

# Optimal Taxation with Private Insurance \*

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## Abstract

We derive a fully *nonlinear* optimal income tax schedule in the presence of private insurance. As in the standard taxation literature without private insurance (e.g., Saez (2001)), the optimal tax formula can still be expressed in terms of sufficient statistics such as the labor supply elasticity. With private insurance, however, the formula involves the statistics that reflect households' savings pattern (the marginal propensity to save) and their interaction with public insurance (crowding in/out elasticity). Since these statistics are neither easy to estimate nor policy-invariant, we obtain them from a structural model calibrated to reproduce salient features of the U.S. economy.

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# 1 Introduction

What is the socially optimal shape of the income tax schedule? This has been one of the classic and central questions in macroeconomics and public finance. Despite significant progress in the literature, surprisingly few studies have investigated the role of private intermediation in the optimal tax system. Understanding the impact of private insurance on the optimal tax is important because, in practice, it is very rare that public insurance can perfectly substitute for a private arrangement. Moreover, even when the government insurance coverage is exactly at the same level that would have been selected by a household from the private market in the absence of government, households may still purchase additional private insurance, if there is moral hazard or a pecuniary externality (see Kaplow (1994)).

In this paper, we study the optimal (fully) nonlinear income tax schedule that highlights the role of the interaction between private and public insurance in determining the optimal tax-and-transfer system. The goal of this paper is to present a framework that is simple but general enough to capture various arrangements in private insurance markets. We study a fully nonlinear schedule but focus on a simple class of tax system that is levied on current income only, which allows a direct comparison of our results to those in classic optimal formulas (Saez (2001), Diamond (1998)). The optimal tax formula is derived using a *variational* approach—the tax schedule is optimal, if there is no welfare gain from a small deviation—as in Piketty (1997) and Saez (2001).

As in Saez (2001), the optimal tax rate can still be expressed in terms of standard statistics—such as the Frisch elasticity of the labor supply, the hazard rate of the income distribution, and marginal social welfare weights. In the presence of a private insurance market, however, the formula also includes additional statistics that reflect households' savings behavior (such as the marginal propensity to save) and their interaction with taxes (crowding in/out elasticities).

These additional terms provide transparent and intuitive insights into the role of the private insurance. First, it represents the substitutability between private and public insurance—for example, a high savings rate leads to a lower marginal tax. Second, the original formula in Saez (2001) needs to be modified to reflect the amplifying (or miti-

gating) factors. Two components are important for this modification: (i) the marginal private savings and (ii) cross-sectional dispersion of consumption. Third, the formula also includes the alignment of public and private insurance. For example, a tax reform is more effective when the response of private intermediation is aligned with that reform.

A key advantage of our approach is that our formula can be applied to a wide class of private market structures, ranging from various incomplete markets to a fully insured complete market.<sup>1</sup> We provide the analysis under specific examples of a private insurance market, including Huggett (1993) and Kehoe and Levine (1993), commonly used incomplete-market models in macroeconomics.

Unfortunately, the additional statistics—e.g., marginal private intermediation and crowding in/out elasticities—are not easy to estimate from the data. First, the formula requires these statistics under the *optimal* steady state. Second, they are not policy invariant in general. Thus, it requires out-of-sample predictions. Given these difficulties, we combine the structural and sufficient-statistics methods following the suggestion by Chetty (2009). We obtain these statistics from quantitative general equilibrium models that are calibrated to resemble some salient features (such as the income and wealth distributions) of the U.S. economy. This allows us to quantify the role of private insurance in determining the optimal tax rate.

While the exact shape of the tax schedule depends on the fundamentals (such as the risk aversion, Frisch elasticity of the labor supply and the nature of the private insurance market), the presence of a private intermediation is quantitatively important. According to our analysis of the baseline economy (self-insurance with exogenous borrowing constraint, Huggett (1993)), the difference in optimal tax rates (with and without a private insurance market) can be as large as 20 percentage points. Moreover, these differences in tax rates do not necessarily exhibit the same sign across incomes. For example, the optimal tax rates are higher than those without private markets for the low-income group—mainly because of the amplification of the original Saez formula. The optimal tax rates are lower (than those without a private market) for the middle- to high-income groups—mainly because of substitution effects and mitigation of the Saez formula. At the very top income group, various forces offset each other, leaving the tax rate similar to that without private

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<sup>1</sup>Our formula is general enough to encompass any private insurance market where the aggregate amount of savings is constant.

insurance. We also compare the quantitative results of our baseline economy to those from an economy with endogenous borrowing constraints (Kehoe and Levine (1993)).

Our paper is most closely related to a literature on optimal labor income taxation using a variational approach, originally pioneered by Piketty (1997) and Saez (2001). In a static model, they express the optimal tax formula in terms of the so-called sufficient statistics (e.g., elasticity of the labor supply and the hazard rate of income), which is obtained by perturbations of a given tax system. This variational approach is a complement to the traditional mechanism-design approach (Mirrlees (1971)) and allows us to understand the key economic forces behind the formula. While this approach has been extended to other contexts such as multi-dimensional screening (Kleven, Kreiner, and Saez (2009)) and dynamic models (Golosov, Tsyvinski, and Werquin (2014), Saez and Stantcheva (2017)), this literature largely abstracts from a private insurance market by assuming that the government is the sole provider of insurance.<sup>2</sup> The paper by Chetty and Saez (2010) is an exception that allows for private insurance, but they assume that both private and public insurance are linear, and thus have limited implications for the interactions between the two types of insurance.

In the alternative Ramsey approach (Ramsey (1927)), which examines the optimal tax schedule within a class of functional forms, many studies have provided quantitative answers to the optimal amount of redistribution in the presence of self-insurance opportunities (e.g., Aiyagari and McGrattan (1998), Conesa and Krueger (2006), Conesa, Kitao, and Krueger (2009), Heathcote, Storesletten, and Violante (2014), and Bhandari, Evans, Golosov, and Sargent (2016)). However, these studies assume a parametric form for the tax schedule—either affine or log-linear. Moreover, they do not particularly focus on how the introduction of private savings affects the optimal tax schedule. While we allow for a fully nonlinear tax system, our analysis provides a transparent comparison to these papers, as we also compute the optimal tax schedule in a general equilibrium incomplete-markets economy—a workhorse model in macroeconomics. Our quantitative analysis shows that the optimal tax schedule is very different from those commonly assumed—an affine or log-linear tax function—in the literature.<sup>3</sup>

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<sup>2</sup>The sufficient statistics approach has been widely used in the taxation literature (e.g., Diamond and Saez (2011), Piketty and Saez (2013b), Piketty, Saez, and Stantcheva (2014), Piketty and Saez (2013a), and Badel and Huggett (2017)).

<sup>3</sup>For example, Heathcote and Tsujiyama (2017) compare three tax systems (affine, log-linear, and

In the *New Dynamic Public Finance* literature, Golosov and Tsyvinski (2007) study optimal taxation in the presence of private insurance, but under a specific market structure—a competitive insurance industry with private information friction.<sup>4</sup> We derive the formula under a very general representation of private intermediation, which can be applied to various market structures, regardless of the source of frictions in private intermediation.

Our baseline quantitative analysis is also related to a recent paper by Findeisen and Sachs (2017), which studies the optimal nonlinear labor income tax and linear capital income tax with self-insurance opportunities. They focus on the interaction between the labor and capital income taxes and there is a very limited interaction between public and private insurance because the interest rate is exogenous. Our paper focuses on how the response of the private insurance market affects the optimal labor income tax schedule, including the general equilibrium effect.

Outside the optimal taxation literature, Attanasio and Ríos-Rull (2000) examine the relationship between compulsory public insurance (against aggregate shocks) and private insurance against idiosyncratic shocks. Krueger and Perri (2011) study the crowding-out effect of a progressive income tax on private risk-sharing under limited commitments.

The remainder of the paper is organized as follows. In section 2, we derive the optimal tax formula. In Section 3, we apply our formula to various arrangements of a private insurance. Section 4 provides a quantitative analysis of the baseline economy. Section 5 compares the results between the two incomplete market models. Section 6 concludes.

## **2 Optimal Nonlinear Tax Formula with Private Insurance**

### **2.1 Restrictions on the Tax System**

While we consider a fully nonlinear income tax system without assuming a functional form, we focus on a restrictive class of tax system. The class of tax system we consider is

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Mirrleesian) and find that the optimal tax schedule is close to a log-linear form. Our analysis shows that under a more realistic productivity distribution and private market structure, the optimal tax schedule is highly nonlinear—quite different from log-linear.

<sup>4</sup>Their questions are centered on the welfare gains from government intervention in the presence of private insurance. Given that the government and private firms face the same information friction, the role of the government is restricted to internalizing the pecuniary externalities, and thus, there are limited implications for the optimal shape of the tax schedule.

a nonlinear labor income tax with a lump-sum transfer.<sup>5</sup> More precisely, (i) we consider a nonlinear labor income tax  $T(z)$  where  $z$  is current labor income; (ii) the tax is levied on the current period's income only (no history dependency); and (iii) the nonlinear tax function  $T(z)$  is age-independent and time invariant.

We impose these restrictions because they allow for a direct comparison to the static Mirrleesian taxation and Ramsey taxation literature. On one hand, in a static Mirrleesian analysis, the labor income tax depends on income only (not on productivity) because of information frictions. However, in a dynamic environment with stochastic productivity—which we study here—the optimal allocation that solves a mechanism design problem with information frictions (as in the New Dynamic Public Finance literature) will depend on the history of incomes. Moreover, it is well known that a tax system that can implement the constrained-efficient allocation is highly complicated, and thus a direct comparison of tax schedules between a static and a dynamic environment is not straightforward, even without a private market.<sup>6</sup> On the other hand, the Ramsey literature focuses on a tax system with particular functional forms. As in the Ramsey literature, our analysis starts with a simple and implementable tax system, but allows for a fully nonlinear functional form. Thus, our analysis provides a transparent comparison to the theoretical results from Mirrleesian taxation as well as those from Ramsey taxation.

## 2.2 Economic Environment with Private and Public Insurance

Consider an economy with a continuum of workers with measure one. Workers face uncertainty about their labor productivity in the future. The individual productivity shock  $x_t$  follows a Markov process (which will be specified below) that has an invariant stationary (cumulative) distribution  $F(x)$  whose probability density is  $f(x)$ .

Individual workers have an identical utility function  $\sum_{t=0}^{\infty} \beta^t E_0[U(c_t, l_t)]$ , where an instantaneous utility  $U(c, l)$  has the following form:  $U(c, l) = u(c - v(l))$ , where  $u(\cdot)$  is concave and increasing in consumption  $c$  and  $v(\cdot)$  is convex and increasing in labor supply  $l$ . We focus on households' preferences that have no wealth effect on the labor supply (the so-called GHH preferences by Greenwood, Hercowitz, and Huffman (1988)). This

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<sup>5</sup>We can allow for capital income tax, but we focus only on the optimal labor income tax for a given capital income tax, without considering joint optimal taxation.

<sup>6</sup>Most studies in the new dynamic public finance literature compare the implicit wedge from a dynamic environment to the marginal tax rate from a static one.

assumption is common in the literature because it significantly simplifies the optimal tax formula. The earnings of a worker whose current productivity is  $x_t$  are  $z_t = x_t l_t$ . The cumulative distribution of earnings is denoted by  $H(z)$  whose density function is  $h(z)$ .

The government provides insurance through a (time-invariant) nonlinear labor income tax and a lump-sum transfer system where the net payment schedule is denoted by  $T(z_t)$ . The after-tax labor income is  $y_t = z_t - T(z_t)$ . Workers can also participate in a private market to insure against their income uncertainty. Denote the individual state in period  $t$  by  $(z_t, \mathbf{s}_t)$ <sup>7</sup>, where  $\mathbf{s}_t = (s_{1,t}, \dots, s_{M,t}) \in R^M$  is the vector of individual state variables other than labor income. For example, if the private insurance market is a Bewley-type incomplete market and consumers can only self-insure themselves by saving and borrowing via a noncontingent bond (e.g., Huggett (1993)), we need only one additional state variable: bond holdings  $a_t$ :  $\mathbf{s}_t = a_t$ .

We denote the net payment from private insurance (payment - receipts) by  $P_t(z_t, \mathbf{s}_t; T)$ . Thus, consumption is  $c_t(z_t, \mathbf{s}_t) = z_t - T(z_t) - P_t(z_t, \mathbf{s}_t; T)$ . This representation is very general, which can be applied to a wide class of private insurance markets with constant aggregate saving. For simplicity, we assume that the sum of the net payment in the private intermediation is constant:  $\int P(\cdot) = 0$ .<sup>8</sup> In a Huggett economy with self-insurance only,  $P_t(z_t, a_t) = a_{t+1}(z_t, a_t) - (1 + r)a_t$  where  $r$  is the rate of return on bond holdings.

The government chooses a tax/transfer schedule  $T(z)$  to maximize the following social welfare function (SWF):  $SWF = E\left[\sum_{t=0}^{\infty} \beta^t G(U(c_t, l_t))\right]$ , where  $G(\cdot)$  is an increasing function that reflects the social preferences for redistribution.

Since the private intermediation  $P(\cdot; T)$  depends on the government tax/transfer schedule  $T$ , the government chooses the optimal  $T$  taking into account this interaction between public and private insurance. From now on, to simplify the notation, we will suppress  $T$  in  $P(\cdot)$  unless necessary. We will also express the private intermediation as a function of after-tax income  $y$ :  $\tilde{P}(y^z, \mathbf{s}; T) = P(z, \mathbf{s}; T)$ , where  $y^z = z - T(z)$ .

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<sup>7</sup>Alternatively, the state can be expressed as  $(x_t, \mathbf{s}_t)$ . With no income effects on labor supply, labor income  $z_t$  and productivity  $x_t$  have a one-to-one relationship and we can use them interchangeably. We also note that even with income effects on labor supply, we can use state variables  $(x_t, \mathbf{s}_t)$  and  $(z_t, \mathbf{s}_t)$  interchangeably because  $z_t(x_t, \mathbf{s}_t) = x_t l_t(x_t, \mathbf{s}_t)$  and  $x_t$  have a one-to-one relationship given  $s_t$ .

<sup>8</sup>That is, we consider a pure insurance market where the aggregate transfer is exactly funded by the aggregate payment in each period.

## 2.3 Deriving an Optimal Formula with Private Insurance

In deriving the optimal tax formula, we apply the variational approach (Piketty (1997); Saez (2001)). Instead of solving a mechanism design problem, we directly find a fully nonlinear marginal tax schedule that maximizes social welfare. That is, we consider a perturbation (a small deviation) from a given nonlinear tax schedule. If there is no welfare-improving perturbation within the class of tax system, the given tax schedule is optimal.

