \Sigma''_j} A''_j = S_j$ or $L_i = (L_i^1, L_i^2) = (A_j, S_j \backslash A_j)$ for some $(\emptyset \neq)\, \Sigma''_j \subseteq \Sigma_j^n$ satisfying $\cup_{A''_j \in \Sigma''_j} A''_j = A_j \subsetneqq S_j$. Therefore, there is $L_i \in \tilde{\mathcal{L}}_i^n$ such that $A_i = C_i(R_i(L_i))$ if and only if $A_i = a_i(\cup_{A''_j \in \Sigma''_j} A''_j) \cap a_i(S_j)$ for some $(\emptyset \neq)\, \Sigma''_j \subseteq \Sigma_j^n$. It now follows from the definition of the operator $\alpha_i(\Sigma'_j)$ that there is $L_i \in \tilde{\mathcal{L}}_i^n$ such that $A_i = C_i(R_i(L_i))$ if and only if $A_i \in \alpha_i(\Sigma_j^n) = \Sigma_i^{n+1}$, which completes the proof. ■

Proof of Proposition 5. Let $\langle \mathcal{L}_1^n, \mathcal{L}_2^n \rangle_{n=1}^{\infty}$ be the sequence of likelihood orderings according to the algorithm for proper rationalizability (cf. Section 3.3). It is sufficient to show, under the assumptions of the proposition, that for every $n \geq 0$ and both players i, it holds that, for every $s_i \in S_i \backslash S_i^{n+1}$, $(s_i, \{a_i\}) \in R_i(\mathcal{L}_i^n)$ for every $a_i \in A_i^{n+1}$. In this case, namely, every properly rationalizable strategy is in $\bigcap_{n=1}^{\infty} S_i^n$. We show by induction that the statement above is true.

Part (i). Let $n = 0$. If $S_i^1 = S_i$, so that there is no $s_i \in S_i \backslash S_i^1$, then the statement is trivially true. If $S_i^1 \neq S_i$, then, by the premise of the proposition, for every $s_i \in S_i \backslash S_i^1$, s_i is weakly dominated by every $a_i \in A_i^1$ on S_j. Hence, by the full support assumption, $(s_i, \{a_i\}) \in R_i(\mathcal{L}_i^*) = R_i(\mathcal{L}_i^0)$, implying that the statement is true also in this case.

Part (ii). Let $n \geq 1$, and assume that, for every $m \in \{0, \ldots, n-1\}$ and both players i, it holds that, for every $s_i \in S_i \backslash S_i^{m+1}$, $(s_i, \{a_i\}) \in R_i(\mathcal{L}_i^m)$ for every $a_i \in A_i^{m+1}$.

Fix a player i. We first make the observation that, for every $m \in \{1, \ldots, n\}$, every $L_i = (L_i^1, \ldots, L_i^K) \in \mathcal{L}_i^m$ satisfies that there exists $k \in \{1, \ldots, K\}$ such that $A_j^m \subseteq L_i^1 \cup \cdots \cup L_i^k \subseteq S_j^m$. This is true by the full support assumption if $S_j^m = S_j$ (and thus $A_j^m = S_j$, by the last bullet point of Proposition 5 and fact that $A_j^0 = S_j$). Assume now that $S_j^m \neq S_j$. By the algorithm for proper rationalizability, every $L_i \in \mathcal{L}_i^m$ respects $R_j(\mathcal{L}_j^{m-1})$, implying $a_j \gg_i s_j$ for every $s_j \in S_j \backslash S_j^m$ and every $a_j \in A_j^m$, and the observation follows also in this case.

If $S_i^{n+1} = S_i$, then the statement is trivially true also for $n \geq 1$.

