Inefficient Sorting under Output Sharing*

Lam, Wing Tung†

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Abstract

I study how output sharing affects sorting in a frictional market. A worker's productivity is privately known while the quality of an asset is public information. Both sides compete for better partners. Asset owners first post their shares of future outputs, and workers then direct their search. The unique equilibrium features inefficient positive assortative matching. The matched pairs of types fully separate into a continuum of sub-markets, where the posted shares and queue lengths yet satisfy the Hosios efficiency condition. The distribution-free result is that all but the highest type workers always pair up with better assets than in the Second Best allocation. There is either excessive entry of workers or insufficient entry of assets. The same conclusion also applies to the dichotomy between the markets where workers buy out the assets upfront and where they pay the posted output shares, and the dichotomy between the cases that workers' types are all public and that theirs are privately known.

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†Division of Social Sciences, Yale–NUS College, Singapore 138527. Contact email: lamwingtunghk@gmail.com

1 Introduction

This paper studies sorting in a frictional market where the two sides can be ranked by particular characteristics, or simply their types. One side competes for partners by offering financial securities or contracts specifying how the payment between the partners is contingent on certain outcomes, say the realized output. An example that has received much attention is the market for top executives. Firms are ranked by their size, while candidates are ranked by their productivity. A larger firm gains more from the CEO's productivity, so that it is efficient for candidates and firms to pair up assortatively. Gabaix and Landier (2008); Terviö (2008) apply a frictionless assignment model to study how positive assortative matching accounts for the empirical distribution of CEOs pay among the largest publicly traded companies in the United States. Frydman and Jenter (2010) document the composition of CEO pay in S&P 500 firms. During the period 2000 to 2008, base salary makes up less than 20% of remuneration, and over half of the pay are option grants and restricted stock grants under which an increase in firm value is shared by the CEO and the shareholders.

In many circumstances, the parties on one side, say candidates for CEO positions, are better informed about their own types. As the remuneration depends on the CEO's performance, candidates of different types expect different amount from the same offer. This in turn affects competition for talents among firms of various size. Little is known about how the offering of contingent payment affects the sorting pattern and the distribution of matching surpluses.

I address this question for the class of output sharing contracts in a competitive search framework. There are double continuums of types of assets and workers. A worker's productivity is privately known, whereas the quality of an asset is publicly observable. Each worker may pair up with an asset. The types on both sides determine the output level. The asset owners first post their shares of future outputs. The workers then decide which type of asset and contract to search for. The meeting is bilateral and subject to search friction. The second best allocation always features positive assortative matching (PAM) despite search friction.[1] I identify a novel source of inefficiency in this environment and analyze the resulting distortion on the sorting pattern.

To better understand the source of inefficiency, let us first consider the result in Eeckhout and Kircher (2010). They study sorting in the described environment where sellers

[1]In the presence of search friction, PAM occurs if better workers always search for better assets.

of heterogeneous goods post fixed prices. The authors show that the equilibria support second best allocations. Furthermore, everyone receives her social value, the shadow price in the utilitarian planner's problem, in equilibrium. I refer to their benchmark by *price competition*. In the current context, this requires that an informed worker, once matched, buys out the asset from the owner at the posted price upfront and assume the residual claim. However, workers' wealth constraints may prevent them from buying the assets upfront. Incentive provision for the asset owners or other stakeholders also undermines the feasibility of the buyout arrangement, calling for the use of contingent payments.

The second best allocations can no longer be decentralized using sharing contracts. A low-type worker pays less than a high-type worker when conceding a larger output share to the asset owner. As such, the offering of sharing contracts handicaps the competition between workers for the *same* type of assets, increasing the expected payoff for the asset owners. Holding the allocation *unchanged*, the shift in the divisions of outputs can be attributed to the linkage principle in auction theory (Milgrom and Weber, 1982; DeMarzo, Kremer, and Skrzypacz, 2005). Assortative matching gives rise to an additional spillover effect. As workers pay more for their partners in the second best allocation, they will find deviation to better assets even more attractive, further intensifying the competition for better assets. As a result, the set of incentive compatible contracts supporting a second best allocation must provide the asset side with higher payoffs than in price competition, which are their social values.

Consequently, the private benefit for an asset to get matched is above the social benefit. The wedge is largest at the top. Facing search friction, owners of the best assets increase their matching probability by inducing an inefficiently long queue of workers. This leads to unraveling of the second best allocation. Inefficiency here is caused by the interplay between three elements: sharing contracts, private types, and search friction.

The preceding discussion begs two questions. Under assortative matching, the pool of workers left to the lower quality assets must deteriorate amid longer queues of workers for the best assets. Despite a higher share of the output, an owner of lower quality asset gains less from a match with a weaker worker, and may even induce a shorter queue of workers instead. Hence, distortions in queue lengths and the sorting pattern are intertwined. The first question is what form of distortion always arises in equilibrium.

The second question is whether there are other sources of inefficiency in the described environment. Unlike a fixed price, asset owners are now concerned about their partners' types which determine their expected payment. An asset owner will take screening into account, and may attempt to poach better workers.[2] Screening by means of rationing has been studied extensively. In particular, Inderst and Müller (2002); Guerrieri, Shimer, and Wright (2010); Guerrieri and Shimer (2014); Chang (2017) study how distortion in queue lengths helps screening in directed search frameworks. To distinguish the channel of inefficiency from those in screening literature, I consider a benchmark setting in which all workers have the same preference over the contract term and the matching probability given the asset quality. This assumption ensures that asset owners never use queue length as an instrument to screen out better workers.

Main results The stylized setting yields a unique equilibrium, which still features PAM. One type of worker pairs up with one type of assets, and vice versa. The matched pairs of types fully separate into a continuum of (sub-)markets. The Hosios (1990) efficiency condition holds in every market. That is, the division of output is given by the elasticity of the matching function, so that the equilibrium payoff of an agent is the reduction in aggregate surplus if she is removed from the population.[3] Looking at each market in isolation, the equilibrium queue length still maximizes the joint payoffs for the pair of types, subject to free entry of the workers at their equilibrium payoff. However, the posted share and the resulting queue length in one market affect the information rent for other types of workers and the remaining pool of workers left to other types of assets. These two kinds of externalities were not considered in Hosios (1990). Therefore, the channel of sorting inefficiency here only arises when the two sides are both heterogeneous and is different from those in search and matching literature.

The equilibrium and second best allocations vary with the entire distribution of types. The key contribution of this paper is the qualitative features of the distortion, which are universal for *all* distributions. Such a "distribution-free" result is of theoretical interest, as the argument illuminates general economic forces that are always at play.

As one may expect, the queue length for the best assets is always inefficiently high. The surprising and novel result is that all but the highest possible type of assets always pair up with weaker workers. Depending on the distribution of types, there is either excessive

[2]An asset owner believes that an off-equilibrium-path contract will only attract the types of workers who accept the lowest matching probability, given all other contract offers. Guerrieri, Shimer, and Wright (2010) motivates this belief restriction using "subgame perfection" with bilateral meeting.