For a given income tax schedule  $T(z)$ , the economy we consider converges to a steady state where the distribution of state variables  $\Phi(z, \mathbf{s})$  is stationary. We assume that in period 0 the economy starts from that steady state. Consider a (revenue-neutral) tax reform that increases the marginal tax rate  $T'(z)$  by  $\delta\tau$  on the income bracket  $[z^*, z^* + dz^*]$ , as in Saez (2001).

In order to analyze the welfare effects of a tax reform (in the presence of a private insurance market), it is important to study its impact on the household's *total intermediation* for insurance: government tax/transfer plus private intermediation. Denote the total intermediation by  $M_t(z, \mathbf{s}) = T(z) + P_t(z, \mathbf{s}; T)$ . Since the government takes into account the response of households, the effects of tax reforms also depend on the following factors: (i) the marginal private intermediation (e.g., the marginal propensity to save)—which we refer as the “substitution” effect and (ii) the response (change) of the private intermediation schedule—which we call the “crowding in/out” effect.

### 2.3.1 Effects on Total Intermediation

The total *marginal* intermediation at income level  $z$  is:

$$M'_t(z, \mathbf{s}) = T'(z) + P'_t(z, \mathbf{s}).$$

Within the income band  $[z^*, z^* + dz^*]$ , where the marginal tax rate is changed, the change in the marginal total intermediation  $dM'$  reflects the change in the marginal tax itself,  $dT'(z^*)$ , and that in the marginal private intermediation,  $dP'(z^*, \mathbf{s})$ . The marginal private intermediation can change via two channels: (i) a change in after-tax income ( $-\delta\tau$ ) and (ii) a change in the marginal private payment schedule (change in  $\tilde{P}'$ ).<sup>9</sup> The sum of these two changes is:

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<sup>9</sup>  $P'(z^*, \mathbf{s}) = \frac{dP}{dy} \frac{dy}{dz} = \tilde{P}' \cdot (1 - T'(z^*))$ , where  $\tilde{P}(y(z), \mathbf{s}; T) = P(z, \mathbf{s}; T)$ .

$$d^o P'_t(z^*, \mathbf{s}) = -\tilde{P}'(y^*, \mathbf{s})\delta\tau + d^o \tilde{P}'_t(y^*, \mathbf{s}) \cdot (1 - T'(z^*)),$$

where  $d^o h(X)$  denotes the change in  $h(X)$  at given  $X$  due to a tax increase in its *own* income bracket—as opposed to the change in  $h(X)$  due to a tax increase in other income brackets. That is,  $d^o \tilde{P}'_t(y^*, \mathbf{s})$  denotes the change in the marginal private payment schedule ( $\tilde{P}'(y^*, \mathbf{s})$ ) for a given after-tax income  $y^* = z^* - T(z^*)$ .

Thus, the change in total marginal intermediation is:

$$d^o M'_t(z^*, \mathbf{s}) = (1 - \tilde{P}'(y^*, \mathbf{s}))\delta\tau + d^o \tilde{P}'_t(y^*, \mathbf{s}) \cdot (1 - T'(z^*)).$$

It is easier to understand these terms in the context of a Huggett (1993) economy with self-insurance. The first term reflects an increase in the total marginal tax at a given marginal propensity to save  $\tilde{P}'_y$ . The second term represents the change in the marginal private savings rate at a given disposable income level—which we refer to as the *own*-crowding in/out effect. We call this the *own* crowding effect because it reflects the change in marginal private savings at the income level where the tax rate is changed. As we will discuss below, the private savings rate may also change when the marginal tax rate in other income levels changes—via (i) changes in permanent income and (ii) general equilibrium effects. We refer to such changes in private intermediation as the *cross*-crowding in/out effects.

Using a similar definition for the elasticity of the *own*-crowding out  $r^o_t(z, \mathbf{s})$ , as in Chetty and Saez (2010),<sup>10</sup> the change in total intermediation  $d^o M'_t(z^*, \mathbf{s})$  can be expressed as:

$$\begin{aligned} d^o M'_t(z^*, \mathbf{s}) &= (1 - r^o_t(z^*, \mathbf{s}))(1 - \tilde{P}'_t(y^*, \mathbf{s}))\delta\tau, \quad \text{where} \\ r^o_t(z, \mathbf{s}) &= -\frac{d^o \log(1 - \tilde{P}'_t(y^z, \mathbf{s}))\big|_{yz}}{d \log(1 - T'(z))}. \end{aligned}$$

To better understand  $d^o M'_t(z^*, \mathbf{s})$ , note that the marginal propensity to consume out of *before-tax* income  $z$  is  $\frac{dc}{dz} = 1 - M'_t(z, \mathbf{s})$ . Thus, an increase in the total marginal intermediation is equivalent to a decrease in marginal consumption. If there is no crowding out of private intermediation ( $r^o = 0$ ), the decrease in marginal consumption is simply the marginal propensity to consume multiplied by the changes in the tax rate  $(1 - \tilde{P}')\delta\tau$ .

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<sup>10</sup>Chetty and Saez (2010) define the degree of crowding out in terms of linear (public and private) savings.

However, because of the crowding-out effect on private intermediation ( $r^o > 0$ ), the increase in total marginal intermediation is smaller than  $(1 - \tilde{P}')\delta\tau$ .

In a dynamic environment, where individual productivity stochastically changes over time, there can be additional effects on private intermediation because of (i) changes in *permanent* income and (ii) *general equilibrium* effects (e.g., changes in interest rate). For example, in an economy with self-insurance, while tax reform is confined to the income region of  $[z^*, z^* + dz^*]$  only, private savings in other income regions also change. We refer to this effect as the *cross* crowding in/out effect. For each income level  $z$ , the change in total marginal intermediation via the cross crowding out effect can be expressed as:

$$\begin{aligned} d^c M_t'(z, \mathbf{s}) &= \frac{d^c \tilde{P}'_t(y^z, \mathbf{s})}{dT'(z^*)} (1 - T'(z)) \delta\tau dz^* \\ &= -r_t^c(z, \mathbf{s}) \frac{(1 - P'_t(z - T(z), \mathbf{s}))(1 - T'(z))}{1 - T'(z^*)} \delta\tau dz^*, \end{aligned}$$

where  $d^c \tilde{P}'_t(y^z, \mathbf{s})$  denotes the changes in the marginal private intermediation schedule due to *cross* crowding out at a given after-tax income  $y^z$  and  $r_t^c(z, \mathbf{s})$  is the elasticity of *cross* crowding out in the marginal private intermediation:

$$r_t^c(z, \mathbf{s}) = - \frac{d^c \log(1 - \tilde{P}'_t(y^z, \mathbf{s}))|_{y^z}}{d \log(1 - T'(z^*))}.$$

To summarize, in an income band where the marginal tax rate has increased, the total marginal intermediation changes due to (i) the increase in the marginal tax rate itself and (ii) the *own* crowding in/out of private intermediation. For *all* income levels, there is an additional change in total marginal intermediation due to the *cross* crowding in/out of private intermediation.

More specifically, for an income level below  $z^*$ —where the tax rate remains unchanged—the change in the level of total intermediation reflects the cross crowding out effect  $d^c M_t(z, \mathbf{s})$  only. For an income level above the band  $[z^*, z^* + dz^*]$ —where the tax payment has increased—the total intermediation changes by  $d^o M_t(z, \mathbf{s}) + d^c M_t(z, \mathbf{s})$  with

$$\begin{aligned} d^o M_t(z, \mathbf{s}) &= (1 - r_t^o(z^*, \mathbf{s}))(1 - \tilde{P}'_t(y^{z^*}, \mathbf{s}))\delta\tau dz^*, \text{ for } z \geq z^* + dz^*, \\ d^c M_t(z, \mathbf{s}) &= d^c P_t(0, \mathbf{s}) + \delta\tau dz^* \int_0^z \frac{d^c \tilde{P}'_t(y^{\tilde{z}}, \mathbf{s})}{dT'(z^*)} (1 - T'(\tilde{z})) d\tilde{z}, \text{ for all } z, \end{aligned}$$

where  $d^c P_t(0, \mathbf{s})$  denotes the change in the intercept of the private intermediation schedule

due to cross crowding out. By integrating the changes in the total intermediation across all households, the *aggregate* change in the total intermediation in period  $t$ ,  $d\bar{M}_t^a$ , is:

$$\begin{aligned} d\bar{M}_t^a &= \int_{z \geq z^*, \mathbf{s}} d^o M_t(z, \mathbf{s}) d\Phi(z, \mathbf{s}) + \int d^c M_t(z, \mathbf{s}) d\Phi(z, \mathbf{s}) \\ &= \delta\tau dz^* \int_{z \geq z^*} \int_{\mathbf{s}} (1 - r_t^o(z^*, \mathbf{s}))(1 - \tilde{P}'_t(y^{z^*}, \mathbf{s})) d\Phi(\mathbf{s}|z) h(z) dz + \int d^c M_t(z, \mathbf{s}) d\Phi(z, \mathbf{s}). \end{aligned}$$

### 2.3.2 Behavioral Response of Labor Supply

The increased marginal tax on an income band  $[z^*, z^* + dz^*]$  affects the (before-tax) labor income via the labor supply through two channels: (i) the direct effects from an exogenous increase in the tax rate itself  $\delta\tau$  and (ii) the indirect effects from the change in labor income along the given tax schedule by  $dz$ , which in turn results in the change in the marginal tax by  $T''(z)dz$ . The change in before-tax income is:

$$dz = -z^* e(z^*) \frac{\delta\tau}{1 - T' + z^* e T''},$$

where  $e(z^*)$  is the Frisch elasticity of the labor supply at income level  $z^*$ . The change in total intermediation from these effects is:

$$\begin{aligned} &\int dz \cdot M'_t(z^*, \mathbf{s}) d\Phi(\mathbf{s}|z^*) h(z^*) dz^* \\ &= - \int \frac{T'(z^*) + \tilde{P}'_t(y^{z^*}, \mathbf{s})(1 - T'(z^*))}{1 - T' + z^* e(z^*) T''} d\Phi(\mathbf{s}|z^*) z^* e(z^*) h(z^*) \delta\tau dz^*. \end{aligned}$$

As in Saez (2001), we introduce the virtual density  $h^*(z)$  to simplify the presentation of the optimal tax formula where  $h^*(z)$  is the density of incomes that would take place at  $z$  if the tax schedule  $T(\cdot)$  were replaced by the linear tax schedule tangent to  $T(\cdot)$  at level  $z$ . When there is no income effect, Lemma 1 of Saez (2001) still applies.

**Lemma 1** (Lemma 1 of Saez (2001)). *For any regular tax schedule  $T$ , the earnings function  $z_x$  is non-decreasing and satisfies the following differential equation:*

$$\frac{\dot{z}_x}{z_x} = \frac{1 + e}{x} - \dot{z}_x \frac{T'_{(x)}}{1 - T'_{(x)}} e.$$

Using Lemma 1 and the fact that  $f(x) = h(z)\dot{z}_x = h^*(z)\dot{z}_x^*$  where  $\dot{z}_x^*$  is the derivative when the linearized tax schedule is in place, we obtain:  $\frac{h^*(z)}{1 - T'(z)} = \frac{h(z)}{1 - T'(z) + ezT''(z)}$ . Then, the aggregate change in the total intermediation via the behavioral response of the labor

supply,  $d\bar{M}_t^b$ , is:

$$d\bar{M}_t^b = - \left\{ \frac{T'(z^*)}{1 - T'(z^*)} + \int \tilde{P}'_t(y^{z^*}, \mathbf{s}) d\Phi(\mathbf{s}|z^*) \right\} z^* e(z^*) h^*(z^*) \delta \tau dz^*.$$

While the Frisch elasticity  $e(z^*)$  may depend on the income level, we will focus on a constant Frisch elasticity  $e$  below, for a simpler representation of the formula.

### 2.3.3 Optimal Tax Formula

Upon the above tax reform, a household pays an extra amount of  $dM_t(z, \mathbf{s}) = d^o M_t(z, \mathbf{s}) + d^c M_t(z, \mathbf{s})$  as a total intermediation in period  $t$ . Using the envelope theorem, this leads to the change in social welfare of:

$$- \int dM_t(z, \mathbf{s}) G'(u(z, \mathbf{s})) u'(c(z, \mathbf{s})) d\Phi(z, \mathbf{s}).$$

Each period the increased (aggregate) total intermediation,  $d\bar{M}_t = d\bar{M}_t^a + d\bar{M}_t^b$ , will be rebated to all households in a lump-sum fashion, which results in the change in social welfare of:<sup>11</sup>

$$d\bar{M}_t \int G'(u(z, \mathbf{s})) u'(c(z, \mathbf{s})) d\Phi(z, \mathbf{s}).$$

The overall change in social welfare from the tax reform is:

$$dSWF = \sum_{t=0}^{\infty} \beta^t d\bar{M}_t \int G'(u(z, \mathbf{s})) u'(c(z, \mathbf{s})) d\Phi(z, \mathbf{s}) - \sum_{t=0}^{\infty} \beta^t \int dM_t(z, \mathbf{s}) G'(u(z, \mathbf{s})) u'(c(z, \mathbf{s})) d\Phi(z, \mathbf{s}).$$

A tax schedule  $T(z)$  is optimal if  $dSWF = 0$  (no improvement in social welfare):

$$\sum_{t=0}^{\infty} \beta^t d\bar{M}_t = \sum_{t=0}^{\infty} \beta^t \int dM_t(z, \mathbf{s}) g(z, \mathbf{s}) d\Phi(z, \mathbf{s})$$

where  $g(z, \mathbf{s}) = \frac{G'(u(z, \mathbf{s})) u'(c(z, \mathbf{s}))}{A}$ ,  $A = \int G'(u(z, \mathbf{s})) u'(c(z, \mathbf{s})) d\Phi(z, \mathbf{s})$ .

By substituting out  $d\bar{M}_t$  and  $dM_t(z, \mathbf{s})$  and rearranging, we obtain the following optimal tax formula.

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<sup>11</sup>More precisely, the change in the aggregate total intermediation  $d\bar{M}_t$  is the sum of the change in the aggregate tax  $d\bar{T}$  and the change in the aggregate private intermediation  $d\bar{P}$ . The change in the aggregate tax  $d\bar{T}$  is rebated as a lump-sum transfer because we consider a revenue-neutral tax reform, and the change in the aggregate private intermediation  $d\bar{P}$  is zero because we consider a pure private insurance market where net payments sum to zero ( $\bar{P} = 0$ ).