If $S_i^{n+1} \neq S_i$, let $(0 \leq)\ m \leq n$ satisfy $S_i^{n+1} = S_i^{m+1} \neq S_i^m$. By a premise of the proposition, for every $s_i \in S_i \backslash S_i^{m+1}$, s_i is weakly dominated by every $a_i \in A_i^{m+1}$ on either (A_j^m and S_j^m) or S_j. If s_i is weakly dominated by a_i on A_j^m and S_j^m, then s_i is weakly dominated by a_i on each strategy set S'_j satisfying $A_j^m \subseteq S'_j \subseteq S_j^m$. By the observation that every $L_i = (L_i^1, \ldots, L_i^K) \in \mathcal{L}_i^m$ satisfies that there exists $k \in \{1, \ldots, K\}$ such that $A_j^m \subseteq L_i^1 \cup \cdots \cup L_i^k \subseteq S_j^m$ it follows that $(s_i, \{a_i\}) \in R_i(\mathcal{L}_i^m)$. If s_i is weakly dominated by a_i on S_j, then by the full support assumption, $(s_i, \{a_i\}) \in R_i(\mathcal{L}_i^*) = R_i(\mathcal{L}_i^0)$. Hence, since the sequence of sets of likelihood orderings is non-increasing, so that $\mathcal{L}_i^n \subseteq \mathcal{L}_i^m \subseteq \mathcal{L}_i^0$ and thus, $R_i(\mathcal{L}_i^n) \supseteq R_i(\mathcal{L}_i^m) \supseteq R_i(\mathcal{L}_i^0)$, for every $s_i \in S_i \backslash S_i^{n+1}$, $(s_i, \{a_i\}) \in R_i(\mathcal{L}_i^n)$ for every $a_i \in A_i^{n+1}$. ■

Proof of Proposition 6. The proof is divided into two parts. In part (i) we show that the strategies in $S_i \backslash (\{k\} \cup \{w\})$ are *not* properly rationalizable. In part (ii) we show that k and w *are* properly rationalizable.

Part (i). Let $\langle \mathcal{L}_1^n, \mathcal{L}_2^n \rangle_{n=1}^{\infty}$ be the sequence of sets of likelihood orderings for the finite version of Ellingsen and Miettinen's (2008, Section I) bilateral commitment bargaining game with zero commitment cost, according to the algorithm for proper rationalizability (cf. Section 3.3). In order to show that the strategies in $S_i \setminus (\{k\} \cup \{w\}) = \{0, 1, \ldots, k-1\}$ are not properly rationalizable, it is sufficient to show that for each player i, it holds that (a) for every $s_i \in \{0, 1, \ldots, \beta_i\}$, $(s_i, \{w\}) \in R_i(\mathcal{L}_i^0)$, and (b) for every $s_i \in \{\beta_i + 1, \beta_i + 2, \ldots, k-1\}$, $(s_i, \{k\}) \in R_i(\mathcal{L}_i^1)$, keeping in mind that the sequence of sets of likelihood orderings is non-increasing, so that $\mathcal{L}_i^n \subseteq \mathcal{L}_i^1 \subseteq \mathcal{L}_i^0$ and thus, $R_i(\mathcal{L}_i^n) \supseteq R_i(\mathcal{L}_i^1) \supseteq R_i(\mathcal{L}_i^0)$ for every $n \geq 1$.

Result (a) follows from the fact that, for each player i and for every $s_i \in \{0, 1, \ldots, \beta_i\}$, w weakly dominates s_i on S_j. (To see this, note that if the opponent chooses w, then player i's payoff by choosing w is β_i, while it is $\{0, 1, \ldots, \beta_i\}$ if player i commits to one of these demands, and if the opponent chooses $s_j \in \{0, 1, \ldots, k\}$, then player i's payoff by choosing w is $1 - s_j$, while it is no more than $1 - s_j$ and sometimes 0 if $s_i \in \{0, 1, \ldots, \beta_i\}$.) Hence, for each player i and for every $s_i \in \{0, 1, \ldots, \beta_i\}$, $(s_i, \{w\}) \in R_i(\mathcal{L}_i^*) = R_i(\mathcal{L}_i^0)$. This result implies that, for each player i, every $L_i = (L_i^1, \ldots, L_i^K) \in \mathcal{L}_i^1 = \mathcal{L}_i^r(R_j(\mathcal{L}_j^0))$ satisfies that there exists $k \in \{1, \ldots, K\}$ such that $\{w\} \subseteq L_i^1 \cup \cdots \cup L_i^k \subseteq \{\beta_j + 1, \beta_j + 2, \ldots, k\} \cup \{w\}$. Result (b) follows from the fact that, for each player i and for every $s_i \in \{\beta_i + 1, \beta_i + 2, \ldots, k-1\}$, k weakly dominates s_i on each strategy set S_j' satisfying $\{w\} \subseteq S_j' \subseteq \{\beta_j + 1, \beta_j + 2, \ldots, k\} \cup \{w\}$. Hence, for each player i and for every $s_i \in \{\beta_i + 1, \beta_i + 2, \ldots, k-1\}$, $(s_i, \{k\}) \in R_i(\mathcal{L}_i^1)$.