[3]The equilibrium allocation here is inefficient. Unlike the textbook setting for Hosios condition, the equilibrium payoff for an agent is no longer the same as her "social value," the maximum increase in the aggregate surplus a utilitarian planner may achieve from assigning a new agent of the same type.

entry of workers or insufficient entry of assets. In comparison with price competition or the complete information case, the best workers suffer while the weakest workers gain from the offering of sharing contracts. The opposite is true for the asset side.

Other applications The preceding discussion adopts the convention that the informed party receives the output and pays the uninformed.[4] In labor market applications, the payment flows in the opposite direction. The buyout arrangement corresponds to high-powered incentive contract whereas the output sharing contract corresponds to low-powered incentive. In this sense, the results here predict that all but the largest firms will hire better workers if high-powered incentive contracts become available, possibly due to improvement in performance measurement. Other applications include the sorting between entrepreneurs and venture capitalists (e.g. Sørensen, 2007). The prospect of the entrepreneurs' projects vary, and entrepreneurs initially know certain aspects of their projects better than the outsiders. Venture capitalists can be ranked by their reputation and obtain financial securities in return for their assistance and initial financing. This paper studies the distortion caused by information asymmetry when forming matches. Another application is sorting in M&A (e.g. Rhodes-Kropf and Robinson, 2008). The assets controlled by acquiring and target firms exhibit complementarity. This paper explains how the sorting pattern depends on whether target firms receive cash or equity shares of the acquiring firms.

Related literature This paper adds to the literature on assortative matching with search friction: Burdett and Coles (1997); Eeckhout (1999); Shimer and Smith (2000); Shi (2001); Shimer (2005); Damiano, Li, and Suen (2005); Smith (2006); Chade (2006); Jacquet and Tan (2007). My setting is built on Eeckhout and Kircher (2010). The authors show that in price competition, the n-root-supermodularity condition is necessary and sufficient for any equilibrium allocation to feature PAM for any distribution of types. In addition, the second best allocations are supported by the equilibria. The production and matching technology used herein satisfy this condition. I address how output sharing distorts assortative matching in equilibrium.

Serfes (2005); Legros and Newman (2007); Kaya and Vereshchagina (2014); Chiappori and Reny (2016) consider the use of contracts in assortative matching with public types. They study how risk sharing or incentive provision shapes the payoff functions for the two

[4] This convention is in line with the existing literature on security-bid auction and assortative matching with private types on one side.

sides, and thus the matching pattern. I explore how the offering of output sharing contracts result in inefficient sorting when combined with search friction and private types.

This paper also contributes to the literature on efficiency in search and matching models: Hosios (1990); Albrecht, Navarro, and Vroman (2010); Julien and Mangin (2017); Mangin and Julien (2018). Guerrieri (2008) studies dynamic efficiency in a directed search model where a worker privately observes his match-specific productivity upon meeting. The worker then weighs the current match against his continuation value in the unemployed pool. Hence, offers from future entrants determine workers' acceptance decisions and information rent in the present period. The author shows that convergence to the steady state is inefficiently slow. The channel of sorting inefficiency here shares the similarity that the asset owners do not internalize the effects of their offers on the markets for other assets.

Guerrieri, Shimer, and Wright (2010) study competitive screening in a competitive search framework with free entry of homogeneous principals. The principals have contract and matching probabilities as screening instruments, and the latter are endogenously determined. The authors characterize the equilibrium and study the form of distortion in various applications. I consider two-sided matching, where both sides compete for better partners from the given pools.[5] The central difference is that the distortion now depends on the entire distribution of types, and I obtain its distribution-free features.

The present paper also extends results in the security bid auction literature (e.g., Hansen, 1985; DeMarzo, Kremer, and Skrzypacz, 2005; Sogo, Bernhardt, and Liu, 2016) to the context of assortative matching with search friction. These authors show that the seller in a private value auction is always better off when the feasible contracts become steeper, handicapping the competition between buyers. In my setting, the insurance motive caused by search friction interacts with the wedge in the divisions of the outputs, resulting in adjustments in the equilibrium allocation. Consequently, change in the form of contingent payments has different distributional implications than those in the literature. Owners of lowest quality assets are worse off because of competition from high quality assets! From this perspective, this paper complements Mailath, Postlewaite, and Samuelson (2016), who study how the wedge in the divisions of matching surpluses affects pre-investment incentives

[5] In Guerrieri, Shimer, and Wright (2010), or the textbook competitive screening models, only the principals compete for informed agents and not the other way around. Consequently, the support of the type distribution alone pins down the set of incentive compatible conditions. The set of separating equilibria and the associated distortion are invariant to the type distribution .

when investment on one side is private information.

Section 2 details the model setting and equilibrium definition. Section 3 covers the benchmark results. That is, the second best allocation is unique and features PAM. It is always decentralized when the asset side posts prices. Section 4 characterizes the unique equilibrium when the asset side offers output shares. Section 5 discusses the form of distortion in equilibrium. Section 6 discusses the assumptions and results. All proofs are relegated to the Appendix.

2 Model Setting

2.1 Production

There are continuums of workers and asset owners. Each asset owner owns a unit of asset. Assets can be ranked according to their publicly known qualities $q \in [0, 1]$. All workers are ex ante homogeneous but differ in their actual productivity $p \in [0, 1]$. Every worker privately knows his productivity. The values of outside options for workers and asset owners are given by $\underline{V}$ and $\underline{U}$ respectively. $\emptyset$ denotes the choice of outside option. All parties are risk neutral and have quasi-linear preferences. I use feminine and masculine pronouns for asset owners and workers, respectively.

A worker may operate an asset to carry out production. Before production takes place, the two parties may enter into a sharing contract $s \in [0, 1]$ where s and $1 - s$ are the shares of output for the asset owner and worker, respectively.

The output level for the pair of types (p, q) is $y(p, q)$, where $y : [0, 1]^2 \to \mathbb{R}_{++}$ is positive, strictly increasing and twice continuously differentiable (C^2) in (p, q). If the output level is stochastic, then $y(p, q)$ denotes its expected value.

Assumption (Y)**.** *The output $y(p, q)$ is strictly log-supermodular (log-SPM) in p and q.*

Assumption (Y) states that better workers can generate a greater percentage increase in output when operating a better asset. This has two important implications in a frictionless setting. First, it is a stronger notion of production complementarity than strict supermodularity (SPM). Without search friction, aggregate surplus is maximized under perfect positive assortative matching. Second, Assumption (Y) is necessary for decentralizing PAM in a frictionless market where the asset side may only post output shares. To elaborate, PAM occurs in such a market if better workers always accept a lower output share than weaker workers for a better asset. This is exactly ensured by Assumption (Y).