**Proposition 2.** *Optimal marginal tax rate at income  $z^*$  should satisfy:*

$$\frac{T'(z^*)}{1 - T'(z^*)} = (1 - \beta) \sum_{t=0}^{\infty} \beta^t \left[ - \int \tilde{P}'(y^{z^*}, \mathbf{s}) d\Phi(\mathbf{s}|z^*) + B_t(z^*) + C_t(z^*) \right] \quad (1)$$

where

$$B_t(z^*) = \frac{1}{e} \frac{1 - H(z^*)}{z^* h^*(z^*)} \int_{z^*}^{\infty} \int (1 - r_t^o(z^*, \mathbf{s})) (1 - \tilde{P}'(y^{z^*}, \mathbf{s})) (1 - g(z, \mathbf{s})) d\Phi(\mathbf{s}|z) \frac{h(z)}{1 - H(z^*)} dz,$$

$$C_t(z^*) = \frac{1}{e} \frac{1}{z^* h^*(z^*)} \int (1 - g(z, \mathbf{s})) \frac{d^c P_t(z, \mathbf{s})}{\delta \tau dz^*} d\Phi(z, \mathbf{s}),$$

$$\text{and} \quad \frac{d^c P_t(z, \mathbf{s})}{\delta \tau dz^*} = \left[ \frac{d^c P_t(0, \mathbf{s})}{\delta \tau dz^*} - \int_0^z r_t^c(\tilde{z}, \mathbf{s}) \frac{(1 - \tilde{P}'(y^{\tilde{z}}, \mathbf{s})) (1 - T'(\tilde{z}))}{1 - T'(z^*)} d\tilde{z} \right].$$

Note that the distributions are time invariant because we consider an economy starting from the steady state and the labor supply adjusts instantaneously (no wealth effect). However, private savings may adjust slowly over time, since asset holdings may change slowly. Thus,  $r_t^o(\cdot)$  and  $r_t^c(\cdot)$  can be time varying.

One of the nice features of Saez's (2001) formula is that the optimal tax schedule can be expressed in terms of "sufficient" statistics. The optimal tax rate ( $T'$ ) is decreasing in (i) the Frisch elasticities of the labor supply,  $e$ , (ii) the hazard rate of the income distributions,  $\frac{z^* h(z^*)}{1 - H(z^*)}$ , and (iii) the average social marginal welfare weight of income above  $z^*$ ,  $E[g(z, \mathbf{s})|z \geq z^*]$ .<sup>12</sup> These channels are still operative in formula (1). However, in the presence of a private insurance market, the standard sufficient statistics are not sufficient to pin down the optimal tax schedule. The optimal tax schedule also depends on how the private insurance market interacts with public savings, such as marginal private savings  $\tilde{P}'(\cdot)$  and crowding in/out elasticities,  $r_t^o(\cdot)$  and  $r_t^c(\cdot)$ .

### 2.3.4 Role of a Private Insurance Market

To better understand the role of private insurance, we re-arrange the formula as follows (by combining the terms related to crowding in/out of private insurance):

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<sup>12</sup>The cost of distortion is proportional to the number of workers ( $z^* h(z^*)$ ) at the margin, while the gain from the tax increase (the increased revenue) is proportional to the fraction of income higher than  $z^*$ :  $1 - H(z^*)$ . Thus, the optimal tax rate is decreasing in the hazard rate ( $\frac{z^* h(z^*)}{1 - H(z^*)}$ ). The term  $1 - g(\cdot)$  measures the net benefit of additional lump-sum transfer (lump-sum transfer for all minus extra tax paid by households whose incomes are above  $z^*$ ) as a result of tax reform. Thus, a larger social welfare weight for households above  $z^*$  leads to a lower tax rate.

$$\begin{aligned}
\frac{T'(z^*)}{1 - T'(z^*)} &= (1 - \beta) \sum_{t=0}^{\infty} \beta^t \left[ - \int \tilde{P}'(y^{z^*}, \mathbf{s}) d\Phi(\mathbf{s}|z^*) \right. \\
&\quad + \frac{1}{e} \frac{1 - H(z^*)}{z^* h^*(z^*)} \int_{z^*}^{\infty} \int (1 - \tilde{P}'(y^{z^*}, \mathbf{s})) (1 - g(z, \mathbf{s})) d\Phi(\mathbf{s}|z) \frac{h(z)}{1 - H(z^*)} dz \\
&\quad \left. + \frac{1}{e} \frac{1}{z^* h^*(z^*)} \int (1 - g(z, \mathbf{s})) \frac{d\tilde{P}_t(y^z, \mathbf{s})}{\delta \tau dz^*} d\Phi(z, \mathbf{s}) \right], \tag{2}
\end{aligned}$$

$$\text{where } d\tilde{P}_t(y^z, \mathbf{s}) = \underbrace{-r_t^o(z^*, \mathbf{s})(1 - \tilde{P}'_t(y^{z^*}, \mathbf{s}))\delta\tau dz^*}_{=d^o\tilde{P}_t(y^z, \mathbf{s})} + \underbrace{d^c M_t(z, \mathbf{s})}_{=d^c\tilde{P}_t(y^z, \mathbf{s})} \tag{3}$$

The first term in the bracket on the right-hand side,  $-\int \tilde{P}'(z^* - T(z^*), \mathbf{s}) d\Phi(\mathbf{s}|z^*)$ , reflects the fact that the two types (public and private) of insurance are substitutes. Thus, the optimal marginal tax is decreasing in marginal private savings  $\tilde{P}'(\cdot)$ .

The second term is identical to the original formula in Saez (2001) except for two aspects. First, the Saez (2001) formula is now multiplied by  $(1 - \tilde{P}')$  because a one-unit increase in tax decreases consumption by  $1 - \tilde{P}'$ . The original Saez effects can be either amplified or mitigated depending on the sign of  $\tilde{P}'$ . Second, the integration is now over the cross-sectional distribution of other state variables as well as income. The shape of  $\Phi(z, \mathbf{s})$  can also amplify (mitigate) the Saez effects through the increase (decrease) in consumption inequality.

The third term in the bracket reflects whether the tax reform is aligned with the change in the cross-sectional pattern of private savings. More precisely, this term captures the interaction of “progressivity” between public and private insurance. To see this, note that the integral in the third term is:

$$\int \left\{ (1 - g(z, \mathbf{s})) \cdot d\tilde{P}_t(y^z, \mathbf{s}) \right\} d\Phi(z, \mathbf{s}) = Cov(1 - g(z, \mathbf{s}), d\tilde{P}_t(y^z, \mathbf{s}))$$

where  $g(z, \mathbf{s}) = \frac{G'(u(z, \mathbf{s}))u'(c(z, \mathbf{s}))}{A}$  is the marginal social welfare weights and the term  $d\tilde{P}_t(y^z, \mathbf{s})$  reflects the response of private savings to a tax reform (crowding in or out). The optimal marginal tax is high when the cross-sectional covariance,  $Cov(1 - g(z, \mathbf{s}), d\tilde{P}_t(y^z, \mathbf{s}))$ , is large. In general,  $1 - g(z, \mathbf{s})$  increases (from a negative to a positive value) with income, because the marginal social welfare weight decreases with income. Thus, a progressive tax reform is desirable when such reform makes the private intermediation more progressive. To be more specific, consider a marginal tax increase—i.e., the tax

schedule becomes more progressive. If  $d\tilde{P}_t(y^z, \mathbf{s})$  increases with income levels, the private intermediation becomes more “progressive” (the rich save more in response to a tax reform). This will generate a positive  $Cov(1 - g(z, \mathbf{s}), d\tilde{P}_t(y^z, \mathbf{s}))$ , which in turn results in a high optimal marginal tax. In other words, a tax reform is effective because private intermediation is aligned with the direction of the reform. On the other hand, if private savings become “regressive” in response to a tax increase (the poor save more), the tax reform is not effective because the progressive tax (for insurance) is partially undone by a regressive private intermediation.

In sum, the optimal marginal tax rate is (i) decreasing in the marginal private savings, (ii) increasing in consumption inequality, and (iii) increasing with the alignment between the two insurances.

### 2.3.5 Special Case: Equivalence Result

We investigate the conditions under which the *total* insurance in the presence of a private market is exactly the same as the optimal public insurance without a private market. Under these conditions, the allocations of the two economies will be identical.

**Proposition 3.** *Suppose that the private market structure has the following properties:*

- (i) *the net payment schedule of the private market  $P_t$  is time invariant and depends on current earnings  $z$  only:  $P_t(z, \mathbf{s}; T) = P(z; T)$ , and*
- (ii) *the elasticity of cross crowding in/out in the marginal private intermediation is zero:*

$$r^c(z) = -\frac{d^c \log(1 - \tilde{P}'(y^z))}{d \log(1 - T'(z))} = 0, \quad \forall z.$$

*Then, the allocation under the optimal tax with a private market is equivalent to that under the optimal tax without a private market, and the optimal tax rate satisfies:*

$$\frac{T'(z^*)}{1 - T'(z^*)} = -\tilde{P}'(y^{z^*}) + \frac{1 - r^o(z^*)}{e_{z,1-T'}} \frac{1 - H(z^*)}{z^* h^*(z^*)} (1 - \tilde{P}'(y^{z^*})) \int_{z^*}^{\infty} (1 - g(z)) \frac{h(z)}{1 - H(z^*)} dz \quad (4)$$

**Proof** See Appendix. ■

Whether introducing a private market will improve or reduce welfare depends on the tools and the frictions that government and the private market have. As we discuss below,

if the private market is complete with fully spanned state-contingent assets, there is no role for government insurance. On the other hand, if the private market is incomplete, social welfare may be lower because of the mis-alignment between the two insurances—caused by the externality from the private insurance. Proposition 3 shows that if the private market and the government have identical tools, the total insurance (private and public) is equivalent to the amount of public insurance without a private market, as in Saez (2001).<sup>13</sup>

### 2.3.6 Comparison to Chetty and Saez (2010)

Chetty and Saez (2010) analyze the optimal tax when both public and private savings are linear. They argue for two general lessons: (1) The formula that ignores the existence of private insurance overstates the optimal tax rate. (2) If private insurance does not create moral hazard (in the labor supply), the optimal tax formula is identical with and without private insurance. Our analysis shows that these two properties do not necessarily hold.

The optimal tax rate with private savings can be either higher or lower than those without. First,  $\tilde{P}'$  can be negative, if households are allowed to borrow for consumption smoothing. Second, the presence of private savings can generate a larger consumption inequality (e.g., incomplete markets), which amplifies the Saez (2001) effects. Third, the *cross* crowding-out effect can lead to a positive alignment between public and private insurances. If these effects are dominant, the standard formula that ignores private savings may understate the optimal tax rate.<sup>14</sup> In fact, our quantitative analysis below shows that there are income regions where the optimal tax rates with private savings are higher than those without.

The appearance of additional terms in the optimal tax formula does not necessarily depend on the existence of moral hazard. For example, under an incomplete capital market with self-insurance, even if households' preferences have no income effect on the labor supply—i.e., the labor supply does not depend on wealth (no moral hazard), the

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<sup>13</sup>Since the government would like to achieve the optimal amount of total insurance, the total marginal intermediation  $M'(z)$  should be set according to the standard formula. Note that the relevant elasticity becomes  $e_{z,1-M'}(z^*) = \frac{e_{z,1-T'}(z^*)}{1-r^o(z^*)}$  because a 1% increase in  $1 - T'$  (without private savings) is equivalent to a  $(1 - r^o)$ % increase in  $1 - M'$ .

<sup>14</sup>In Chetty and Saez (2010) where both the tax rate and private savings are linear, the standard tax formula always overstates the degree of public insurance because of the positive savings rate and *own* crowding-out effect.

optimal formula still retains the additional terms that reflect the interaction between private and public insurance. The terms associated with marginal private intermediation ( $\tilde{P}'$ ) should appear in the formula, as long as the private savings schedule depends on after-tax income. On the other hand, the terms associated with the crowding in/out effects will appear when the private insurance is not optimally chosen from the perspective of the government.<sup>15</sup>

### 3 Examples of a Private Insurance Market

Our representation of a private insurance market with  $P(z, \mathbf{s}; T)$  is very general and can capture many different market structures. In this section, we provide several examples of a private insurance market: the complete markets and two stylized incomplete markets. We also illustrate how to extend our formula when there are market failures in private insurance due to preexisting information.

#### 3.1 Two Extreme Cases: Autarky and Complete Insurance

In an autarky economy, every individual consumes after-tax income:  $c_t(z_t) = z_t - T(z_t)$ , and  $P_t(z_t; T) = 0, \forall z_t$ . Thus,  $P'_t(\cdot) = r_t^o(\cdot) = r_t^c(\cdot) = 0$  and  $\Phi(z, \mathbf{s}) = h(z)$ . Then, our optimal tax formula (1) goes back to that in Saez (2001) despite the stochastic productivity shock. Thus, our formula naturally nests the standard optimal tax formula without a private market as a special case.

The opposite extreme of the private insurance market is the complete market with fully spanned state-contingent assets. Then, the private insurance market can achieve full insurance for any tax schedule. More precisely, with the preferences without income effects on the labor supply, consumption net of the labor supply cost is constant across states under any tax schedule:

$$c(z) - v(l(z)) = \tilde{c}, \quad \forall z, \quad \text{for some constant } \tilde{c}.$$

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<sup>15</sup>This inefficiency of private insurance can arise from various sources. For example, if the government has a strong taste for redistribution with concave  $G(\cdot)$ , the social welfare function will not be aligned with the expected utility of each individual. Even if the government maximizes the utilitarian social welfare function, the private insurance decision can generate externalities (e.g., pecuniary externalities or externalities through the change in outside options). But the exact types of externalities do depend on the structure of the private insurance market.

Note that income  $z$  is the sufficient state variable. Using the definition of the private intermediation  $P(z; T) = z - T(z) - c(z)$ , we can represent the complete insurance market by  $P(z) = z - T(z) - \tilde{c} - v(l(z))$  for some constant  $\tilde{c}$ , and thus  $\tilde{P}(y) = y - \tilde{c} - v(l(y))$ . Then, we can easily show that the marginal propensity to save out of after-tax income in the complete market is zero:

$$\tilde{P}'(y) = 1 - v'(l)l'(y) = 1 - \frac{v'(l(y))}{x(1 - T'(xl))} = 0,$$

where the second equality is obtained by applying the inverse function theorem to  $y(l) = xl - T(xl)$ , and the last equality is obtained by the intratemporal optimality condition of the household. The following corollary summarizes the optimal tax formula in these two opposite extreme insurance market examples.

**Corollary 4.** *The optimal tax formula (1) can be simplified in the autarky and the complete market.*

1. *In an autarky economy, our formula goes back to that in Saez (2001):*

$$\frac{T'(z^*)}{1 - T'(z^*)} = \frac{1}{e} \frac{1 - H(z^*)}{z^* h^*(z^*)} \int_{z^*}^{\infty} \left( 1 - \frac{G'(u(z))u'(c(z))}{A} \right) \frac{h(z)}{1 - H(z^*)} dz. \quad (5)$$

2. *Suppose that the private market is complete with fully spanned state-contingent assets. Then, the optimal tax rate is:  $T'(z) = 0, \quad \forall z$ .*

**Proof** We only prove the case with complete insurance market. The first term in the optimal tax formula (1) is zero, because  $\tilde{P}'(y^{z^*}) = 0, \quad \forall z^*$ . Since  $c - v(l)$  is constant,  $G'(u(c(z) - v(l(z))))u'(c(z) - v(l(z))) = A, \forall z$ , for some constant  $A$ . This implies that the marginal social welfare weight is one:  $g(z) = 1, \forall z$ , resulting in  $B_t(z^*) = C_t(z^*) = 0, \forall z^*$  in our optimal tax formula (1). ■

### 3.2 Two Popular Cases: Huggett (1993) and Kehoe and Levine (1993)

In reality, the private insurance market probably lies in between the two extreme cases described above: various factors cause market failures that result in incomplete private insurance. Here, we consider two models of incomplete markets that are widely used in macroeconomics: Huggett (1993) and Kehoe and Levine (1993) to illustrate how to apply our formula (1).