Part (ii). We establish that k and w *are* properly rationalizable in the finite version of Ellingsen and Miettinen's (2008, Section I) bilateral commitment bargaining game with zero commitment cost, by showing that both k and w can be used with positive probability in a proper equilibrium; thus, they are properly rationalizable (Asheim, 2001, Proposition 2). To prove this claim, consider the likelihood orderings

$$L_1 = \{\{w\}, \{1\}, \{2\}, \ldots, \{\beta_2 - 1\}, \{k\}, \{k-1\}, \ldots, \{\beta_2 + 1\}, \{\beta_2\}, \{0\}\},$$
$$L_2 = \{\{k\}, \{k-1\}, \ldots, \{\beta_1 + 1\}, \{w\}, \{\beta_1\}, \{\beta_1 - 1\}, \ldots, \{1\}, \{0\}\}.$$

Since each element in either of these partitions contains only one strategy, they determine a pair of LPSs. It is straightforward to check that this pair of LPSs determines a proper equilibrium, according to Blume et al.'s (1991b, Proposition 5) characterization, where player 1 chooses k with probability 1 and player 2 chooses w with probability 1. ■

Claim *Consider the finite version of Ellingsen and Miettinen's (2008, Section I) bilateral commitment bargaining game with zero commitment cost. Assume that $x_1(s_1, s_2) = s_1$ and $x_2(s_2, s_1) = s_2$ if $s_1 + s_2 \leq k$.*

(i) There exists a proper equilibrium where both players assign probability 1 to k.

(ii) For both players i and any strategy $\ell \in \{\beta_i + 1, \beta_i + 2, \ldots, k-1\}$, there exists a perfect equilibrium where player i assigns positive probability to both w and ℓ and player j assigns probability 1 to k.

Proof. *Part (i).* Consider the LPSs

$$\lambda_1 = \{\lambda_1^1, \ldots, \lambda_1^{k+1}\}$$
$$\lambda_2 = \{\lambda_2^1, \ldots, \lambda_2^{k+1}\},$$

where for both players i and each $\ell \in \{1, \ldots, k+1\}$, the support of λ_i^ℓ is included in $\{w, k+1-\ell\}$ for $\ell \in \{1, \ldots, \beta_j+1\}$, $\{w, 1\}$ for $\ell = \beta_j+2$, $\{w, k+2-\ell\}$ for $\ell \in \{\beta_j+3, \ldots, k\}$, and $\{w, 0\}$ for $\ell = k+1$. Let, for each $\ell \in \{1, \ldots, k+1\}$, λ_i^ℓ be determined by $u_i(w, \lambda_i^\ell) = u_i(k-1, \lambda_i^\ell)$. This means that

$$\begin{array}{ll}
\lambda_i^1(w) = 0 & \lambda_i^1(k) = 1 \\
\lambda_i^2(w) = \frac{1}{\beta_j} & \lambda_i^2(k-1) = \frac{\beta_j-1}{\beta_j} \\
\lambda_i^3(w) = \frac{2}{\beta_j+1} & \lambda_i^3(k-2) = \frac{\beta_j-1}{\beta_j+1} \\
\ldots & \ldots \\
\lambda_i^{\beta_j+1}(w) = \frac{\beta_j}{2\beta_j-1} & \lambda_i^{\beta_j+1}(\beta_i) = \frac{\beta_j-1}{2\beta_j-1} \\
\lambda_i^{\beta_j+2}(w) = 0 & \lambda_i^{\beta_j+2}(1) = 1 \\
\lambda_i^{\beta_j+3}(w) = \frac{\beta_j+1}{2\beta_j} & \lambda_i^{\beta_j+3}(\beta_i - 1) = \frac{\beta_j-1}{2\beta_j} \\
\ldots & \ldots \\
\lambda_i^k(w) = \frac{k-2}{\beta_j+k-3} & \lambda_i^k(2) = \frac{\beta_j-1}{\beta_j+k-3} \\
\lambda_i^{k+1}(w) = \frac{1}{\beta_j} & \lambda_i^{k+1}(0) = \frac{\beta_j-1}{\beta_j}
\end{array}$$