Example (O-ring production technology). *Assumption (Y) is satisfied if the conditional distribution of output* $Y|(p,q)$ *is a Bernoulli distribution with support* $\{\underline{y}, \overline{y}\}$, *where* $\overline{y} > \underline{y} > 0$ *and* $\Pr(Y = \overline{y}|p,q) = pq$: *Production is composed of two tasks. The probability of success for the first task is* p, *and that of the second task is* q. *Production yields a high output* $\overline{y}$ *if both tasks are successful. Otherwise, only the base output* $\underline{y}$ *is produced. Such production technology is proposed by Kremer (1993).*

The types on both sides are continuously distributed with support $[0,1] \times [0,1]$. $F(p)$ denotes the measure of workers of productivity below p and $G(q)$ is the measure of assets with qualities below q. The total measure of assets and workers, $G(1)$ and $F(1)$, may differ. F and G are C^2 and their derivatives are denoted by f and g, respectively. f and g are positive and bounded over $[0,1]$.

2.2 Matching

There are continuums of (sub-)markets indexed by $(q,s) \in [0,1]^2$. An owner of asset quality q may participate in one of the markets (q,s) while a worker may participate in any one of the markets. This market structure reflects the assumption that the quality of an asset is public whereas a worker's type is private information.

The timing of the events is as follows. Asset owners first make their participation decisions simultaneously. Observing the measure of asset owners in every market, the workers simultaneously make their participation decisions. The two sides of the market then pair up randomly. Everyone may meet at most one participant from the other side.

Define the queue length $\lambda \in [0,\infty]$ as the ratio of workers to asset owners in the market. A worker gets matched with probability $\eta(\lambda)$ while the matching probability for an asset owner is $\delta(\lambda)$. Meeting is bilateral, so $\delta(\lambda) \leq min\{\lambda, 1\}$ and $\lambda\eta(\lambda) = \delta(\lambda)$. The payoffs for those who are left unmatched are normalized to zero.

η is a strictly decreasing function. $\delta : [0,\infty] \to [0,1]$ is C^2, strictly increasing and strictly concave. These properties jointly imply the following. For any positive $\lambda \in (0,\infty)$, $\frac{d\ln\delta}{d\ln\lambda} \in (0,1)$, and so $1 > \eta(\lambda) > \delta'(\lambda)$.

Assumption (M). $\frac{d\ln\delta}{d\ln\lambda}$, *the elasticity for* $\delta(\lambda)$, *is decreasing.*

Following Eeckhout and Kircher (2010), I assume a decreasing elasticity for $\delta(\lambda)$.[6] The presence of search friction gives rise to an insurance motive against the risk of being

[6]Eeckhout and Kircher (2010) assume a strictly decreasing elasticity for $\delta(\lambda)$. Their results remain valid here because I strengthen the assumption on output $y(p,q)$ to strict log-SPM.

unmatched. Assumption (M) states that an asset owner's marginal gain in her matching probability from an increase in the queue length is diminishing. As $1 = \frac{d\ln\delta}{d\ln\lambda} + \frac{d\ln\eta}{d\ln\frac{1}{\lambda}}$, the workers also see a diminishing marginal gain from a decrease in the queue length.

Assumption (M) is equivalent to a unit upper bound on the elasticity of substitution of the aggregate matching function. Let $M(l,k)$ denote the number of matches in a market with l workers and k assets, that is $M(l,k) = k\delta(\frac{l}{k})$. Then, Assumption (M) can be rewritten as

$$-\frac{d\ln\lambda}{d\ln\frac{M_l(\lambda,1)}{M_k(\lambda,1)}} = \frac{M_l(\lambda,1)M_k(\lambda,1)}{M_{lk}(\lambda,1)M(\lambda,1)} \leq 1.$$

Example (Random matching)**.** *Assumption (M) is satisfied for* $\delta(\lambda) = \frac{\lambda}{\lambda+1}$ *: All participants on both sides are pooled together to form pairs randomly. The pair may carry out production only when it consists of a worker and an asset owner in the spirit of Kiyotaki and Wright (1993).*

Example (Urn-ball matching)**.** *Assumption (M) is satisfied for an Urn-ball matching function,* $\delta(\lambda) = 1 - \exp(-\lambda)$ *: Every worker approaches one asset owner without coordination. If an owner is approached by at least one worker, then she will randomly pair up with one of these workers, utilizing her asset. This form of matching function is proposed by Peters (1991).*

Participation in matching is costly because the agent has to forgo her outside option. Suppose l workers of type p and k assets of quality q form matches in a market, I define the net expected surplus for the pair of types by $M(l,k)y(p,q) - l\underline{V} - k\underline{U}$. I assume that the maximal net expected surplus at the top is positive,

$$\max_{\lambda\geq 0}[\delta(\lambda)y(1,1) - \lambda\underline{V} - \underline{U}] > 0, \tag{1}$$

so that it is always efficient to have the best agents on both sides searching for partners.

Example (Random matching)**.** *Given* $\delta(\lambda) = \frac{\lambda}{\lambda+1}$, *condition (1) is satisfied if and only if* $y(1,1) > (\sqrt{\underline{U}} + \sqrt{\underline{V}})^2$.

Example (Urn-ball matching)**.** *Given* $\delta(\lambda) = 1 - \exp(-\lambda)$, *condition (1) is satisfied if* $(1 - \sqrt{e})y(1,1) > \frac{1}{2}\underline{U} + \underline{V}$.

2.3 Equilibrium definition

$K(q,s)$ is the measure of asset owners participating in the markets $(q',s') \leq (q,s)$. $L(p,q,s)$ is the measure of workers with types $p' \leq p$ participating in the markets $(q',s') \leq (q,s)$. The marginal distributions are denoted with the corresponding variables as subscripts. For example, $L_{qs}(q,s)$ is the total measure of workers in the markets $(q',s') \leq (q,s)$ and $L_p(p)$ is the measure of workers with types $p' \leq p$ participating in any of the markets.

Definition. (K,L) *is feasible if* $K_q \leq G$ *and* $L_p \leq F$*.*

$G(q) - K_q(q)$ and $F(p) - L_p(p)$ are respectively the measures of assets of quality below q and workers with productivity below p taking the outside option. The support of K is denoted by Ψ. A market is active if it is in Ψ; otherwise, it is inactive. As participation is costly, it is never optimal for workers to visit a market with no asset owners. Therefore, L_{qs} is required to be absolutely continuous w.r.t. K.

The equilibrium concept here follows the literature on large games e.g., Mas-Colell (1984). The payoff for every single agent depends on her own decision, and the decisions of all others only through K and L. K and L in turn are consistent with the optimal decisions of all individual agents.

Each market (q,s) is associated with a queue length $\Lambda(q,s;K,L)$ and a distribution of participating workers $R(q,s;K,L)$, where $R(.|q,s;K,L)$ is the C.D.F. for workers type. The environment is competitive in the sense that everyone takes Λ and R as given. For active markets, Λ is the Radon-Nikodym derivative, $\frac{dL_{qs}}{dK}$ and R is derived using Bayes' law.[7] By participating in an active market (q,s), a worker of type p receives an expected payoff

$$\eta(\Lambda(q,s;K,L))(1-s)y(p,q), \tag{2}$$

while an asset owner receives an expected payoff

$$\delta(\Lambda(q,s;K,L))s\int y(p,q)dR(p|q,s;K,L). \tag{3}$$

I now extend the payoff functions to the inactive markets. I will elaborate on the belief restriction underlying the payoffs functions in subsection 2.3.1.