In Huggett (1993), households can trade state-noncontingent assets only and face an (ad hoc) exogenous borrowing constraint. The relevant state variables of a household are current income and asset:  $(z, a)$ . The budget constraint of a household becomes  $c(z, a) = z - T(z) + (1+r)a - a'(a, z)$  where  $a'(a, z)$  is the asset holding in the next period and  $r$  is the equilibrium interest rate. Thus, the private intermediation can be represented by  $P_t(z, a; T) = a'(z, a; T) - (1+r)a$ . Then,  $\tilde{P}'_t$  reflects the marginal propensity to save and  $d\tilde{P}'_t$  measures how the household's savings schedule changes in response to a tax reform taking into account general equilibrium effects. This market structure will be used for our baseline model for the quantitative analysis in Section 4.

In Kehoe and Levine (1993), households can trade fully state-contingent assets but they will face an endogenously determined borrowing constraint due to a limited commitment. Households can deviate from the current contract any time. However, if they default on their private debt, they are banned from the private insurance markets forever—they have to live with their labor incomes only. While this (indefinite autarky) is a common assumption in the literature, our formula can be applied to a more general assumption on the deviation from the contract. For example, we can allow households to trade state-noncontingent assets at the market interest rate even after default.

In this environment, in order to prevent households from deviating, the value of staying in the insurance contract should be higher than the value of default  $U^{aut}(x_t)$  at any time and history (the participation constraint). Since the equilibrium with limited commitment is constrained efficient,<sup>16</sup> the allocation of an equilibrium solves the following planner's problem given the tax schedule  $(T(z))$ :

$$\max \quad \sum_t \sum_{x^t} \beta^t \pi_t(x^t) u(c(x^t) - v(l(x^t))) \quad (6)$$

$$(\beta^t \pi_t(x^t) \lambda(x^t)) \quad \sum_{s \geq t} \sum_{x^s} \beta^{s-t} \pi(x^s | x^t) u(c(x^s) - v(l(x^s))) \geq U^{aut}(x_t) \quad (7)$$

$$(\beta^t \gamma_t) \quad \sum_{x^t} c_t(x^t) \pi_t(x^t) = \sum_{x^t} \{x_t l_t(x^t) - T(x_t l_t(x^t))\} \pi_t(x^t).$$

As in Marcet and Marimon (2016), we define the sum of Lagrange multipliers associated

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<sup>16</sup>A competitive equilibrium given  $\{T(\cdot)\}$  consists of Arrow-Debreu prices of consumption  $\{P_t(x^t)\}$  and allocations  $\{c(x^t), l(x^t)\}$  such that (i) given prices, the allocation solves the households' problem — maximize (6) subject to the participation constraints (7) and the budget constraint  $(\sum_t \sum_{x^t} P_t(x^t) c(x^t) \leq \sum_t \sum_{x^t} P_t(x^t) (x_t l_t(x^t) - T(x_t l_t(x^t))))$ , and (ii) the market clears.

with the participation constraints over the history by  $\xi_t(x^t) = 1 + \sum_{x^s \prec x^t} \lambda(x^s)$ .<sup>17</sup> Then, the consumption allocation should satisfy the optimality condition  $\xi_t(x^t)u'(c(x^t) - v(l_t(x^t))) = \gamma_t$  and the labor supply should satisfy the usual intra-temporal condition  $v'(l_t(x_t)) = x_t(1 - T'(x_t l_t(x_t)))$ . Under the CRRA utility function,  $u(c - v(l)) = \frac{(c - v(l))^{1-\sigma}}{1-\sigma}$ , the consumption net of the cost from the labor supply,  $h_t(x^t) \equiv c_t(x^t) - v(l_t(x_t))$ , is:  $h_t(x^t) = \omega(x^t)H_t$ , where the individual weight is  $\omega(x^t) \equiv \frac{\xi(x^t)^{\frac{1}{\sigma}}}{\sum_{x^t} \xi(x^t)^{\frac{1}{\sigma}}}$  and the aggregate net consumption is  $H_t \equiv \sum_{x^t} h_t(x^t) = \sum_{x^t} (c_t(x^t) - v(l_t(x_t))) = \sum_{x^t} \{x_t l_t(x_t) - T(x_t l_t(x_t)) - v(l_t(x_t))\}$ . Consumption is then determined by  $c_t(x^t) = h_t(x^t) + v(l_t(x^t))$ .

The steady-state allocation can be represented by the state variables  $(x, \omega)$ —or equivalently by  $(z, \omega)$  under the preferences without income effects on the labor supply:  $c(z, \omega) = \omega'(z, \omega) \cdot H + v(l(z))$ , where  $\omega$  and  $\omega'$  are the individual weights in the previous period and current period, respectively. Thus, we can represent the private intermediation by  $P(z, \omega; T) = z - T(z) - c(z, \omega)$ . Here,  $\tilde{P}'$  reflects the binding pattern of the participation constraints, and  $d\tilde{P}$  measures whether the tax reform relaxes or tightens the participation constraints. We also perform a quantitative analysis based on this alternative market structure for private insurance in Section 6 below.

### 3.3 Pre-Existing Information

So far, we have derived and analyzed the optimal tax formula (1) in an economy with *ex ante* identical households. Both the private market and the government provide insurance against the *ex post* income risks. We now extend our formula to allow for a private market with pre-existing information—another type of market failure in private insurance.

As in Chetty and Saez (2010), we consider a simple case of a linear private insurance. Households are heterogeneous *ex ante*—e.g., with respect to the initial wealth or the region of birth—and can be partitioned into  $K$  groups. The invariant distribution of earnings in group  $k$  is  $h_k(z)$  and group  $k$  has a fraction  $\pi_k$  of the population so that  $h(z) = \sum_k \pi_k h_k(z)$ . Private insurance is provided within each group  $k$ , but there is no private insurance against the pre-existing information. With a linear private insurance schedule within group  $k$ , the consumption of individual with income  $z$  is  $c(z, k) = (1 - p_k)y^z + p_k \bar{y}_k$ , where  $p_k$  is

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<sup>17</sup>There are two recursive representations of the market allocation using (i) the promise utility and (ii) the recursive Lagrange multiplier. We illustrate the representation based on the recursive Lagrange multiplier.

constant for given tax schedule and  $y^z = z - T(z)$ . By considering the same tax reform as above, we can extend the optimal formula in the presence of a private insurance market to allow for pre-existing information.

**Proposition 5.** *Consider an economy with ex ante heterogeneous  $K$  groups, and there is a linear private insurance within the group. Then the optimal marginal tax rate at income  $z^*$  should satisfy:*

$$\begin{aligned} \frac{T'(z^*)}{1 - T'(z^*)} = & - \sum_k \pi_k \frac{A_k}{A} p_k \frac{h_k(z^*)}{h(z^*)} \\ & + \frac{1}{e} \frac{1 - H(z^*)}{z^* h^*(z^*)} \sum_k \pi_k (1 - p_k) \frac{(1 - H_k(z^*))}{1 - H(z^*)} \int_{z^*}^{\infty} \left( \frac{A_k}{A} - g(z, k) \right) \frac{h_k(z)}{1 - H_k(z^*)} dz \\ & - \frac{1}{e} \frac{1}{z^* h^*(z^*)} \sum_k \pi_k \frac{(1 - p_k) r_k}{1 - T'(z^*)} \int \left( \frac{A_k}{A} - g(z, k) \right) z h_k(z) dz \\ & + \frac{1}{e} \frac{1}{z^* h^*(z^*)} \sum_k \pi_k \frac{A_k}{A} (H_k(z^*) - H(z^*)), \end{aligned} \quad (8)$$

where  $A_k = \int G'(u) u'(c) h_k(z) dz$ ,  $g(z, k) = \frac{u(c(z, k))}{A}$ ,  $r_k = -\frac{d(1 - p_k)}{d(1 - T')} \frac{1 - T'}{1 - p_k}$ .

**Proof** See the appendix. ■

The first three terms (8) are similar to those in (1) except that they now represent the welfare effects within groups. Note that in the third term the change in the private insurance schedule  $\frac{d\tilde{P}_t(y^z, k)}{\delta \tau dz^*}$  is exactly  $-\frac{(1 - p_k) r_k}{1 - T'(z^*)}$ . In the economy with pre-existing information, however, there is a new term—the last line of (8)—that reflects the redistribution across the groups.

## 4 A Quantitative Analysis

### 4.1 Structural Sufficient-Statistics Approach

A powerful feature of Saez (2001) is that the optimal tax schedule can be expressed in terms of “sufficient” statistics—such as the Frisch elasticity of the labor supply and the cross-sectional distributions of income and marginal utility—which can be estimated or imputed from the data. In the presence of a private market, however, it is far more challenging because the formula includes additional statistics that capture the interaction between private and public insurance, which are difficult to obtain from the available data.

Most important, the formula requires the relevant statistics and the distribution of the economy at the *optimal steady state*, which is hard to observe, unless the current tax schedule is already optimal. While the same is true in Saez (2001), given the elasticity of the labor supply, one can still infer the optimal distribution of hours and consumption from an exogenously given distribution of productivity and tax schedule in a static environment. This is no longer the case in a dynamic environment with private savings. We need to know the consumption *rule* and distribution over individual states (e.g., productivity and assets) under the optimal tax. Moreover, these statistics are not policy invariant in general. Thus, it requires out-of-sample predictions. Second, the optimal tax formula involves very detailed micro estimates —e.g., marginal private savings across individual state variables.<sup>18</sup> The formula also requires the crowding in/out elasticities,  $r_t^o(z, \mathbf{s})$  and  $r_t^c(z, \mathbf{s})$ , along the transition path of each alternative tax reform.

Faced with these difficulties, we combine the structural and sufficient-statistics methods, following the suggestion by Chetty (2009). We compute the optimal tax schedule using quantitative general equilibrium models calibrated to match some salient features of the U.S. economy. We consider two incomplete markets that are widely used in macroeconomic analysis: Huggett (1993) and Kehoe and Levine (1993).

## 4.2 Baseline Model: Huggett (1993)

We consider a variant of Huggett (1993) for two reasons. First, it is widely used in many macroeconomic analyses. Second, the analytical formula requires the aggregate private intermediation to be a constant. In Huggett (1993), the aggregate private saving is zero (e.g.,  $\int P = 0$ ): a pure insurance market. In this environment, the private savings market is incomplete in two senses: (i) the only asset available for private insurance is a state-noncontingent bond  $a_t$ , and (ii) there is an exogenous borrowing limit:  $a_{t+1} \geq \underline{a} (< 0)$ .

In this economy, the individual state variables are asset holdings  $a$  and productivity  $x$ . The consumption of a worker with asset  $a$  and productivity  $x$  is  $c(a, x) = xl - T(xl) + (1 + r)a - a'$ , where  $a'$  is the asset holdings in the next period. Private savings of a worker

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<sup>18</sup>While there are empirical analyses on the marginal propensity to consume (MPC)—e.g., Jappelli and Pistaferri (2014) and Sahm, Shapiro, and Slemrod (2010), these estimates are available for the average or coarsely defined groups of households only.

with asset  $a$  and productivity  $x$  are  $P(a, xl) = xl - T(xl) - c(a, x) = a'(a, x) - (1 + r)a$ . The aggregate private savings sum to zero in equilibrium:  $\int a' = 0$ .

The government in our model economy spends its tax revenue on (i) purchasing goods  $E$  (which does not enter into the households' utility) and (ii) a lump-sum transfer. When we consider a revenue-neutral tax reform, we assume that the government purchase  $E$  remains unchanged. Then, the government's budget constraint is:  $\int T(z)h(z)dz = \bar{E}$ .

We assume that the individual productivity  $x$  can take values from a finite set of  $N$  grid points  $\{x_1, x_2, \dots, x_N\}$  and follows a Markov process that has an invariant distribution. We approximate an optimal nonlinear tax and private intermediation with a piecewise-linear over  $N$  grid points.<sup>19</sup>

### 4.3 Calibration

#### *Preferences, Government Expenditure, and Borrowing Constraints*

The utility function of households is assumed to be a constant relative risk aversion (CRRA):

$$u(c, l) = \frac{(c - v(l))^{1-\sigma}}{1-\sigma}, \quad v(l) = \frac{l^{1+1/e}}{1+1/e},$$

where  $\sigma = 1.5$  and the Frisch elasticity of the labor supply ( $e$ ) is 0.5. We choose the discount factor ( $\beta$ ) so that the rate of return from asset holdings is 4% in the steady state. The government purchase  $\bar{E}$  is chosen so that the government expenditure-GDP ratio is 0.188 under the current U.S. income tax schedule (approximated by a log-linear functional form:  $T(z) = z - \lambda z^{1-\tau}$ ) as in Heathcote, Storesletten, and Violante (2014).<sup>20</sup> The exogenous borrowing constraint ( $\underline{a} = -86.87$ ) is set to the average earnings of our model economy under the current U.S. tax schedule.<sup>21</sup> Under this borrowing limit, 9.7% of households are credit-constrained in the steady state. Finally, we assume that the social

<sup>19</sup>More precisely,  $T(z) = T(0) + \sum_{k=1}^{i-1} T'_k(z_{x_k} - z_{x_{k-1}}) + T'_i(z - z_{x_{i-1}})$ ,  $z_{x_{i-1}} < z \leq z_{x_i}$ , and  $\tilde{P}(y) = \tilde{P}(y_{x_0}) + \sum_{k=1}^{i-1} \tilde{P}'_k(y_{x_k} - y_{x_{k-1}}) + \tilde{P}'_i(y - y_{x_{i-1}})$ ,  $y_{x_{i-1}} < y \leq y_{x_i}$ , where  $z_{x_0} = 0$  and  $y_{x_0} = -T(0)$ . Consider a tax reform with an alternative marginal tax rate—suggested by the right-hand side of optimal tax formula (1)—on a grid point  $T'_i$ ,  $i = 1, \dots, N$ . If the tax reform for every grid point no longer improves social welfare—i.e., Equation (1) is satisfied, the optimal tax schedule is found.

<sup>20</sup>Given the estimated value for the progressivity,  $\tau^{US} = 0.161$  from Heathcote, Storesletten, and Violante (2014), we set  $\lambda$  to match the government expenditure-GDP ratio ( $\frac{\bar{E}}{Y}$ ).