The LPSs λ_1 and λ_2 determine the following likelihood orderings:

$$L_1 = \{\{k\}, \{w, k-1\}, \{k-2\}, \ldots, \{\beta_1+1\}, \{\beta_1\}, \{1\}, \{\beta_1 - 1\}, \ldots, \{2\}, \{0\}\},$$
$$L_2 = \{\{k\}, \{w, k-1\}, \{k-2\}, \ldots, \{\beta_2+1\}, \{\beta_2\}, \{1\}, \{\beta_2 - 1\}, \ldots, \{2\}, \{0\}\}.$$

It can be checked that L_1 respects the preference restrictions that u_2 and λ_2 give rise to, and L_2 respects the preference restrictions that u_1 and λ_1 give rise to. To see this in the case of L_1 (the demonstration for L_2 is symmetric), note:

(a) Player 2 ranks the commitment strategies 0, 2, 3, …, k according to size since $u_2(s_2, \lambda_2^1) = 0$ and $u_2(s_2, \lambda_2^2) = s_2/\beta_1$ for $s_2 \in 0, 2, 3, \ldots, k$.

(b) Player 2 is indifferent between the commitment strategy $k-1$ and waiting w since, by construction, $u_2(w, \lambda_2^\ell) = u_2(k-1, \lambda_2^\ell)$ for all $\ell \in \{1, \ldots, k+1\}$.

(c) Player 2 ranks the commitment strategy 1 between the commitment strategies β_1 and $\beta_1 - 1$ since

$$u_2(\beta_1, \lambda_2^1) = u_2(1, \lambda_2^1) = u_2(\beta_1 - 1, \lambda_2^1) = 0,$$
$$u_2(\beta_1, \lambda_2^2) = u_2(1, \lambda_2^2) = 1 > u_2(\beta_1 - 1, \lambda_2^2) = \tfrac{\beta_1-1}{\beta_1},$$
$$u_2(\beta_1, \lambda_2^3) = \tfrac{2\beta_1}{\beta_1+1} > u_2(1, \lambda_2^3) = 1,$$

since $\beta_1 > 1$ and $x_2(1, k-2) = 1$.

It follows from Blume et al.'s (1991b, Proposition 5) characterization that $(\lambda_1^1, \lambda_2^1)$, where λ_2^1 is the mixed strategy of player 1 and λ_1^1 is the mixed strategy of player 2, is a proper equilibrium. Note that, for both players i, $\lambda_i^1(k) = 1$.

Part (ii). Let ℓ be any player 1 strategy in $\{\beta_1+1, \beta_1+2, \ldots, k-1\}$. Consider the LPSs $\lambda_1 = \{\lambda_1^1, \ldots, \lambda_1^{k+1}\}$ and $\lambda_2 = \{\lambda_2^1, \lambda_2^2\}$ defined by

$$
\begin{array}{lll}
\lambda_1^1(w) = 0 & \lambda_1^1(k) = 1 & \lambda_2^1(w) = \frac{\beta_2}{k} \qquad \lambda_2^1(\ell) = 1 - \frac{\beta_2}{k} \\
\lambda_1^2(w) = 0 & \lambda_1^2(k-\ell) = 1 & \lambda_2^2(s_1) = \frac{1}{k} \text{ for all } s_1 \in S_1 \backslash \{w, \ell\} \\
\lambda_1^3(w) = \frac{1}{\ell-\beta_1+1} & \lambda_1^3(k-1) = \frac{\ell-\beta_1}{\ell-\beta_1+1} & \\
\ldots & \ldots & \\
\lambda_1^{\ell+1}(w) = \frac{\ell-1}{2\ell-\beta_1-1} & \lambda_1^{\ell+1}(k-\ell+1) = \frac{\ell-\beta_1}{2\ell-\beta_1-1} & \\
\lambda_1^{\ell+2}(w) = \frac{1}{\ell-\beta_1+1} & \lambda_1^{\ell+2}(k-\ell-1) = \frac{\ell-\beta_1}{\ell-\beta_1+1} & \\
\ldots & \ldots & \\
\lambda_1^{k}(w) = \frac{k-\ell-1}{k-\beta_1-1} & \lambda_1^{k}(1) = \frac{\ell-\beta_1}{k-\beta_1-1} & \\
\lambda_1^{k+1}(w) = \frac{k-\ell}{k-\beta_1} & \lambda_1^{k+1}(0) = \frac{\ell-\beta_1}{k-\beta_1}\,, &
\end{array}
$$

with, for each level of these LPSs, zero probability assigned to other strategies.