A worker will never get matched if visiting an inactive market unilaterally. A worker

[7] Formally, $\int_\psi \Lambda(q,s;K,L)dK = \int_\psi dL_{qs}$, and $\int_\psi R(p'|q,s;K,L)dL_{qs} = \int_{\{p\leq p'\}\times\psi} dL$ for any measurable subset $\psi \subseteq \Psi$ and any $p' \in [0,1]$. $\frac{dL_{qs}}{dK}$ is well defined as L_{qs} is absolutely continuous w.r.t. K.

of type p can, at most, attain

$$V(p;K,L) = \max\{\underline{V}, \sup\{\eta(\Lambda(q,s;K,L))(1-s)y(p,q), (q,s) \in \Psi\}\}.$$

$V(.;K,L)$ then determines the deviating payoff for asset owners. For any inactive market,

$$\Lambda(q,s;K,L) = \inf\{\lambda \in [0,\infty] : V(p) \geq \eta(\lambda)(1-s)y(p,q), \forall p \in [0,1]\}, \tag{4}$$

and $R(q,s;K,L)$ is degenerate at

$$\inf\{p \in [0,1] : V(p;K,L) \leq \eta(\Lambda(q,s;K,L))(1-s)y(p,q)\}. \tag{5}$$

The definition (4) and (5) represent two conditions. First, if $V(p;K,L) > \eta(0)(1-s)y(p,q)$ for all $p \in [0,1]$, then $\Lambda(q,s;K,L) = 0$ and $R(q,s;K,L)$ is degenerate at $p = 0$ for such an inactive market. Second, for any market, be it active or inactive,

$$V(p;K,L) \geq \eta(\Lambda(q,s;K,L))(1-s)y(p,q)$$

for all types of workers, and equality holds for any p in support of $R(q,s;K,L)$ if $\Lambda(q,s;K,L) > 0$.

Facing $\Lambda(q,s;K,L)$ and $R(q,s;K,L)$, an owner of asset quality q can receive

$$U(q;K,L) = \max\{\underline{U}, \sup\{\delta(\Lambda(q,s;K,L))s \int y(p,q)dR(p|q,s;K,L), (q,s) \in [0,1]^2\}\}.$$

Definition. *An equilibrium is a pair of feasible distributions (K,L) satisfying:*

- *Asset owners' optimality: $(q,s) \in \Psi$ only if s maximizes the asset owner's expected payoff (3). $K'_q(q) = g(q)$ if $U(q;K,L) > \underline{U}$.*
- *Workers' optimality: (p,q,s) is in support of L only if $(q,s) \in \Psi$ and maximizes the worker's expected payoff (2). $L'_p(p) = f(p)$ if $V(p;K,L) > \underline{V}$.*

The optimality conditions are interpreted as follows: Asset owners and workers take the queue length and composition of workers in every market as given. If participating, they go to the markets where they receive the highest expected payoff. They never take their outside options if a higher expected payoff can be obtained by participating one of the markets. In particular, $f(p) - L'_p(p)$ indicates some workers of type p take their outside options, and the same for the asset side.

Fix an equilibrium (K,L), then $V(p;K,L)$ and $U(q;K,L)$ are the equilibrium payoffs for workers and asset owners respectively. The arguments K and L will be omitted from the equilibrium objects if no confusion arises.

The notions of incentive compatibility and voluntary participation are useful for discussing constraints in the utilitarian planner's problem.

Definition. *A pair of distributions (K, L) is incentive compatible if it satisfies workers' optimality condition in the definition of an equilibrium.*

Definition. *A pair of distributions (K, L) induces voluntary participation of the asset side if for any $(q, s) \in \Psi$, asset owners' expected payoff in (3) is no less than $\underline{U}$.*

The analysis remains the same if the asset owners may post menus of output shares. This is because an asset owner meets at most one worker who always prefers a higher share for himself regardless of his type.

2.3.1 Belief restriction

As there are continuums of workers and assets, switching between active markets or taking the outside option by a single party has negligible impact. The same is true when a worker unilaterally switches to an inactive market. The focus here is on the deviation to some inactive market by an asset owner. For an inactive market (q, s), $\Lambda(q, s; K, L)$ and $R(q, s; K, L)$ are interpreted as the public belief regarding the queue length and the composition of workers attracted to that market after an owner of asset quality q deviates to it. An advantage of this notation is that it eliminates the distinction between deviations to active or inactive markets by an asset owner.

Fix a pair of feasible distributions (K, L). Suppose an owner of asset quality q deviates to post a contract s. If $V(p; K, L) \geq \eta(0)(1-s)y(p, q)$ for all types, then no workers will ever profit from accepting the deviating offer. The asset owner believes such an offer will attract no workers and $R(q, s; K, L)$, which has no bearing in such case, is set to be degenerate at $p = 0$. Consider the case that $V(p; K, L) < \eta(0)(1 - s)y(p, q)$ for some types. Then, $\Lambda(q, s; K, L)$ is uniquely determined by the lowest matching probability some workers are willing to endure. The asset owner believes that only the lowest type among these workers will be attracted.

The restriction on the "off-equilibrium-path" belief here is often motivated by the "subgame perfection" on the workers' side in the competitive search literature. Suppose only ϵ-measure of the owners of asset quality q deviate to some inactive market (q, s). Observing the measure of asset owners in every market, a worker has to anticipate his matching probability in each of the markets and adjust his participation decision accordingly. When

$\epsilon \to 0^+$, no types of workers can strictly gain from participating in the market (q, s) in the equilibrium of this "subgame". Otherwise, workers of all such types will turn up in this market but only ϵ-measure of them will get matched, resulting in an expected payoff below their outside option. It follows that any workers attracted to the market (q, s), if any, are those willing to endure the lowest matching probability. By continuity, workers' payoffs in the equilibrium of this "subgame" must converge to $V(p; K, L)$ in the limit $\epsilon \to 0^+$. This justifies the belief restriction discussed.

The belief restriction here closely follows Guerrieri, Shimer, and Wright (2010). Eeckhout and Kircher (2010) adopt the same restriction on queue length. As asset owners post prices and are indifferent to their partner's type in their setting, they leave out the off-equilibrium-path belief on the worker's type.

2.4 Assortative Matching

Definition. *A pair of distributions features positive assortative matching (PAM) if there exists a pair of threshold types* $(\underline{p}, \underline{q}) < (1, 1)$ *and an increasing function* $\kappa : [\underline{p}, 1] \to [\underline{q}, 1]$ *such that* $\kappa(\underline{p}) = \underline{q}$ *and* $L_{pq}(p, \kappa(p)) = F(p) - F(\underline{p})$.