<sup>21</sup>This is largely in line with consumer credit card limits (which is around 50% - 100% of average earnings) in the data. For example, according to Narajabad (2012), based on the 2004 Survey of Consumer Finances data, the mean credit limit of U.S. households is \$15,223 measured in 1989 dollars.

welfare function is utilitarian:  $G(\cdot)$  is linear. Table 1 summarizes the parameter values in our benchmark case. In Section 4.6 and the appendix, we do sensitivity analysis.

Table 1: Benchmark Parameter Values

Parameter	Description
$\sigma = 1.5$	Relative Risk Aversion
$\beta = 0.9032$	Discount Factor
$e = 0.5$	Frisch Elasticity of Labor Supply
$\underline{a} = -86.87$	Borrowing Constraint
$\frac{\bar{E}}{\bar{Y}} = 0.181$	Government Expenditure to GDP Ratio under U.S. Tax
$G''(\cdot) = 0$	Utilitarian Social Welfare Function
$\rho_x = 0.92$	Persistence of Log Productivity (before modification)
$\sigma_x = 0.561$	S.D. of Log Productivity
$\frac{xf(x)}{1-F(x)} = 2$	Hazard Rate at Top 5% of Wage (Income) Distribution

### *Productivity Process*

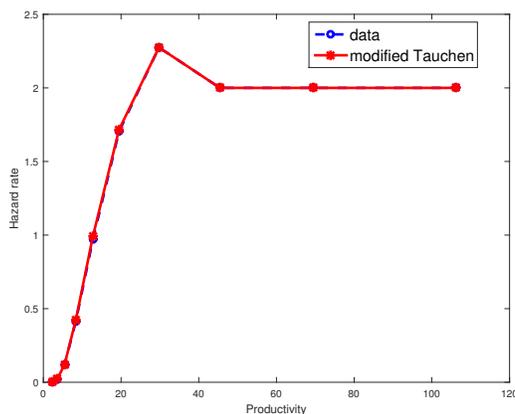
As shown in formula (2), the shape of the income distribution (which is dictated by the stochastic process of a productivity shock under our preferences with no wealth effect in the labor supply) is crucial for the optimal marginal tax schedule. We generate an empirically plausible distribution of productivity as follows. Consider an AR(1) process for log productivity  $x$ :  $\ln x' = (1-\rho)\mu + \rho \ln x + \sigma_\epsilon \epsilon'$ , where  $\epsilon$  is distributed normally with mean zero and variance one. The cross-sectional standard deviation of  $\ln x$  is  $\sigma_x = \frac{\sigma_\epsilon}{\sqrt{1-\rho^2}}$ . While this process leads to stationary log-normal distributions of productivity and earnings, it is well known that the actual distributions of productivity (wages) and earnings have much fatter tails than a log-normal distribution.<sup>22</sup>

We modify the Markov transition probability matrix to generate a fatter tail as follows. First, we set the persistence of the productivity shock to be  $\rho = 0.92$  following Floden and Linde (2001), which is based on PSID wages and largely consistent with other estimates in the literature. We obtain a transition matrix of  $x$  in a discrete space using the Tauchen

<sup>22</sup>Saez (2001) and Heathcote, Storesletten, and Violante (2014) estimate the earnings distribution and use tax data to obtain the underlying skill distribution, while Mankiw, Weinzierl, and Yagan (2009) use the wage distribution as a proxy for the productivity distribution.

(1986) method, with  $N = 10$  states and  $(\mu, \sigma_x) = (2.757, 0.5611)$ , which are Mankiw, Weinzierl, and Yagan’s (2009) estimates from the U.S. wage distribution in 2007. We set the end points of the grid at  $(x_1, x_N) = (\exp(\mu - 3.4\sigma_x), \exp(\mu + 3.4\sigma_x))$ .<sup>23</sup> Second, in order to generate a fat right tail, we modify the transition matrix.<sup>24</sup> Third, we also increase the transition probability of the lowest grid,  $\pi(x_1|x)$ , so that the stationary distribution has a little bit fatter left tail.<sup>25</sup> As Figure 1 shows, the hazard rates of the productivity distribution from our model almost exactly match those in the wage distribution in the data. In Section 4.6, we also study the model economy under a simple log-normal distribution of productivity to examine the impact of fat tails.

Figure 1: Hazard Rates of Wage (Productivity)



*Note:* The hazard rates are from Mankiw, Weinzierl, and Yagan (2009).

#### 4.4 Indirect Diagnostics

We will compute the values of these additional statistics—the marginal private savings ( $\tilde{P}'$ ), the cross-sectional distribution of income and assets ( $\Phi$ ), and the crowding in/out elasticities ( $r^c$  and  $r^o$ ) under the optimal tax schedule—using our quantitative model. Unfortunately, we cannot directly evaluate the empirical fit of these statistics because they are impossible to observe, unless the current tax is already optimal. However, it might still be of interest to compare these statistics in our model under the *current*

<sup>23</sup>We set the highest grid point to 3.4 standard deviations of log-normal so that the highest productivity is the top 1% of the productivity distribution in Mankiw, Weinzierl, and Yagan (2009).

<sup>24</sup>We increase the transition probability  $\pi(x'|x)$  of the highest 3 grids so that the hazard rate of the stationary distribution is  $\frac{xf(x)}{1-F(x)} = 2$  for the top 5% of productivities. This hazard rate of 2 for the top 5% is based on the empirical wage distribution in Mankiw, Weinzierl, and Yagan (2009).

<sup>25</sup>The bottom tail of the productivity distribution should take into account disabled workers or those not employed.

tax schedule to the available estimates in the literature as an indirect diagnostic of our quantitative model.

### Distribution of $\tilde{P}'$

First, we compare the marginal propensity to consume ( $\text{MPC} = 1 - \tilde{P}'$ ) in our model (under the current U.S. tax schedule approximated by the HSV form) to the existing empirical values in the literature. While there are ample empirical studies on the MPC, the MPCs across detailed income and asset levels are not available. Most estimates of MPC are based on the 2001 and 2008 tax rebate policies (e.g., Johnson, Parker, and Souleles (2006) and Sahm, Shapiro, and Slemrod (2010) among others). The estimated MPCs in the literature vary between 0.2 and 0.4. Using a quantile regression method, Misra and Surico (2011) report a wide range of heterogeneity in MPC across households. Jappelli and Pistaferri (2014) also provide detailed MPCs by income and financial assets using the 2010 Italian Survey of Household Income and Wealth.

The average MPC in our benchmark model is 0.85, much higher than the 0.48 reported by Jappelli and Pistaferri (2014) or the 0.33 in Sahm, Shapiro, and Slemrod (2010). This gap seems inevitable because the income process is highly persistent in our model—making the MPC close to 1, whereas most empirical estimates are based on idiosyncratic events associated with temporary changes in income, such as tax rebates, which typically imply a small MPC.<sup>26</sup> For this reason, it is not fair to directly compare the levels of MPC between the model and the available estimates. Thus, we rather focus on the *relative* MPCs across different income and asset groups, for which the model is not very far from the data. Table 2 compares the MPCs in our model to those in Jappelli and Pistaferri (2014)—the average and those at the 1st and 5th quintiles (the bottom and top 20%) in the income and asset distributions. The MPCs in the data at the 1st and 5th quintiles are computed using the regression coefficients on dummy variables for the corresponding group (from Table 4 in Jappelli and Pistaferri (2014)). For example, according to Jappelli and Pistaferri (2014), the households in the 1st quintile of the income distribution exhibit MPCs that are 9 to 12% higher than the average MPC of the entire sample, whereas in our model their average MPC is 17% larger than the population average. The households

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<sup>26</sup>In fact, the average MPC of a Huggett-style incomplete market New Keynesian model (Dupor, Karabarbounis, Kudlyak, and Mehkari (2017)) is 0.22 for a one time unexpected increase in income, which is well within the range of empirical estimates.

Table 2: Relative MPC by Income and Assets

	<u>By Income</u>		<u>By Assets</u>	
	Data	Model	Data	Model
Bottom 20%	+9 ~ +12%	+17%	+22 ~ +25%	+14%
Top 20%	-14 ~ -11%	-16%	-30 ~ -22%	-12%

*Notes:* The numbers represent the average MPC of each group relative to the entire sample mean (0.48 in the data and 0.85 in the model). The data statistics are based on Jappelli and Pistaferri (2014).

at the 5th quintile show the MPCs that are 11 to 14% smaller than the entire sample average in the data, whereas they are 16% smaller than the average in our model. Thus, the model generates MPCs that are a little bit more dispersed than those in the data. By assets, the model generates MPCs that are somewhat less dispersed than those in Jappelli and Pistaferri (2014).

### Distributions of Income and Assets

Our model is designed to match the income distribution of the U.S. economy fairly well because we calibrate the stochastic process of productivity to exactly match the hazard rates of the wage distribution in Mankiw, Weinzierl, and Yagan (2009) (shown in Figure 1). Table 3 shows that the Gini coefficient of earnings in our model is 0.48, not far from those of the U.S. (0.53 – 0.67). The distribution of assets is not necessarily close to that in the data. While the Gini coefficient of wealth in our model is 0.9, even higher than those in the data (0.76-0.86), this comparison is misleading. Given that our model requires zero aggregate savings in equilibrium, there are a large number of households with negative assets. Thus, the Gini is not an appropriate measure and we need a dispersion measure that can accommodate a large fraction of the population with negative values. Instead we report the relative dispersion such as  $\frac{a_{80}-a_{20}}{a_{60}-a_{40}}$  where  $a_{80}$  is asset holdings at the 80th percentile of the asset distribution. According to Table 3, the model generates an asset distribution whose dispersion is fairly close to that in the data for a wide range of distributions. For example, the relative dispersions in the model are  $\frac{a_{90}-a_{10}}{a_{60}-a_{40}} = 3.9$ ,  $\frac{a_{90}-a_{10}}{a_{60}-a_{40}} = 8.0$ , and  $\frac{a_{95}-a_{05}}{a_{60}-a_{40}} = 15.2$ , fairly close to 3.9, 8.6, and 17.3, respectively, in the data. But the dispersion at the tail 2% of the asset distribution,  $\frac{a_{99}-a_{01}}{a_{60}-a_{40}}$ , is only 34 in the

model, much smaller than the 72 in the data. As is well known, this type of incomplete-markets models has difficulty in generating super-rich households. The income and assets are somewhat more strongly correlated in the model (with a correlation coefficient of 0.68) than they are in the data (0.53).

Table 3: Distribution of Assets

	Data	Model
Gini (earnings)	0.53-0.67	0.48
Gini (assets)	0.76-0.86	0.90
corr(assets, earnings)	0.53	0.69
$\frac{a_{80}-a_{20}}{a_{60}-a_{40}}$	3.9	3.9
$\frac{a_{90}-a_{10}}{a_{60}-a_{40}}$	8.6	8.0
$\frac{a_{95}-a_{05}}{a_{60}-a_{40}}$	17.3	15.2
$\frac{a_{99}-a_{01}}{a_{60}-a_{40}}$	72.8	34.3

*Notes:* The data statistics are based on Ríos-Rull and Kuhn (2016) and Chang and Kim (2006).  $a_{80}$  denotes asset holdings at the 80th percentile of the asset distribution.

## 4.5 Optimal Tax Schedule

In our quantitative analysis, for computational convenience, we focus on the tax rates that maximize the steady-state social welfare only, then the tax formula is simplified to<sup>27</sup>

$$\begin{aligned}
 \frac{T'(z^*)}{1 - T'(z^*)} &= -E \left[ \tilde{P}'(z^* - T(z^*), a) | z^* \right] \\
 &+ \frac{1 - H(z^*)}{z^* h(z^*)} \frac{1}{e} E \left[ (1 - g(z, a)) (1 - \tilde{P}'(y^z, a)) | z \geq z^* \right] \\
 &+ \frac{1}{z^* h(z^*)} \frac{1}{e} E \left[ (1 - g(z, a)) \frac{d\tilde{P}(y^z, a)}{\delta \tau dz^*} \right].
 \end{aligned} \tag{9}$$

Using this formula, we compute the optimal tax schedule quantitatively. We start with a given  $T'$ . We compute the competitive equilibrium for given  $T'$  and the schedule with the tax reform to get the statistics  $(\tilde{P}', dP)$ , distribution, and consumption across states. We then use the formula to compute the new vector of  $T'$ . More precisely, we use the formula with respect to the exogenous productivity distribution—see Appendix A. We

<sup>27</sup>If households can adjust their asset holdings without much friction, the steady-state comparison will be similar to that taking into account the transition path.

repeat the algorithm until the vector converges to a fixed point. This algorithm is a modification of the one used in Brewer, Saez, and Shephard (2010).

Figures 2 and 3 show the optimal marginal tax schedule across productivity and income, respectively, with and without a private insurance market. We normalize the units of quantities in our model so that the average productivity (wage) is \$20 and the average labor income is \$40,000 (comparable to those in 2015 in the U.S.). Without a private insurance market (dotted line), the optimal marginal tax schedule exhibits a well-known U-shape as in the standard Mirrleesian taxation literature (Diamond (1998), Saez (2001)). High marginal tax rates at the very low income levels indicate that net transfers to low-income households should quickly phase out. As seen in Figure 1, the hazard rate of productivity sharply increases, implying that the cost of distorting the labor supply quickly increases (relative to the benefit): the optimal marginal tax rate should start decreasing with income. As income increases, the marginal social welfare weight gradually diminishes—which eventually becomes a dominant factor and results in a higher marginal tax at the high-income group.