These LPSs imply that player 1 is indifferent between w and ℓ and that player 1 prefers each of these strategies to any strategy in $S_1 \backslash \{w, \ell\}$, and that player 2 prefers k to any strategy in $S_2 \backslash \{k\}$. To see this, note:

(a) It follows that player 1 strictly prefers each of w and ℓ to any strategy in $S_1 \backslash \{w, \ell\}$ since $u_1(s_1, \lambda_1^1) = 0$ for all $s_1 \in S_1$ and $u_1(w, \lambda_1^2) = u_1(\ell, \lambda_1^2) = \ell$, while $u_1(s_1, \lambda_1^2) = s_1 < \ell$ if s_1 is a commitment strategy in $\{1, 2, \ldots, \ell-1\}$ and $u_1(s_1, \lambda_1^2) = 0 < \ell$ if s_1 is a commitment strategy in $\{\ell+1, \ell+2, \ldots, k\}$. It follows that player 1 is indifferent between w and ℓ since $\lambda_1^3, \lambda_1^4, \ldots, \lambda_1^{k+1}$ have been constructed so that $u_1(w, \lambda_1^m) = u_1(\ell, \lambda_1^m)$ for each $m \in \{3, 4, \ldots, k+1\}$.

(b) It follows that player 2 strictly prefers k to any strategy in $S_2 \backslash \{k\}$ since $u_2(k, \lambda_2^1) = \beta_2$ and $u_2(s_2, \lambda_2^1) < \beta_2$ for all $s_2 \in S_2 \backslash \{k\}$.

Since both λ_1 and λ_2 have full support on the set of opponent strategies, it follows from Blume et al.'s (1991b, Proposition 4) characterization that $(\lambda_1^1, \lambda_2^1)$, where λ_2^1 is the mixed strategy of player 1 and λ_1^1 is the mixed strategy of player 2, is a perfect equilibrium where player 1 assigns positive probability to both w and ℓ and player 2 assigns probability 1 to k.

In a simular fashion we can show that, for any player 2 strategy $\ell \in \{\beta_2+1, \beta_2+2, \ldots, k-1\}$, there exists a perfect equilibrium where player 1 assigns probability 1 to k and player 2 assigns positive probability to both w and ℓ. ■

B Tables

Table 1: The functioning of the algorithms in game G_1.

Permissibility

$\mathcal{L}_1^0 = \tilde{\mathcal{L}}_1^*$	$\mathcal{L}_2^0 = \tilde{\mathcal{L}}_2^*$
$\mathcal{L}_1^\infty = \tilde{\mathcal{L}}_1^*$	$\mathcal{L}_2^\infty = \{(\{U\},\{M,D\}),(\{M\},\{U,D\}),(\{U,M\},\{D\})\}$

Iterated admissibility

$\mathcal{L}_1^0 = \mathcal{L}_1^*$	$\mathcal{L}_2^0 = \mathcal{L}_2^*$
$\mathcal{L}_1^1 = \mathcal{L}_1^*$	$\mathcal{L}_2^1 = \{(\{U,M\},\{D\}),(\{U\},\{M\},\{D\}),(\{M\},\{U\},\{D\})\}$
$\mathcal{L}_1^2 = \{(\{L\},\{R\})\}$	$\mathcal{L}_2^2 = \{(\{U,M\},\{D\}),(\{U\},\{M\},\{D\}),(\{M\},\{U\},\{D\})\}$
$\mathcal{L}_1^\infty = \{(\{L\},\{R\})\}$	$\mathcal{L}_2^\infty = \{(\{U\},\{M\},\{D\})\}$

Proper rationalizability

$\mathcal{L}_1^0 = \mathcal{L}_1^*$	$\mathcal{L}_2^0 = \mathcal{L}_2^*$
$\mathcal{L}_1^\infty = \mathcal{L}_1^*$	$\mathcal{L}_2^\infty = \{(\{U\},\{M,D\}),(\{U,M\},\{D\}),$ $(\{U\},\{M\},\{D\}),(\{U\},\{D\},\{M\}),(\{M\},\{U\},\{D\})\}$