$\kappa(p)$ denotes the quality of the asset assigned to a worker of type p. The above definition of PAM requires not only the participants to match assortatively, but also every worker above the threshold type to participate in matching. This is not restrictive, as a better worker always gains more when entering the same market, so only the lowest types may take their outside options in equilibrium and in any efficient allocations.

The above observation does not automatically extend to the asset side. Even when posting the same contract, an owner of a better asset may end up attracting weaker workers, gaining less from participation. In the subsequent sections, I show that monotonic participation for both sides indeed occurs in any efficient allocations and equilibria. In this case, κ is bijective and strictly increasing. The inverse of κ is well defined and denoted by $r : [\underline{q}, 1] \to [\underline{p}, 1]$. $r(q)$ is the type of worker assigned to the asset of quality q.

3 Second Best Allocation

Consider a utilitarian planner has complete information and may dictate the participation decision for each type. Nevertheless, search friction remains in the matching process. Her problem is to choose a pair of feasible distribution to maximize aggregate surplus,

$$\max_{K,L} \int \eta(\frac{dL_{qs}}{dK})y(p,q)dL + [F(1) - L_p(1)]\underline{V} + [G(1) - K_q(1)]\underline{U}$$

subject to

$$K_q \leq G \text{ and } L_p \leq F.$$

The first tern in the maximand is the total amount of output produced by the matched pairs. The second and third term is the total payoff for the workers and assets assigned to their outside option.

The presence of search friction introduces an insurance motive, which is conducive to negative assortative matching. This is because the most efficient way to increase the matching probability for high types, which produce more output, is assigning them to a market flooded with low types from the other side. The more responsive the matching probabilities to a change in the queue length, the greater the insurance motive. Assumption (M) caps the improvement in matching probability yielded by negative assortative matching, while Assumption (Y) puts a lower bound on the gain from production complementarity. They jointly ensure that production complementarity always outweighs the insurance motive, and the utilitarian planner's solution must feature PAM.

Theorem (Eeckhout and Kircher, 2010)**.** *Under Assumption (Y) and (M), second best allocations feature PAM for all distributions of types.*

Proof. The n-root-supermodularity condition is met Under Assumption (Y) and (M). See Proposition 4 in Eeckhout and Kircher (2010). □

The efficient allocation requires monotonic participation because the matching surplus is strictly increasing in types. The strict concavity of $\delta(\lambda)$ implies that it is efficient to pool same pairs of matched types into one market. The contract term s can be omitted, as it does not affect the gain from a match. As a result, the utilitarian planner's problem can be simplified as

$$\max_{\underline{p},\underline{q},r,\lambda} \int_{\underline{q}}^{1} \delta(\lambda(q))y(r(q),q)dG(q) + F(\underline{p})\underline{V} + G(\underline{q})\underline{U}$$

subject to the boundary conditions

$$r(\underline{q}) = \underline{p}, r(1) = 1, \tag{6}$$

and for $q \geq \underline{q}$,

$$\int_q^1 \lambda(q')dG(q') = F(1) - F(r(q)). \tag{7}$$

The condition (7) captures monotonic participation on both sides, restricting the queue length $\lambda(q)$.

Abusing the terminology, a solution, denoted by $(r_{SB}, \lambda_{SB}, \underline{p}_{SB}, \underline{q}_{SB})$, is called a second best (SB) allocation.[8] The utilitarian planner's problem can be reformulated as an optimal control problem with r as the state variable and λ as the control variable. Differentiating (7), the law of motion is given by

$$r'(q) = \frac{g(q)}{f(r(q))}\lambda(q). \tag{8}$$

Introducing the co-state variable $\tau(q)$, the Hamiltonian is given by

$$\mathcal{H}(q, r, \lambda, \tau) = g(q)[\delta(\lambda)y(r, q) - \tau\frac{\lambda}{f(r)}].$$

Eeckhout and Kircher (2010) show that the first order conditions (FOC) can be written as

$$v_{SB}(r_{SB}(q)) = \delta'(\lambda_{SB}(q))y(r_{SB}(q), q), \tag{9}$$

$$\frac{dv_{SB}(r_{SB}(q))}{dp} = \eta(\lambda_{SB}(q))\frac{\partial y(r_{SB}(q), q)}{\partial p}, \tag{10}$$

where $\tau(q) = f(r_{SB}(q))v_{SB}(r_{SB}(q))$. In particular, r_{SB} and λ_{SB} are continuously differentiable, C^1. $v_{SB}(p)$ is the shadow value for a worker of type $p \geq \underline{p}_{SB}$. When comparing $v_{SB}(p)$ and $v'_{SB}(p)$, it is noteworthy that $\delta'(\lambda) < \eta(\lambda) < 1$ for $\lambda > 0$. The gap between $\delta'(\lambda)$ and $\eta(\lambda)$ reflects a better worker's benefit from a better asset.

One can derive the shadow value for an asset of quality $q \geq \underline{q}_{SB}$ by a symmetric approach,

$$u_{SB}(q) = [\delta(\lambda_{SB}(q)) - \lambda_{SB}(q)\delta'(\lambda_{SB}(q))]y(r_{SB}(q), q),^{9} \tag{11}$$

and $u_{SB}(q)$ is strictly increasing. The boundary conditions at the bottom are given by

$$\underline{q}_{SB}[u_{SB}(\underline{q}_{SB}) - \underline{U}] = \underline{p}_{SB}[v_{SB}(\underline{p}_{SB}) - \underline{V}] = 0. \tag{12}$$

The above set of conditions, (6)-(12), defines a boundary value problem, which admits a unique solution $(\underline{p}_{SB}, \underline{q}_{SB}, r_{SB}, \lambda_{SB}, v_{SB}, u_{SB})$ under Assumptions (Y) and (M). The assumption in (1) ensures participation at the top, $\underline{p}_{SB} < 1$ and $\underline{q}_{SB} < 1$. The shadow

[8] There is a continuum of (K, L) with the same $(r, \lambda, \underline{p}, \underline{q})$ but different divisions of outputs.

[9] Note that $\delta - \lambda\delta' = \frac{d\eta(\lambda)}{d\lambda^{-1}}$.

value of an agent below the threshold type is simply the value of her outside option. I extend the definition of v_{SB} and u_{SB} to their entire type space, with $v_{SB}(p) = \underline{V}$ for $p < \underline{p}_{SB}$ and $u_{SB}(q) = \underline{U}$ for $q < \underline{q}_{SB}$.[10]

Remark 1. *The second best allocation is unique. For any p, q and λ,*

$$u_{SB}(q) + \lambda v_{SB}(p) \geq \delta(\lambda) y(p, q) \tag{13}$$

with equality if and only if $\lambda = \lambda_{SB}(q)$ and $p = r_{SB}(q)$.

The inequality in (13) is the counterpart of the well-known condition for stable matching in Koopmans and Beckmann (1957). Without search friction, $\delta(\lambda) = min\{\lambda, 1\}$, the set of inequalities (13) collapses to $u_{SB}(q) + v_{SB}(p) \geq y(p, q)$ for any pair of types.