Figure 2: Optimal Marginal Tax by Productivity

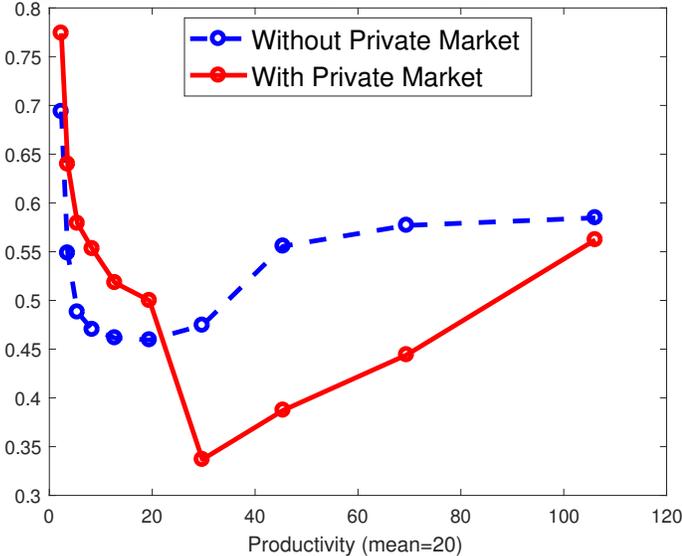
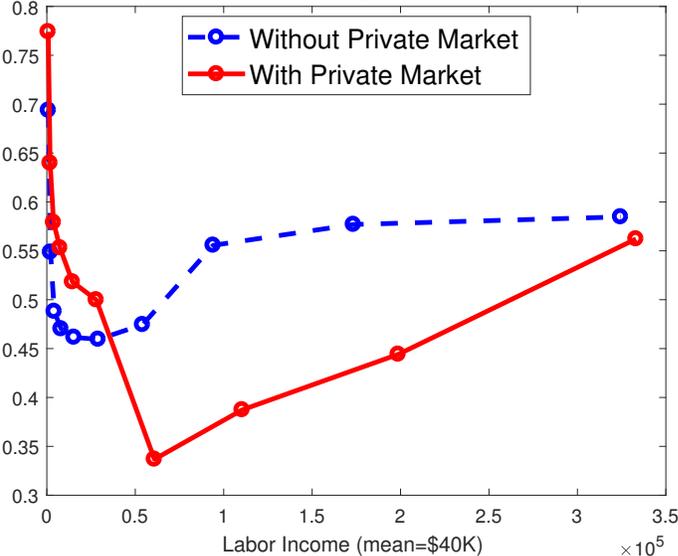


Figure 3: Optimal Marginal Tax by Income



While the same driving forces are operative in an economy with a private insurance market, there are additional factors that make the optimal tax schedule different from that without a private insurance market. Looking at Figures 2 and 3 again, the optimal tax rates in the presence of private insurance (solid line) are higher than those without a private market (dotted line) at the low- and middle-income group (wage rates less than

\$25). For the upper-middle and high-income groups (wage rates between \$25 and \$100), the optimal tax rates are lower than those without private insurance. For the income group at the top (wage rates above \$100, which is approximately the 95th percentile of the distribution), the tax rates with and without private insurance are similar.

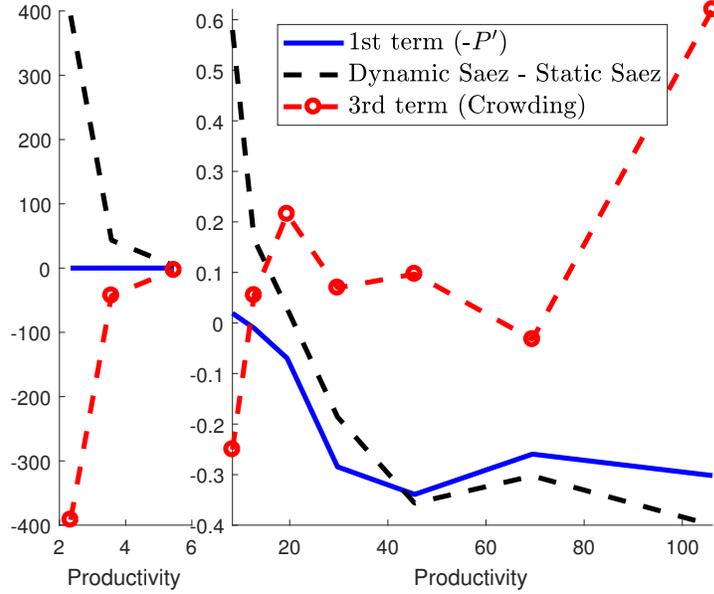
We now examine the factors that account for the difference in optimal tax rates with and without private insurance in detail. Comparing our optimal tax formula (9) to that of Saez (2001), the difference between the two formulas is made up of three components. The first component simply reflects the fact that private savings and government tax/transfer are substitutes in insuring against future income uncertainty—the first line in our formula ( $-\tilde{P}'$ ). The second component is the difference between the second term in our formula and the original Saez formula—i.e., “dynamic Saez vs. static Saez.” The third component reflects the fact that a tax reform is more effective when the response of private savings is aligned with such reform—the last term in our formula ( $E[(1 - g)\frac{d\tilde{P}}{\delta\tau dz^*}]$ ).

As we discussed in section 2.3.4, the sign and magnitude of these three components are determined by marginal private savings (MPS)  $\tilde{P}'$ , consumption inequality across income and asset, and the crowding in/out of private savings. Figure 4 plots each of these three components. The solid line represents the first term (substitution effect). The dashed line represents the difference between the dynamic Saez and static Saez. The dashed line with circles represents the third term, the crowding effect. Since the magnitude of the first and second terms is huge at the very low productivity level, we plot these terms separately in two productivity groups: productivity below \$8 and above.

At the low-income group (whose wages are less than \$8), marginal private savings are negative ( $\tilde{P}' < 0$ ), as shown in Figure 5. Low-income households would like to borrow more at the margin. This has two impacts. First, the government increases the marginal tax rate to achieve the optimal total savings (the first term in the formula). Second, the negative marginal private savings rate amplifies the Saez effects (the second term). While the third component (dashed line with circles) shows negative values—the private saving schedule becomes less progressive, the first and second components dominate this third component. Thus for this bottom income group, optimal tax rates are higher than those without private insurance.

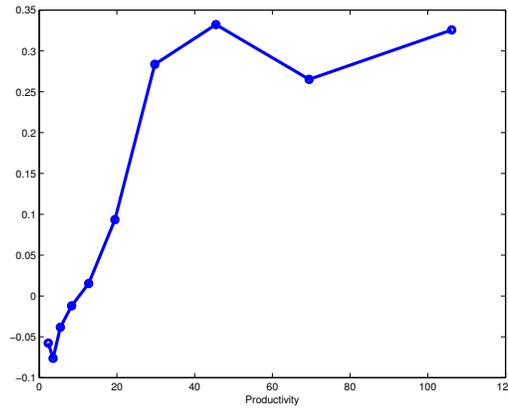
At the low-middle-income group (whose wages are between \$8 and \$25) marginal private

Figure 4: Decomposition of the Difference in  $\frac{T'}{1-T'}$  with and without private insurance



*Note:* The “1st term ( $-P'$ )” (solid line) represents the first term in our optimal tax formula (9). “Dynamic Saez - Static Saez” is the second term in our formula (9) minus the original Saez. The “3rd term (crowding)” represents the third term in our formula (9).

Figure 5: Marginal Private Savings  $P'$



savings now become positive: workers would like to save with an additional income. This would lower the optimal marginal tax. However, the second component (dynamic Saez - static Saez) is still positive and big because of the large consumption inequality generated by an unequal wealth distribution. The third component now turns from a negative to a positive value — the private insurance schedule becomes more “progressive”—which calls for a higher tax rate. Overall, the second and third components dominate the first

component, and thus, the optimal tax rate is higher in the presence of private insurance.

At the upper-middle and high-income groups (whose wages are higher than \$25), marginal private savings are always positive, which calls for a lower optimal tax rate as private saving is a good substitute for public insurance. Moreover, the difference between the dynamic and static Saez terms is now negative — the Saez effects are mitigated:  $(1 - P') < 1$ . While the third term (the alignment of private savings) is positive — private insurance becomes more progressive — the first and second components dominate the third component.<sup>28</sup> At the very top income group (whose wage rates are higher than \$100), these effects almost offset each other as the alignment term gets stronger, leaving the optimal tax rate similar to that without a private insurance market.<sup>29</sup>

In sum, the difference in tax rates with and without a private insurance market is quantitatively important, as the difference between the two can be as large as almost 20 percentage points. According to our benchmark model, for the low- and lower-middle-income group, the optimal tax rate is higher in the presence of a private market, mainly because of the amplified Saez effects. Higher consumption inequality driven by asset inequality makes the Saez effects stronger. Moreover, for the very low-income group, households try to borrow at the margin ( $1 - P' > 1$ ), which amplifies the Saez effects even more. For the middle- to high-income groups where marginal savings is positive, the substitution effect (the first term) and mitigating effect in Saez (the second term) dominate the crowding in/out effects (the third term), resulting in lower tax rates in the presence of a private market. At the very top income distribution, these forces almost cancel out each other, leaving the tax rate with private savings close to that without.

Figure 6 shows the marginal total intermediation across productivity levels,  $E[M'(z, a)|z] = T'(z) + E[P'(z, a)|z]$ . It clearly shows that the total marginal intermediation is larger in

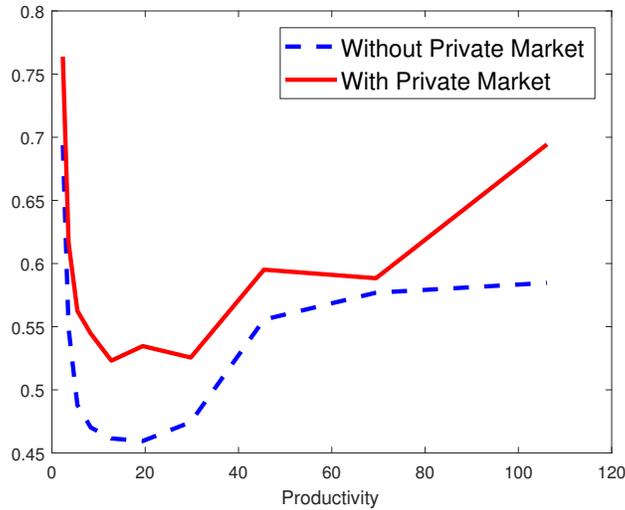
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<sup>28</sup>At the productivity level between \$50 and \$70, the marginal savings rate slightly falls. This pattern is due to our modification of the transition probability to match the fat tail at the top. Recall that we generated a fatter tail by increasing the probability of drawing higher productivity at the two highest productivity grids. This leads to a decrease in marginal private savings because once you reach this level of productivity, it is more likely to stay at a high productivity level (less need for savings). In Section 4.6 below, this pattern disappears when we use a pure log-normal productivity process.

<sup>29</sup>The third term is highly positive at the top, because increasing the tax rate at the top — which leads to a more progressive tax system — makes the rich save more than the poor. More public insurance reduces the precautionary savings, especially for low-income households that are credit constrained. This reduced precautionary saving leads to a higher interest rate in equilibrium, which induces saving for the rich and reduces borrowing for the poor.

the presence of a private market. This is mainly driven by the high marginal propensity to save from the precautionary savings motive in our model economy. This implies that in the presence of a private market, the government cannot perfectly reproduce the optimal insurance in an economy without a private market, because households (with an individual-specific history of productivity shocks) have different precautionary motives for saving. This also indicates that social welfare in the economy with a private market can be lower than that in the economy without a private market.

Figure 6: Marginal Total Intermediation: (Tax + Savings)



We now compare social welfare under three tax schedules: the optimal tax with a private market (based on our formula), the optimal tax without a private market (Saez (2001)) and the current U.S. tax system (approximated by a log-linear form as in HSV). Social welfare is compared based on a constant-compensating differential in (steady-state) consumption. The welfare cost (or gain) of a tax system  $T$  relative to our optimal tax formula ( $T^*$ ) is  $\Delta$  that satisfies:

$$\int V(x, a; \Delta) d\Phi(x, a; T) = SWF(T^*)$$

$$V(x, a; \Delta) = u((1 + \Delta)c(x, a) - v(l(x, a))) + \beta E[V(x', a'(x, a); \Delta) | x]$$

where  $\Phi(a, x, T)$  is the steady-state distribution under tax system  $T$  and  $SWF(T^*)$  is the steady-state social welfare under our optimal tax (with a private market) schedule  $T^*$ . According to this measure, the welfare cost of the U.S. tax system compared to our optimal tax schedule is 8%. As we discussed above, it may generate higher welfare if the

government simply shuts down the private insurance market and adopts the optimal tax without a private market (i.e., the original Saez (2001) formula). Indeed, social welfare under the optimal tax without a private market is higher than that with a private market by 8.5% (i.e.,  $\Delta = -8.5\%$ ).

This result is not actually surprising, given that the market structure of private intermediation assumed in our quantitative analysis is rather primitive—the only asset available for households is a noncontingent bond for self-insurance. If the private market has richer tools for intermediation and faces less frictions than the government does, the welfare results can be quite different: social welfare under the optimal tax in the presence of a private market could be higher than that under the optimal tax without a private market.

Finally, we compare our optimal tax schedule to the current U.S. income tax rates. Figure 7 compares the optimal marginal tax rates implied by our model (solid line) to the current U.S. income tax schedule approximated by the HSV functional form (dotted line). First, the optimal marginal tax rates are higher than the current rates in the U.S. for the low and top income groups.<sup>30</sup> However, for the middle-income group (i.e., individual income ranges between \$70K and \$250K), the current tax rates are close to optimal. This result is very different from those without private insurance seen in Figure 3, where the optimal tax rates are much higher than the current ones.

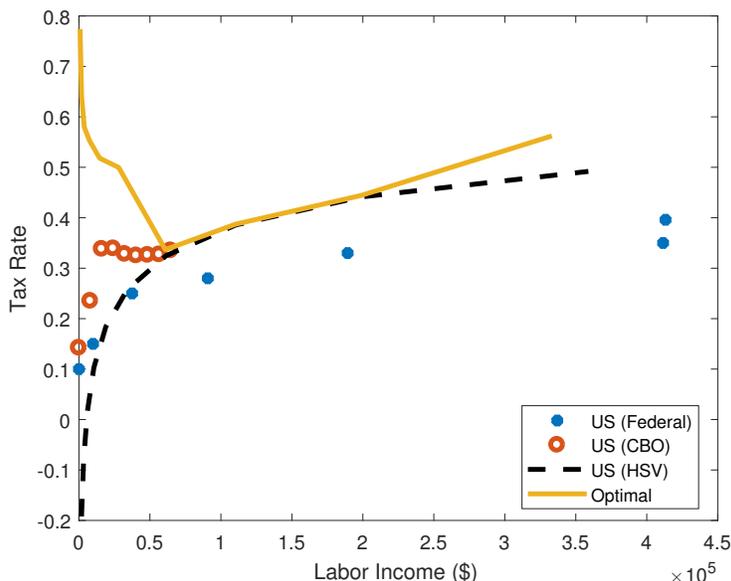
## 4.6 Comparative Statics

In this section, we investigate how the optimal tax schedule changes with respect to different specifications on (i) the right tail of the income distribution (log-normal rather than Pareto) and (ii) the persistence of productivity shocks. For each alternative specification, we find a new value for the time discount factor  $\beta$  to clear the private insurance market at the given interest rate  $r = 4\%$  under the current U.S. tax schedule (approximated by a log-linear form as in HSV). Simultaneously, we recalibrate the exogenous borrowing limit  $\underline{a}$  so that about 10% of households are credit-constrained in the steady state. In the

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<sup>30</sup>Heathcote and Tsujiyama (2017) find that the optimal tax schedule is close to a log-linear form. There are at least two important differences between the results of Heathcote and Tsujiyama (2017) and ours. First, we match the exact shape of the hazard rate of the productivity distribution in the data, while they approximate the productivity distribution by an exponentially modified Gaussian. Second, they assume a complete separation between perfectly insurable and noninsurable productivity shocks, whereas we assume a partial insurance market.