Full permissibility

$\mathcal{L}_1^0 = \tilde{\mathcal{L}}_1^*$	$\mathcal{L}_2^0 = \tilde{\mathcal{L}}_2^*$
$\mathcal{L}_1^1 = \tilde{\mathcal{L}}_1^*$	$\mathcal{L}_2^1 = \{(\{U\},\{M,D\}),(\{M\},\{U,D\}),(\{U,M\},\{D\})\}$
$\mathcal{L}_1^2 = \{S_2,(\{L\},\{R\})\}$	$\mathcal{L}_2^2 = \{(\{U\},\{M,D\}),(\{M\},\{U,D\}),(\{U,M\},\{D\})\}$
$\mathcal{L}_1^\infty = \{S_2,(\{L\},\{R\})\}$	$\mathcal{L}_2^\infty = \{(\{U\},\{M,D\}),(\{U,M\},\{D\})\}$

Table 2: The functioning of the algorithms in game G_2.

Permissibility

$\mathcal{L}_1^0 = \tilde{\mathcal{L}}_1^*$	$\mathcal{L}_2^0 = \tilde{\mathcal{L}}_2^*$
$\mathcal{L}_1^\infty = \tilde{\mathcal{L}}_1^*$	$\mathcal{L}_2^\infty = \{(\{U\},\{M,D\}),(\{M\},\{U,D\}),(\{U,M\},\{D\})\}$

Iterated admissibility

$\mathcal{L}_1^0 = \mathcal{L}_1^*$	$\mathcal{L}_2^0 = \mathcal{L}_2^*$
$\mathcal{L}_1^1 = \mathcal{L}_1^*$	$\mathcal{L}_2^1 = \{(\{U,M\},\{D\}),(\{U\},\{M\},\{D\}),(\{M\},\{U\},\{D\})\}$
$\mathcal{L}_1^2 = \{(\{L\},\{R\})\}$	$\mathcal{L}_2^2 = \{(\{U,M\},\{D\}),(\{U\},\{M\},\{D\}),(\{M\},\{U\},\{D\})\}$
$\mathcal{L}_1^\infty = \{(\{L\},\{R\})\}$	$\mathcal{L}_2^\infty = \{(\{U\},\{M\},\{D\})\}$

Proper rationalizability

$\mathcal{L}_1^0 = \mathcal{L}_1^*$	$\mathcal{L}_2^0 = \mathcal{L}_2^*$
$\mathcal{L}_1^1 = \mathcal{L}_1^*$	$\mathcal{L}_2^1 = \{(\{U\},\{M,D\}),(\{U,M\},\{D\}),$ $(\{U\},\{M\},\{D\}),(\{U\},\{D\},\{M\}),(\{M\},\{U\},\{D\})\}$
$\mathcal{L}_1^2 = \{(\{L\},\{R\})\}$	$\mathcal{L}_2^2 = \{(\{U\},\{M,D\}),(\{U,M\},\{D\}),$ $(\{U\},\{M\},\{D\}),(\{U\},\{D\},\{M\}),(\{M\},\{U\},\{D\})\}$
$\mathcal{L}_1^\infty = \{(\{L\},\{R\})\}$	$\mathcal{L}_2^\infty = \{(\{U\},\{D\},\{M\})\}$

Full permissibility

$\mathcal{L}_1^0 = \tilde{\mathcal{L}}_1^*$	$\mathcal{L}_2^0 = \tilde{\mathcal{L}}_2^*$
$\mathcal{L}_1^1 = \tilde{\mathcal{L}}_1^*$	$\mathcal{L}_2^1 = \{(\{U\},\{M,D\}),(\{M\},\{U,D\}),(\{U,M\},\{D\})\}$
$\mathcal{L}_1^2 = \{S_2,(\{L\},\{R\})\}$	$\mathcal{L}_2^2 = \{(\{U\},\{M,D\}),(\{M\},\{U,D\}),(\{U,M\},\{D\})\}$
$\mathcal{L}_1^3 = \{S_2,(\{L\},\{R\})\}$	$\mathcal{L}_2^3 = \{(\{U\},\{M,D\}),(\{U,M\},\{D\})\}$
$\mathcal{L}_1^4 = \{(\{L\},\{R\})\}$	$\mathcal{L}_2^4 = \{(\{U\},\{M,D\}),(\{U,M\},\{D\})\}$
$\mathcal{L}_1^\infty = \{(\{L\},\{R\})\}$	$\mathcal{L}_2^\infty = \{(\{U\},\{M,D\})\}$

Table 3: The functioning of the algorithms in game G_3.