When defining the second best allocation, it is assumed that the utilitarian planner knows workers' types. One may wonder if this is an appropriate benchmark when workers' types are privately known. Now suppose the utilitarian planner observes only the types of assets and may restrict the set of markets available. In essence, she may dictate the sharing contracts $s(q)$ for each type of asset, subject to their voluntary participation. The planner can induce the second best allocation by excluding all assets below $\underline{q}_{SB}$ from participation and imposing $\widehat{s}(q)$ for $q \geq \underline{q}_{SB}$, where $\widehat{s}(q)$ satisfies

$$1 - \widehat{s}(\underline{q}_{SB}) = \frac{d \ln \delta(\lambda_{SB}(\underline{q}_{SB}))}{d \ln \lambda}; \text{and}$$

$$\frac{d}{dq}[\ln(1 - \widehat{s}(q))\eta(\lambda_{SB}(q))y(p, q)]\bigg|_{p=r_{SB}(q)} = 0, q > \underline{q}_{SB}.$$

For the pair of threshold types, their equilibrium payoffs are the same as their shadow values. Under Assumption (Y), the second condition ensures that workers of type $r_{SB}(q)$ strictly prefer the market $(\widehat{s}(q), q)$ to all other markets, receiving an expected payoff of $\widehat{V}(r_{SB}(q)) = \eta(\lambda_{SB}(q))(1 - \widehat{s}(q))y(r_{SB}(q), q)$. By construction, $v_{SB}(p) \geq \widehat{V}(p) \geq v_{SB}(\underline{p}_{SB})$ for $p \geq \underline{p}_{SB}$ and equalities hold only at $p = \underline{p}_{SB}$.[11] This ensures incentive compatibility on the workers' side and voluntary participation for asset owners of $q \geq \underline{q}_{SB}$. Therefore, the second best allocation can be supported by $\widehat{s}(q)$.[12]

[10] By an abuse of notation, v_{SB} and u_{SB} denote their respective restrictions over $[\underline{p}_{SB}, 1]$ and $[\underline{q}_{SB}, 1]$ when referring to the solution of the boundary value problem.

[11] $\frac{\partial}{\partial p} v_{SB}(p) > \frac{\partial}{\partial p} \widehat{V}(p)$ for $p \geq \underline{p}_{SB}$. This also implies that $\widehat{s}(.) < 1$ and hence is well defined.

[12] The corresponding pair of distributions (K, L) can be recovered from the second best allocation and the set of active markets $\{(q, \widehat{s}(q)) : q \in [\underline{q}_{SB}, 1]\}$ and check the conditions formally. The construction of (K, L) mirrors that in the proof of Proposition 1 in the Appendix.

3.1 Price competition

Price competition refers to the benchmark setting whereby asset owners may post prices, and workers buy out the asset up front. When participating in a market with posted price t and queue length λ, a worker receives an expected payoff $\eta(\lambda)[y(p,q)-t]$ while an asset owner receives an expected payoff of $\delta(\lambda)t$. Eeckhout and Kircher (2010) defines an equilibrium in price competition, which is in line with the equilibrium definition when output shares are offered.

Theorem (Eeckhout and Kircher, 2010)**.** *Under Assumptions (Y) and (M), the second best allocation can always be decentralized in price competition.*

Proof. Again by Proposition 4 in Eeckhout and Kircher (2010). □

For $(\underline{p}_{SB}, \underline{q}_{SB}, r_{SB}, \lambda_{SB}, v_{SB}, u_{SB})$ satisfying (6)-(12), the authors construct an equilibrium supporting the second best allocation $(r_{SB}, \lambda_{SB}, \underline{p}_{SB}, \underline{q}_{SB})$ and the equilibrium payoffs for the two sides are given by v_{SB} and u_{SB}. Let $t_{SB}(q)$ denote the price posted by the owners of asset quality $q \geq \underline{q}_{SB}$ in equilibrium. $t_{SB}(q)$ is pinned down by the FOC (9),

$$v_{SB}(r_{SB}(q)) = \delta'(\lambda_{SB}(q))y(r_{SB}(q),q) = \eta(\lambda_{SB}(q))[y(r_{SB}(q),q) - t_{SB}(q)],$$

so that

$$t_{SB}(q) = \left(1 - \frac{d\ln\delta(\lambda_{SB}(q))}{d\ln\lambda}\right) y(r_{SB}(q),q). \tag{14}$$

This is known as the Hosios condition, for which a worker's share of the output is given by the elasticity of $\delta(\lambda)$ at the efficient queue length for the asset he searches for. Furthermore, Eeckhout and Kircher (2010) show that $t_{SB}(q)$ is increasing in q. The corresponding pair of distributions (K, L) can again be readily recovered from the second best allocation and $t_{SB}(q)$.

The FOC (10) ensures

$$v_{SB}(r_{SB}(q)) = \max_{q' \in [\underline{q}_{SB},1]} \{\eta(\lambda_{SB}(q'))[y(r_{SB}(q),q') - t_{SB}(q')]\}.$$

Together with the boundary condition (12), the incentive compatibility for workers is met.

Given the type of her potential partner $r_{SB}(q)$, an owner of asset quality q cannot profit from adjusting the price if and only if the Hosios condition holds. This is the standard result in settings with homogeneous workers. With heterogeneous workers,

$$\Lambda(q,t) = \inf_{\lambda \in [0,\infty]} \{v_{SB}(p) \geq \eta(\lambda)[y(p,q) - t], p \in [0,1]\},$$

so a deviating offer may attract a longer queue of workers of other types. Yet, Assumptions (Y) and (M) ensure such an offer will not be profitable for the asset owners. The price paid by the worker is independent of his type. Rearranging the inequality (13),

$$u_{SB}(q) \geq \max_{p,\lambda}[\delta(\lambda)y(p,q) - \lambda v_{SB}(p)] = \max_{t} \delta(\Lambda(q,t))t,$$

and equality holds for $q \geq \underline{q}_{SB}$. Figure 1 depicts workers' and asset owners' indifference curves, which yield their equilibrium payoffs, for assets q' in price competition. Taking workers' equilibrium payoff as given, the set of prices and resulting queue lengths available to an asset owner is the lower envelope of all workers' indifference curves. Note that the asset owners' indifference curve applies to any type of worker, as her payoff does not depend on the latter. The equilibrium queue length and price is then the tangent point of the indifference curves for the asset owners and workers of type $r_{SB}(q')$. This is a graphical representation of the Hosios condition.