Figure 7: Marginal Tax Rate: Current vs. Optimal



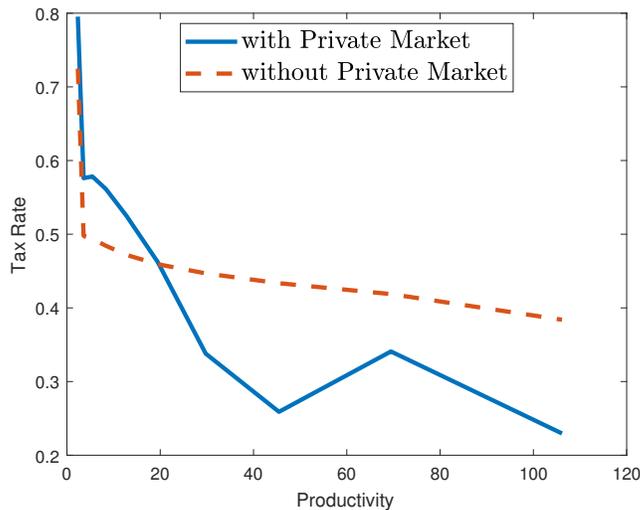
*Note:* “US (Federal)” reflects the statutory federal income tax rates for singles in 2015. “US (CBO)” shows the median of effective marginal tax rates for low- and moderate-income workers (single parent with one child) in 2016 published by the CBO.

appendix, we also carry out the sensitivity analysis with respect to other parameters of the model economy such as relative risk aversion, the Frisch elasticity, and the borrowing constraint. The results are consistent with our economic priors. The optimal tax rates are higher when (i) households are more risk averse, (ii) the labor supply is inelastic, and the role of private insurance is less significant when the borrowing constraint is tighter.

#### 4.6.1 Log-normal Distribution of Income: Effects of Fat Tails

In the benchmark analysis, we have modified the transition probability (from the discretized log-normal distribution) to match the fat tail in the income (and wage) distribution in the data. To examine the role of the fat tail, we compute the optimal tax under a pure log-normal productivity process without modification. The hazard rate  $\frac{xf(x)}{1-F(x)}$  of the log-normal distribution monotonically increases. This results in the monotonically decreasing tax rate without a private insurance market in Figure 8. The pattern prevails in the presence of private insurance, suggesting that the fat tail is crucial for the U-shaped optimal marginal tax schedule.

Figure 8: Log-normal Distribution of Productivity



#### 4.6.2 Persistence of Productivity Shock

Note that the persistence of the productivity shock,  $\rho$ , does not appear in the optimal tax formula (9) because we restrict our tax system to be noncontingent on history and the government maximizes steady-state welfare only in the benchmark analysis. However, the persistence of shocks affects households' savings pattern and, as a result, the optimal tax rate in the presence of a private insurance market. We examine the model with  $\rho = 0.8$  (lower persistence). We recalibrate the standard deviation to the innovation  $\sigma_\epsilon$  to obtain the same standard deviation of log productivity,  $\sigma_x = 0.561$ , in the benchmark. We also modify the transition probability matrix at both ends of the productivity distribution to match the hazard rates in the data, as we did in our benchmark case.

Figure 9 shows that the optimal tax rates under  $\rho = 0.8$  are smaller than those in the benchmark model except for the very low-income (productivity) group. This is mainly driven by the marginal private savings ( $P'$ ) shown in Figure 10. At the low-income levels (wage rate below \$10), workers save less (borrow more) at the margin under  $\rho = 0.8$  because they expect that productivity will increase in the near future. This leads to higher tax rates via substitution effects and amplification of the Saez effects. For higher-income levels (wage rate above \$10), the optimal tax rates under  $\rho = 0.8$  are higher because these workers save more at the margin, which leads to a lower marginal tax via substitution effects and mitigation of the Saez effects.

Figure 9: Optimal Tax with Private Market

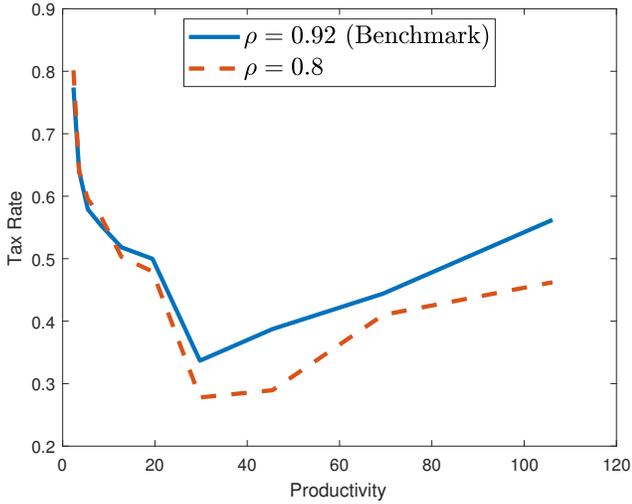
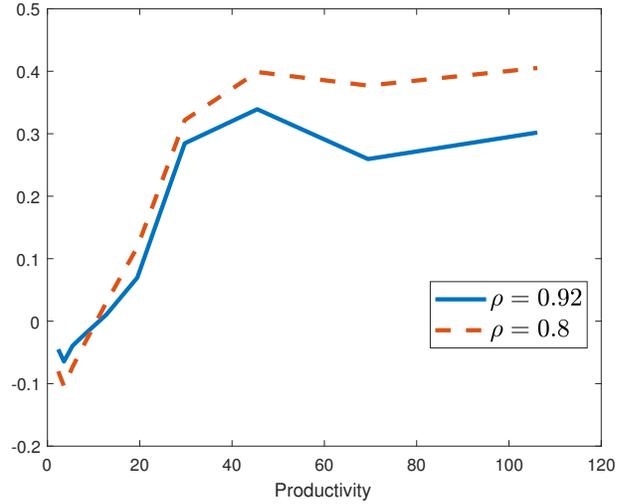


Figure 10: Marginal Private Savings



## 5 Alternative Market Arrangement

In our baseline economy (Huggett (1993)) of the quantitative analysis, the private insurance market is not complete for two reasons: the lack of state-contingent assets and an exogenous (ad hoc) borrowing constraint. Thus, the government tax reform cannot affect the incompleteness of the market. In this section, we relax these assumptions by considering an alternative market arrangement—Kehoe and Levine (1993). We think this is an interesting case to study because it allows for interaction between the government tax and the degree of market incompleteness. By comparing the optimal tax schedule under two different market arrangements, we can learn how much the optimal schedule depends on the market arrangement or vice versa. Using the decomposition of the formula, we can further analyze through which channel the market structure affects the optimal tax schedule the most.

We consider a version of Kehoe and Levine (1993). As we briefly discussed in section 3.2, households can purchase a complete set of Arrow-Debreu state-contingent consumption claims at period 0 to insure against their future idiosyncratic risk. However, the risk sharing is incomplete due to a limited commitment—households have the option to renege on a risk-sharing contract at any time. We assume that after default, all remaining assets are seized but households can retain their labor incomes. They have to pay taxes according to the same tax schedule because the government cannot discriminate between households based on a default history. We also assume that these households can save by purchasing

state-noncontingent assets at the equilibrium interest rate, but cannot access the private insurance market forever.<sup>31</sup>

As we showed in section 3.2, in this economy, the state variables are productivity  $x$  and individual weights in the previous period  $\omega$ . The consumption net of the disutility from labor supply,  $c - v(l)$ , is determined by the individual weight:  $h(x, \omega) = \omega'(x, \omega)H$ , where  $\omega'(x, \omega)$  is the current individual weight that captures the history of a binding participation constraint.<sup>32</sup> Consumption is thus determined by  $c(x, \omega) = h(x, \omega) + v(l(x))$ , and the private intermediation of a household with productivity  $x$  and individual weight  $\omega$  is  $P(xl, \omega) = xl - T(xl) - c(x, \omega)$ .

The participation constraint—Equation (7)—is typically interpreted as an endogenous borrowing constraint (Alvarez and Jermann (2000)) and this borrowing constraint can be relaxed, if the tax reform decreases the value of deviation relative to staying in a contract. This implies that the market incompleteness depends on the tax schedule, and thus the substitution effects and the crowding effects can play larger roles than in the economy with an exogenous borrowing constraint (such as our baseline model).

We calibrate the discount factor  $\beta$  so that the state-noncontingent interest rate is 4% in the steady state.<sup>33</sup> The average marginal propensity to save in our model is about 0.25 under the current U.S. tax schedule (approximated by a log-linear function), which is somewhat higher than that in the baseline Huggett economy (0.15) due to higher risk sharing in an endogenous borrowing constraint.

Figure 11 compares the optimal marginal tax schedules from both private insurance markets: Huggett and Kehoe-Levine. The optimal tax rates in the Kehoe-Levine economy

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<sup>31</sup>It is well known that the Kehoe-Levine model with the standard autarky assumption—i.e., households cannot save at all in autarky—tend to generate a significant amount of risk sharing. In order to avoid too much risk sharing, the discount factor  $\beta$  is often set to a small value in a quantitative analysis. Thus, increasing the value of deviation (by allowing the savings in autarky as we assume) can avoid the excessive risk sharing.

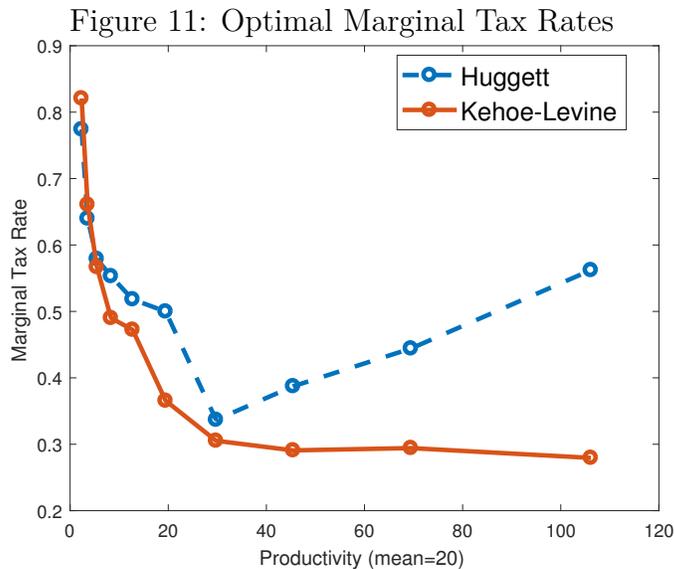
<sup>32</sup>The individual weight  $\omega'(x, \omega)$  is determined by the following rule:

$$\omega'(x, \omega) = \begin{cases} \omega/g & \text{if the participation constraint does not bind} \\ \underline{\omega}(x) & \text{if the participation constraint binds} \end{cases}$$

where  $g = \frac{\sum_{x^{t+1}} \xi(x^{t+1})^{\frac{1}{\sigma}} \pi(x^{t+1})}{\sum_{x^t} \xi(x^t)^{\frac{1}{\sigma}} \pi(x^t)}$ , and  $\underline{\omega}(x)$  is the cutoff weight for a household with productivity  $x$ .

<sup>33</sup>We can compute the state-noncontingent interest rate from the state-contingent Arrow-Debreu prices:  $\frac{P(x^{t+1})}{P(x^t)} = \frac{\pi(x_{t+1}|x_t)}{R}$ . The gross interest rate  $R$  is such that  $R = \frac{1}{\beta g^\sigma}$ , where  $g$  is the average growth rate of the Lagrange multiplier on the participation constraint. The calibrated  $\beta$  in our model is 0.8197.

are lower than those in the Huggett across all productivity levels except for the very low. As a result, the income tax schedule in a Kehoe-Levine economy is less progressive.



First, in the Kehoe-Levine economy, households can insure against their idiosyncratic shock using the state-contingent claims, and thus the amount risk sharing is relatively larger than that in the Huggett. This leads to a higher savings rate on average in the Kehoe-Levine economy (shown in Figure 12), which, in turn, contributes to lower marginal income tax rates—i.e., less need for public insurance.

A less progressive tax schedule in the Kehoe-Levine economy is mainly driven by the third term in formula (1), which reflects the alignment between private and public insurances. As shown in Figure 13, the third term of the formula in the Kehoe-Levine economy is positive and large for the low-income level. This implies that increasing the tax rate for low-income households results in more private insurance (thus a high marginal tax is effective). On the other hand, the third term is small (even negative) at high-income levels, implying that a tax increase crowds out private insurance (thus making a tax increase ineffective). This result is consistent with Krueger and Perri (2011), who show that a progressive tax crowds out private insurance in a Kehoe-Levine economy. That is, providing more public insurance by increasing the progressivity improves the deviation value more than the value of staying in a contract because households have less insurance opportunities after default, and thus the optimal schedule is less progressive.

Our quantitative analysis shows that the shape of the optimal tax schedule depends on

Figure 12: Marginal Private Savings

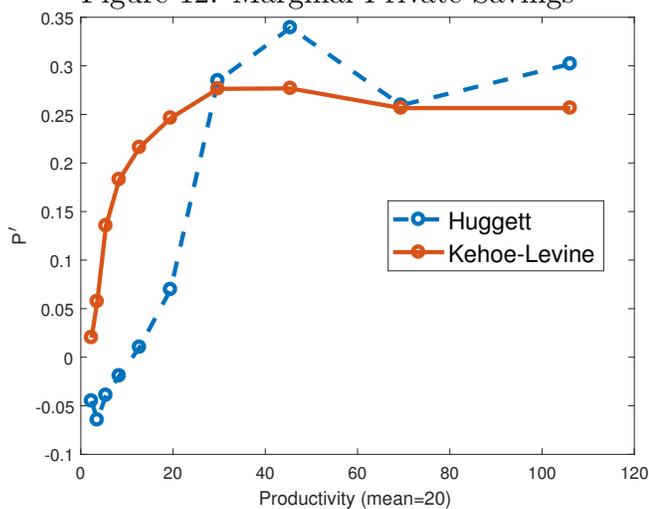
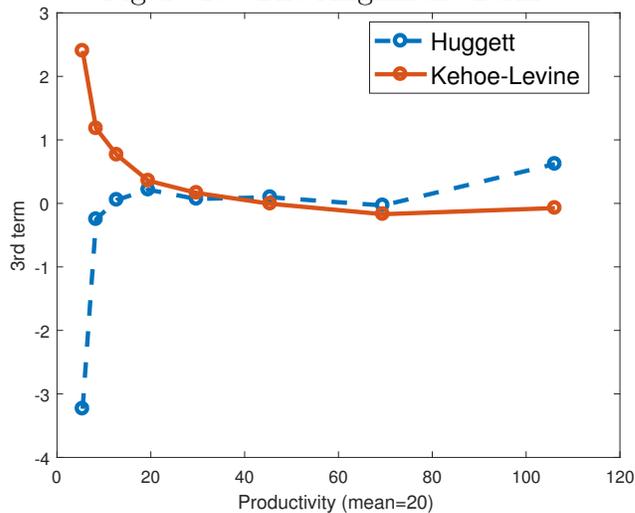


Figure 13: The Alignment Term



the market structure. However, the two models we used may not be perfect representations of the insurance market in the real world. For example, by comparing the cross-sectional dispersion of consumption, Krueger and Perri (2006) show that the exogenous incomplete-market model tends to understate the provision of private insurance compared to the data, whereas the endogenous incomplete-market model tends to overstate the provision of insurance. Nevertheless, our quantitative analysis illustrates that the optimal tax schedule with a private market can be significantly different from that without a private market.