Permissibility	
$\mathcal{L}_1^0 = \tilde{\mathcal{L}}_1^*$	$\mathcal{L}_2^0 = \tilde{\mathcal{L}}_2^*$
$\mathcal{L}_1^1 = \{(\{d\},\{fd,ff\}),$ $(\{fd\},\{d,ff\}),(\{d,fd\},\{ff\})\}$	$\mathcal{L}_2^1 = \tilde{\mathcal{L}}_2^*$
$\mathcal{L}_1^\infty = \{(\{d\},\{fd,ff\}),$ $(\{fd\},\{d,ff\}),(\{d,fd\},\{ff\})\}$	$\mathcal{L}_2^\infty = \{(\{D\},\{FD,FF\}),$ $(\{FD\},\{D,FF\}),(\{D,FD\},\{FF\})\}$
Iterated admissibility	
$\mathcal{L}_1^0 = \mathcal{L}_1^*$	$\mathcal{L}_2^0 = \mathcal{L}_2^*$
$\mathcal{L}_1^1 = \{(\{d,fd\},\{ff\}),$ $(\{d\},\{fd\},\{ff\}),(\{fd\},\{d\},\{ff\})\}$	$\mathcal{L}_2^1 = \mathcal{L}_2^*$
$\mathcal{L}_1^2 = \{(\{d,fd\},\{ff\}),$ $(\{d\},\{fd\},\{ff\}),(\{fd\},\{d\},\{ff\})\}$	$\mathcal{L}_2^2 = \{(\{D,FD\},\{FF\}),$ $(\{D\},\{FD\},\{FF\}),(\{FD\},\{D\},\{FF\})\}$
$\mathcal{L}_1^3 = \{(\{d\},\{fd\},\{ff\})\}$	$\mathcal{L}_2^3 = \{(\{D,FD\},\{FF\}),$ $(\{D\},\{FD\},\{FF\}),(\{FD\},\{D\},\{FF\})\}$
$\mathcal{L}_1^\infty = \{(\{d\},\{fd\},\{ff\})\}$	$\mathcal{L}_2^\infty = \{(\{D\},\{FD\},\{FF\})\}$
Proper rationalizability	
$\mathcal{L}_1^0 = \mathcal{L}_1^*$	$\mathcal{L}_2^0 = \mathcal{L}_2^*$
$\mathcal{L}_1^1 = \{(\{fd\},\{d,ff\}),(\{d,fd\},\{ff\}),$ $(\{d\},\{fd\},\{ff\}),(\{fd\},\{d\},\{ff\}),$ $(\{fd\},\{ff\},\{d\})\}$	$\mathcal{L}_2^1 = \mathcal{L}_2^*$
$\mathcal{L}_1^2 = \{(\{fd\},\{d,ff\}),(\{d,fd\},\{ff\}),$ $(\{d\},\{fd\},\{ff\}),(\{fd\},\{d\},\{ff\}),$ $(\{fd\},\{ff\},\{d\})\}$	$\mathcal{L}_2^2 = \{(\{FD\},\{D,FF\}),(\{D,FD\},\{FF\}),$ $(\{D\},\{FD\},\{FF\}),(\{FD\},\{D\},\{FF\}),$ $(\{FD\},\{FF\},\{D\})\}$
$\mathcal{L}_1^3 = \{(\{d\},\{fd\},\{ff\})\}$	$\mathcal{L}_2^3 = \{(\{FD\},\{D,FF\}),(\{D,FD\},\{FF\}),$ $(\{D\},\{FD\},\{FF\}),(\{FD\},\{D\},\{FF\})\}$ $(\{FD\},\{FF\},\{D\})\}$
$\mathcal{L}_1^\infty = \{(\{d\},\{fd\},\{ff\})\}$	$\mathcal{L}_2^\infty = \{(\{D\},\{FD\},\{FF\})\}$
Full permissibility	
$\mathcal{L}_1^0 = \tilde{\mathcal{L}}_1^*$	$\mathcal{L}_2^0 = \tilde{\mathcal{L}}_2^*$
$\mathcal{L}_1^1 = \{(\{d\},\{fd,ff\}),$ $(\{fd\},\{d,ff\}),(\{d,fd\},\{ff\})\}$	$\mathcal{L}_2^1 = \tilde{\mathcal{L}}_2^*$
$\mathcal{L}_1^2 = \{(\{d\},\{fd,ff\}),$ $(\{fd\},\{d,ff\}),(\{d,fd\},\{ff\})\}$	$\mathcal{L}_2^2 = \{(\{D\},\{FD,FF\}),$ $(\{FD\},\{D,FF\}),(\{D,FD\},\{FF\})\}$
$\mathcal{L}_1^3 = \{(\{d\},\{fd,ff\}),(\{d,fd\},\{ff\})\}$	$\mathcal{L}_2^3 = \{(\{D\},\{FD,FF\})$ $(\{FD\},\{D,FF\}),(\{D,FD\},\{FF\})\}$
$\mathcal{L}_1^\infty = \{(\{d\},\{fd,ff\}),(\{d,fd\},\{ff\})\}$	$\mathcal{L}_2^\infty = \{(\{D\},\{FD,FF\}),(\{D,FD\},\{FF\})\}$