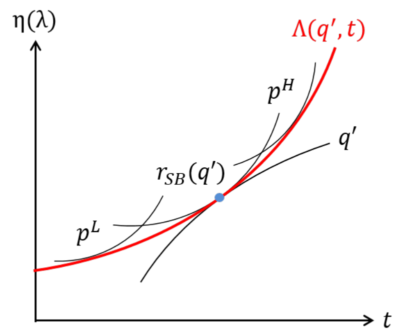

Figure 1: Graphical representation of the Hosios condition in price competition

4 Equilibrium Characterization

I now turn to the set of equilibria when only output sharing contracts are feasible. Workers' expected payoff can be separated into $y(p,q)$ and $\eta(\lambda)(1-s)$, and only the former depends on the private type. This has two important implications. First, workers' preferences over q and $\eta(1-s)$ satisfy the strict single crossing property (SCP), as the output exhibits strict log-SPM. This reduces the multidimensional sorting into a familiar one-dimensional case. Second, fix the quality of an asset, workers of all types share the same preference over their matching probability and the contract term. The corresponding sets of indifference curves for workers are illustrated in Figure 2

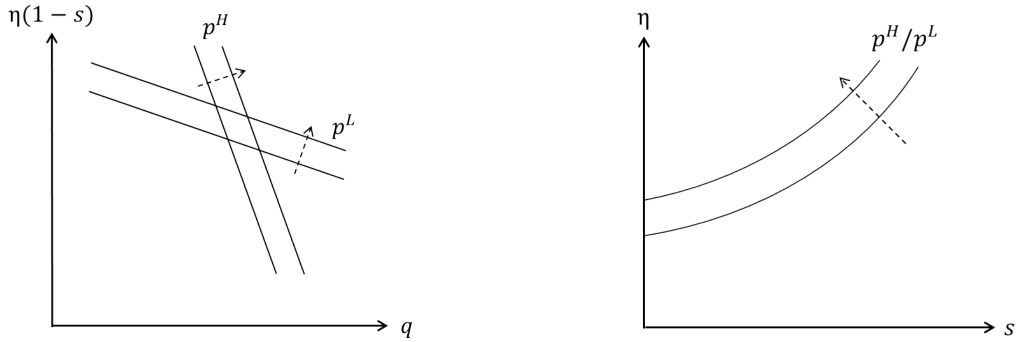

Figure 2: Properties of workers' indifference curves

The first property implies that if a worker prefers a market with better assets to another market with lower quality assets, then all better workers strictly prefer the market for better assets, and vice versa. This holds regardless of queue lengths and contract terms in these markets. Therefore, the participants must match assortatively in any equilibrium.

This property also ensures monotonic participation on the asset side in any equilibrium. Suppose some type of workers participate in an active market (q^L, s^L), an owner of a better asset q^H can find a less generous contract s^H, leaving these workers indifferent about accepting the two contracts. Hence, queue length in market (q^H, s^H) will be no lower than that in the active market (q^L, s^L). The strict SCP then implies that all weaker workers strictly prefer the latter market to the former. As such, posting the contract s^H provides an asset owner of q^H an expected payoff above the equilibrium payoff for her peers of q^L. I conclude that the equilibrium payoff for the asset side is increasing.

Let $(\underline{\widetilde{p}}, \underline{\widetilde{q}}, \widetilde{\kappa})$ represent PAM in the equilibrium under consideration, and $\widetilde{r}$ denotes inverse of $\widetilde{\kappa}$.

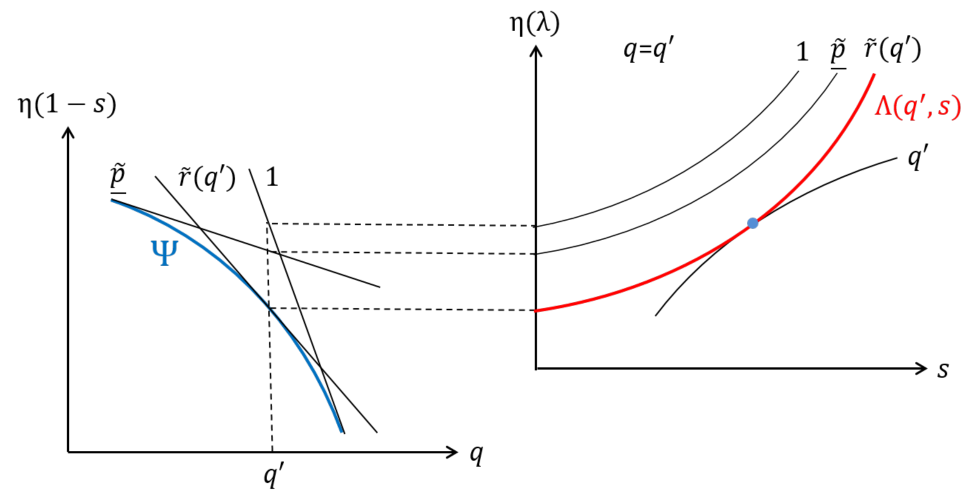

Figure 3: Graphical characterization of active markets and the Hosios condition

The indifference curves over q and $\eta(1-s)$ of the participating workers, which yield their equilibrium payoff, are plotted in the left panel of Figure 3. To satisfy the incentive compatibility for workers, the lower envelope of all of these indifference curves must be $\eta(\Lambda(q,s))(1-s)$ for the set of active markets. Furthermore, the workers will have a strict preference over the resulting set of active markets. Put differently, the workers of $\widetilde{r}(q')$ accept a lower matching probability in market (q', s') than anybody else.

As all workers have the same preference over their matching probability and the contract term for assets q', only workers of $\widetilde{r}(q')$ will accept the lowest matching probability for any contract (q', s). In equilibrium, the competition between workers of $\widetilde{r}(q')$ alone will result in an adjustment in queue length fully offsetting the change in the posted share. For a given asset, the share and the matching probability are perfect substitutes to the workers. Hence, a deviating asset owner of $q' > \underline{\widetilde{q}}$, if gets matched, always pairs up with the same type of worker in equilibrium. She only trades off between her matching probability and her output share. This is illustrated in the right panel of Figure 3. Suppose we fix the pair of types $(\widetilde{r}(q'), q')$ and plot the indifference curves over $\eta(\lambda)$ and $(1-s)$. Taking workers' equilibrium payoff as given, the tangent point of the indifference curves of both sides is the optimal contract, and the associated queue length, for the asset owners. This is exactly the Hosios condition in a market where the two sides are homogeneous. Observe that the joint payoffs for the pair of types attains its maximum at such a queue length, subject to the free entry of workers at their equilibrium payoff.[13] In summary, owners of the same asset quality q, if participating, will post the same share

$$s = 1 - \frac{d \ln \delta(\widetilde{\lambda}(q))}{d \ln \lambda}$$

where $\widetilde{\lambda}(q)$ is the resulting queue length. Hence, $\widetilde{r}$ and $\widetilde{\lambda}$ must satisfy the law of motion in (8). Abusing the terminology, I call $(\underline{\widetilde{p}}, \underline{\widetilde{q}}, \widetilde{\lambda}, \widetilde{r})$ the equilibrium allocation.