## 6 Conclusion

We study a fully nonlinear optimal income tax schedule in the presence of a private insurance market. As in Saez (2001), the optimal tax formula includes standard statistics such as the Frisch elasticity of the labor supply, and the distributions of income and consumption. In the presence of a private market, however, these statistics are no longer sufficient. The optimal tax formula also depends on private households' savings behavior and their interaction with public insurance (e.g., crowding in/out elasticity). For example, a tax reform is more effective when the response of private savings is aligned with that reform.

Given that these additional statistics are neither easy to measure nor policy invariant, we compute the optimal tax schedule based on a structural model—by obtaining those

hard-to-measure statistics from general-equilibrium model economies (Huggett (1993) and Kehoe and Levine (1993)) calibrated to resemble the U.S. income distribution. The presence of a private market is quantitatively important, since the difference in optimal tax rates (with and without private insurance) can be as large as 20 percentage points. For the low-income group, the optimal tax rate is higher in the presence of a private market. For the middle- and high-income groups, tax rates are lower in the presence of a private market. Interestingly, for households whose income ranges between \$70K and \$250K, the current tax rates are close to optimal. This result is very different from those in the existing literature where the optimal tax rates are much higher than the current U.S. rates.

While our analysis helps us to better understand the role of a private market in the optimal tax schedule, our analysis has several limitations. First, there is no wealth effect in the labor supply. Second, we consider a tax system that does not depend on history. Third, the market structure of private insurance is rather primitive. Fourth, the transition dynamics are not considered in the welfare analysis. We leave a more thorough analysis that delves into these aspects for future work.

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## Appendix (For Online Publication)

### A Rewriting Optimal Formula w.r.t. Productivity Distribution

The optimal tax formula (2) involves an endogenous income distribution. For the quantitative analysis below, it is useful to express the formula with respect to the exogenous productivity distribution.

We know that income density and skill density are related through the equation  $h^*(z)\dot{z}_x^* = f(x)$  and using lemma 1, we get

$$f(x) = (1 + e)\frac{\dot{z}_x}{x}h^*(z_x). \quad (10)$$

With a slight abuse of notation we denote the joint distribution of skill and other state variables by  $\Phi(x, \mathbf{s})$  and its density by  $\phi(x, \mathbf{s})$ , where  $\phi(x, \mathbf{s}) = \phi(\mathbf{s}|x)f(x)$ . By combining (10) with (2), we obtain the following proposition.

**Proposition 6.** *The optimal marginal tax rate of the government should satisfy the following:*

$$\begin{aligned} \frac{T'(xl_x)}{1 - T'(xl_x)} &= (1 - \beta) \sum_{t=0}^{\infty} \beta^t \left[ - \int \tilde{P}'(xl_x, \mathbf{s}) d\Phi(\mathbf{s}|x) \right. \\ &+ \frac{1 - F(x)}{xf(x)} \left(1 + \frac{1}{e}\right) \int_x^{\infty} \int (1 - g(m, \mathbf{s})) (1 - \tilde{P}'(y_x), \mathbf{s}) d\Phi(\mathbf{s}|m) \frac{f(m)}{1 - F(x)} dm \\ &\left. + \frac{1}{xf(x)} \left(1 + \frac{1}{e}\right) \int (1 - g(m, \mathbf{s})) \frac{d\tilde{P}_t(y^m, \mathbf{s})}{\delta\tau dz^*} d\Phi(m, \mathbf{s}) \right], \end{aligned} \quad (11)$$

where  $g(m, \mathbf{s}) = \frac{G'(u(m, \mathbf{s}))u'(c(m, \mathbf{s}))}{A}$ .

### B Proof of the Main Text

#### B.1 Proof of Proposition 3

Since the private market payment schedule  $P$  depends only on current earnings  $z$ , the first term of the tax formula (1) boils down to  $-\tilde{P}'(y^{z^*})$ . The condition (i) of the private

market payment schedule implies that the elasticity of own crowding out in the marginal private intermediation is time invariant and depends only on current earnings  $z$ :

$$r_t^o(z) = -\frac{d^o \log(1 - \tilde{P}'(y^z))}{d \log(1 - T'(z))} = r^o(z), \quad \forall t, \forall z.$$

It is straightforward to see that the third term of the tax formula (1) is zero by the condition (ii) of the private market payment schedule. Thus, we get the optimal tax formula (4).

We now show that an optimal allocation in an economy without a private market — allocation under the formula in Saez (2001)—can be reproduced in an economy with a private market. For this, we only need to show that the optimal total marginal intermediation  $M'(z) = T'(z) + \tilde{P}'(y^z)(1 - T'(z))$  generated by setting the tax rate to (4) is equal to the optimal tax formula without a private market. We can rewrite the formula (4) by combining the terms associated with the marginal private intermediation with the marginal tax rate:

$$\frac{M'(z^*)}{1 - M'(z^*)} = \frac{T'(z^*) + \tilde{P}'(y^{z^*})(1 - T'(z^*))}{(1 - T'(z^*))(1 - \tilde{P}'(y^{z^*}))} = \frac{1 - r^o(z^*)}{e_{z,1-T'}} \frac{1 - H(z^*)}{z^* h^*(z^*)} \int_{z^*}^{\infty} (1 - g(z)) \frac{h(z)}{1 - H(z^*)} dz.$$

By exploiting  $\frac{d(1-M')}{1-M'} = (1 - r^o) \frac{d(1-T')}{1-T'}$ , we obtain the following relationship:

$$e_{z,1-M'}(z^*) = \frac{e_{z,1-T'}(z^*)}{1 - r^o(z^*)},$$

and thus the sum of private and public insurance  $M'(z)$  satisfies the standard formula without a private market.

## B.2 Proof of Proposition 5

The private intermediation can be represented by the function  $P(z, k) = p_k(z - T(z)) - p_k \bar{z}$ . Note that there is no time index in the private insurance schedule  $P$  because the private insurance rate  $p_k$  only depends on time-invariant tax schedule  $T$ . Consider the same tax reform as in the main text. Then, we can easily show that the effects on total intermediation due to mechanical and crowding effects are

$$dM(z, k) = (1 - p_k) \delta \tau dz^* 1_{z \geq z^*} - \delta \tau \frac{dz^*(1 - p_k)}{1 - T'(z^*)} r_k z.$$

The aggregate change in the tax receipt via behavioral response of labor supply is  $-\frac{T'(z^*)}{1-T'(z^*)}ez^*\delta\tau h(z^*)dz^*$  and the aggregate change in the private market payment via the behavior response of the labor supply in group  $k$  is  $-p_k ez^*\delta\tau h_k(z^*)dz^*$ .

The aggregate change in the tax receipt that is rebated to all households is  $d\bar{T} = \delta\tau dz^*(1 - H(z^*)) - \frac{T'(z^*)}{1-T'(z^*)}ez^*\delta\tau h(z^*)dz^*$ . Since private insurance only exists within group  $k$ , the aggregate change in group  $k$   $d\bar{P}_k$  is rebated to all households in group  $k$ , where

$$d\bar{P}_k = -p_k\delta dz^*(1 - H(z^*)) - p_k\delta\tau dz^*(1 - H_k(z^*)) - \frac{r_k\delta\tau dz^*(1 - p_k)\bar{z}_k}{1 - T'(z^*)}.$$

Thus the overall change in social welfare from the tax reform is:

$$\begin{aligned} dSWF = & \sum_{t=0}^{\infty} \beta^t \sum_k \pi_k [d\bar{T} + d\bar{P}_k] \int G'(u(z, k))u'(c(z, k))h_k(z)dz \\ & - \sum_{t=0}^{\infty} \beta^t \sum_k \pi_k \int G'(u(z, k))u'(c(z, k))dM(k, z)h_k(z)dz \end{aligned}$$

At the optimum,  $dSWF = 0$ . By rearranging the terms, we get the formula.

## C Additional Comparative Statics

### C.1 Risk Aversion and Labor Supply Elasticity

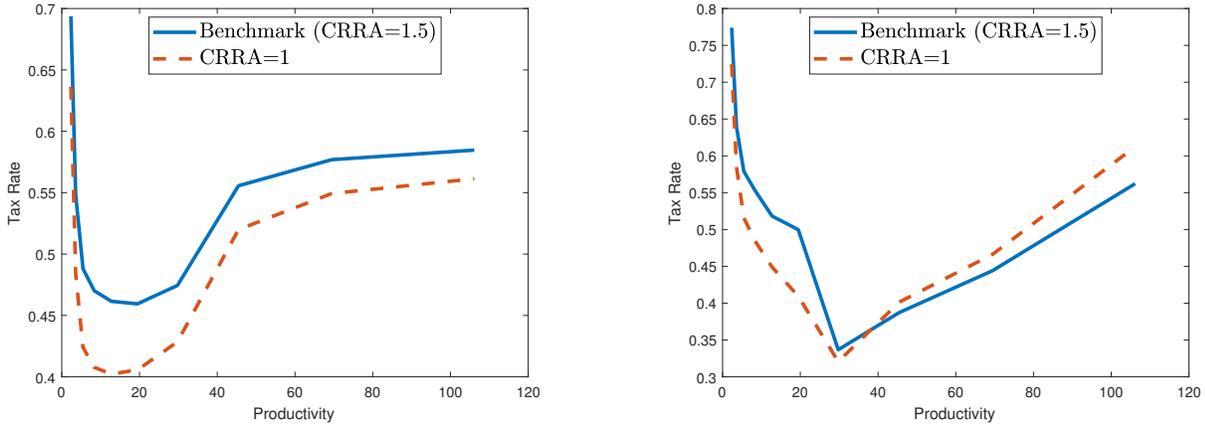
We consider the relative risk aversion  $\sigma = 1$  (lower than the benchmark risk aversion:  $\sigma = 1.5$ ). Figure 14 shows the optimal tax rates without a private insurance market for  $\sigma = 1$  and 1.5. With a smaller risk aversion, the optimal tax rates are lower than those in our benchmark at all productivity levels because there is less need for insurance.<sup>34</sup> This driving force is still present in an economy with a private market.

In the presence of a private insurance market, however, the optimal tax rates at the upper-middle and high-income groups are in fact higher when  $\sigma = 1$  (lower risk aversion) in Figure 15. This is because of the general equilibrium effect. With a smaller risk aversion, households save less (a weaker precautionary savings motive). The real interest

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<sup>34</sup>Under utilitarian social welfare, for example, the social welfare weights at high-income levels ( $\frac{u'(c)}{A}$ ) are relatively high when risk aversion is low.

Figure 14: Optimal Tax without Private Insurance Figure 15: Optimal Tax with Private Insurance



rate has to increase to clear the private savings market. For example, the real interest rate net of discounting  $\beta(1+r)$  increases from 0.9647 (under  $\sigma = 1.5$ ) to 0.9808 (under  $\sigma = 1$ ). This encourages high-income households to save more and makes the marginal private savings ( $P'$ ) schedule steeper, which leads to a larger cross-sectional dispersion in assets and consumption. Thus, the second term and the third term in our formula (amplified Saez effects and aligned private progressivity effects) become larger. For upper-middle and high-income groups (wages above \$35), this general equilibrium effect (larger second and third terms in the formula) dominates a weaker precautionary savings (smaller first term in the formula), resulting in higher marginal tax rates.

Next, we consider a smaller Frisch elasticity of the labor supply ( $e = 0.25$ ). Figure 16 shows that for all income levels the optimal tax rates under  $e = 0.25$  are higher than those in our benchmark ( $e = 0.5$ ) because an inelastic labor supply is associated with a smaller cost of distortion.

## C.2 Borrowing Constraints

In the benchmark economy, we set the borrowing limit  $\underline{a} = -86.55$ , which is the average annual earnings in the steady state under the current U.S. tax schedule (approximated by a log-linear form as in HSV). Under this borrowing limit, 9.7% of households are credit-constrained under the current U.S. tax schedule. We consider a tighter borrowing limit, which is half of our benchmark case ( $\underline{a} = -43.3$ ) so that workers can borrow one-half of the average earnings in the economy. With this tighter borrowing limit, about 34% of

Figure 16: Optimal Tax with Private Market

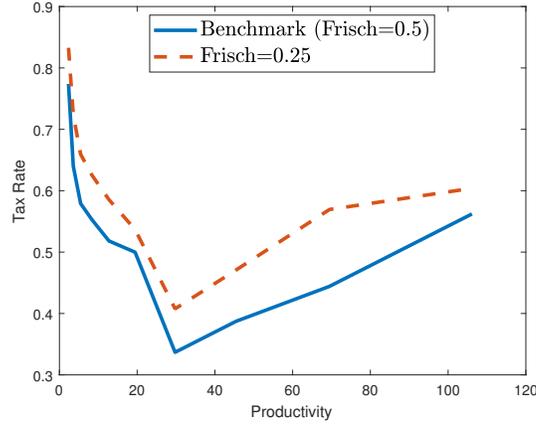
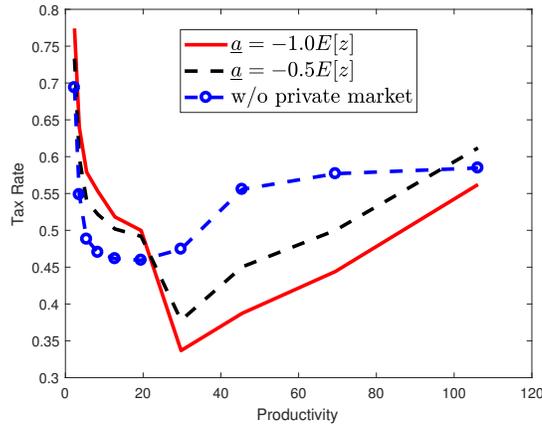


Figure 17: Optimal Tax under Tighter Borrowing Constraint



the population is credit-constrained under the current U.S. tax schedule in our model. Figure 17 shows the optimal tax rate schedules under this tighter borrowing constraint. The optimal tax rates are roughly between those in the benchmark and those without a private insurance market, except for the highest productivity group.

Under the tighter borrowing constraint, households tend to save more due to a stronger precautionary savings motive. To clear the private savings market, the equilibrium interest rate has to decrease. This increases the MPS of the low-income group and decreases the MPS of the high-income group. This will lead to a lower marginal tax for the low-income group and a higher marginal tax for the high-income group. In addition, the crowding in/out effects become larger under the tighter borrowing constraint as the tax reform induces a more progressive response of private savings. At the top income group, this effect is strong enough to generate an even higher optimal tax rate.