Table 4: The functioning of the algorithms in game G_4.

Permissibility

$\mathcal{L}_1^0 = \tilde{\mathcal{L}}_1^*$ $\qquad$ $\mathcal{L}_2^0 = \tilde{\mathcal{L}}_2^*$

$\mathcal{L}_1^\infty = \tilde{\mathcal{L}}_1^*$ $\qquad$ $\mathcal{L}_2^\infty = \{(\{U\},\{M,D\}),(\{D\},\{U,M\}),(\{U,D\},\{M\})\}$

Iterated admissibility

$\mathcal{L}_1^0 = \mathcal{L}_1^*$ $\qquad$ $\mathcal{L}_2^0 = \mathcal{L}_2^*$

$\mathcal{L}_1^1 = \mathcal{L}_1^*$ $\qquad$ $\mathcal{L}_2^1 = \{(\{U,D\},\{M\}),(\{U\},\{D\},\{M\}),(\{D\},\{U\},\{M\})\}$

$\mathcal{L}_1^2 = \{(\{L\},\{R\})\}$ $\qquad$ $\mathcal{L}_2^2 = \{(\{U,D\},\{M\}),(\{U\},\{D\},\{M\}),(\{D\},\{U\},\{M\})\}$

$\mathcal{L}_1^\infty = \{(\{L\},\{R\})\}$ $\qquad$ $\mathcal{L}_2^\infty = \{(\{U\},\{D\},\{M\})\}$

Proper rationalizability

$\mathcal{L}_1^0 = \mathcal{L}_1^*$ $\qquad$ $\mathcal{L}_2^0 = \mathcal{L}_2^*$

$\mathcal{L}_1^\infty = \mathcal{L}_1^*$ $\qquad$ $\mathcal{L}_2^\infty = \{(\{D\},\{U,M\}),(\{U,D\},\{M\}),$
$(\{U\},\{D\},\{M\}),(\{D\},\{U\},\{M\}),(\{D\},\{M\},\{U\})\}$

Full permissibility

$\mathcal{L}_1^0 = \tilde{\mathcal{L}}_1^*$ $\qquad$ $\mathcal{L}_2^0 = \tilde{\mathcal{L}}_2^*$

$\mathcal{L}_1^1 = \tilde{\mathcal{L}}_1^*$ $\qquad$ $\mathcal{L}_2^1 = \{(\{U\},\{M,D\}),(\{D\},\{U,M\}),(\{U,D\},\{M\})\}$

$\mathcal{L}_1^2 = \{S_2,(\{L\},\{R\})\}$ $\qquad$ $\mathcal{L}_2^2 = \{(\{U\},\{M,D\}),(\{D\},\{U,M\}),(\{U,D\},\{M\})\}$

$\mathcal{L}_1^3 = \{S_2,(\{L\},\{R\})\}$ $\qquad$ $\mathcal{L}_2^3 = \{(\{U\},\{M,D\}),(\{U,D\},\{M\})\}$

$\mathcal{L}_1^4 = \{(\{L\},\{R\})\}$ $\qquad$ $\mathcal{L}_2^4 = \{(\{U\},\{M,D\}),(\{U,D\},\{M\})\}$

$\mathcal{L}_1^\infty = \{(\{L\},\{R\})\}$ $\qquad$ $\mathcal{L}_2^\infty = \{(\{U\},\{M,D\})\}$