The Hosios condition for pair $(\widetilde{r}(q), q)$ can be rearranged as

$$\delta'(\widetilde{\lambda}(q)) = \eta(\widetilde{\lambda}(q))(1 - s'),$$

and thus,

$$V(\widetilde{r}(q)) = \delta'(\widetilde{\lambda}(q)) y(\widetilde{r}(q), q).$$

This is the same as the condition (9) for the second best allocation. I simply refer to the condition (9), which holds for *all* $q \geq \underline{\widetilde{q}}$, as the Hosios condition, unless confusion arises. Violation of the Hosios condition explains why $\widehat{s}(q)$ in Section 3 cannot be supported by an equilibrium.

The incentive compatibility (IC) condition for workers above the threshold type can also be rewritten as

$$V(\widetilde{r}(q)) = \max_{q' \in [\underline{\widetilde{q}}, 1]} \delta'(\widetilde{\lambda}(q')) y(\widetilde{r}(q), q'). \tag{15}$$

After accounting for the contract posting decisions, sorting of workers is induced by variation in the queue length. The better the asset, the longer the queue in the active market. Under Assumption (M), the output shares posted by asset owners' increase with their asset quality.

Applying the envelope theorem to (15), I obtain

$$\frac{dV(\widetilde{r}(q))}{dp} = \delta'(\widetilde{\lambda}(q)) \frac{\partial y(\widetilde{r}(q), q)}{\partial p}. \tag{16}$$

Under the strict SCP over q and $\eta(1-s)$, the conditions (15) and (16) are in fact equivalent. Abusing the terminology, I call the latter as workers' IC condition.

The strict SCP also implies that the active market for asset quality $\underline{\widetilde{q}}$ is the most profitable deviation for workers of the type below $\underline{\widetilde{p}}$. Therefore, workers of the threshold type $\underline{\widetilde{p}}$ must be indifferent about participation. This yields the boundary condition for the workers side, $\underline{\widetilde{p}}(V(\underline{\widetilde{p}}) - \underline{V}) = 0$.

[13] To be precise, $\widetilde{\lambda(q)}$ maximizes $\delta(\lambda) y(\widetilde{r}(q), q) - \lambda V(\widetilde{r}(q))$.

The boundary condition for the asset side is more complicated, as we have to determine the deviating payoff for owners of asset quality below $\underline{\widetilde{q}}$. Suppose an owner of asset quality $q' < \underline{\widetilde{q}}$ posts a deviating offer s'. We again observe the lower envelope of workers' indifference curves over q and $\eta(1-s)$, which yield their equilibrium payoff, including those below $\underline{\widetilde{p}}$, in Figure 4 The indifference curve for the workers attracted must be tangent to the lower envelope, which pins down $\eta(\Lambda(q', s'))(1-s')$.

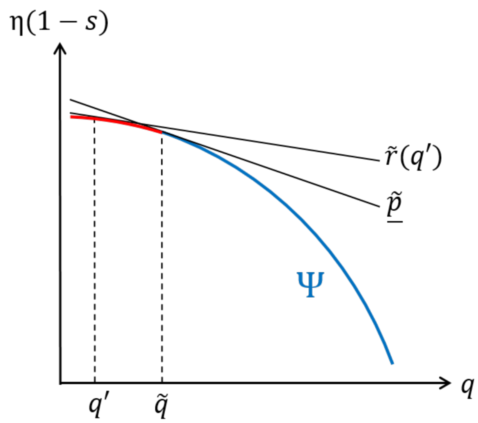

Figure 4: Graphical characterization of inactive markets for assets below threshold type

Under the strict SCP, workers' threshold type $\underline{\widetilde{p}}$ is the highest type a deviating offer may attract. As such, the asset owner can never make more than her peers of the threshold type $\underline{\widetilde{q}}$. However, posting the same output share provides an asset owner slightly below $\underline{\widetilde{q}}$ a deviating payoff very close to the equilibrium payoff of the threshold type $\underline{\widetilde{q}}$. The latter is given by $U(\underline{\widetilde{q}}) = [\delta(\widetilde{\lambda}(\underline{\widetilde{q}})) - \delta'(\widetilde{\lambda}(\underline{\widetilde{q}})\widetilde{\lambda}(\underline{\widetilde{q}})]y(\underline{\widetilde{p}}, \underline{\widetilde{q}})$ under the Hosios condition. As such, the boundary condition for the asset side, $\underline{\widetilde{q}}(U(\underline{\widetilde{q}}) - \underline{U}) = 0$, mirrors that for the workers. Note that the boundary conditions at the bottom are the same as those for the second best allocation in (12).

The set of equilibrium conditions (6)-(9), (11), (12), and (16) defines a boundary value problem, for which the set of equilibria can be recovered from the solutions. I analyze this boundary value problem in the Appendix. I establish the existence and uniqueness of its solution, and hence the equilibrium. As such, Proposition 1 fully characterizes the set of equilibria.

Proposition 1. *There exists a unique equilibrium. This equilibrium supports PAM and has the following properties:*

1. *Asset owners (workers) participate if and only if* $q \geq \underset{\sim}{q}$ $(p \geq \underset{\sim}{p})$, *and*

2. *Workers of* $p \geq \underset{\sim}{p}$ *have equilibrium payoffs* $\widetilde{v}(p)$, *and*

3. *The set of active markets* Ψ *is given by* $\left\{(q,s) : q \in [\underset{\sim}{q}, 1], s = 1 - \frac{d \ln \delta(\widetilde{\lambda}(q))}{d \ln \lambda}\right\}$, *with* $\Lambda(q,s) = \widetilde{\lambda}(q)$ *for* $(q,s) \in \Psi$, *and*

4. *For any* $s \in (0,1)$, $R(q,s)$ *is degenerate at* $\widetilde{r}(q)$ *if* $q \geq \underset{\sim}{q}$ *and* $\Lambda(q,s) > 0$.

$\widetilde{r} : [\underset{\sim}{q}, 1] \to [\underset{\sim}{p}, 1]$, $\widetilde{\lambda} : [\underset{\sim}{q}, 1] \to \mathbb{R}_{++}$, *and* $\widetilde{v} : [\underset{\sim}{p}, 1] \to \mathbb{R}_{++}$ *are all continuously differentiable and strictly increasing. Together with the pair of threshold types* $(\underset{\sim}{p}, \underset{\sim}{q})$, *they satisfy the conditions*

$$\begin{cases} \widetilde{r}(\underset{\sim}{q}) = \underset{\sim}{p}, \widetilde{r}(1) = 1, \\ \quad 0 = \underset{\sim}{q}[(\delta(\widetilde{\lambda}(\underset{\sim}{q})) - \delta'(\widetilde{\lambda}(\underset{\sim}{q}))\widetilde{\lambda}(\underset{\sim}{q}))y(\underset{\sim}{p}, \underset{\sim}{q}) - \underline{U}], \\ \quad 0 = \underset{\sim}{p}(\widetilde{v}(\underset{\sim}{p}) - \underline{V}), \\ \widetilde{r}'(q) = \dfrac{g(q)}{f(r(q))}\widetilde{\lambda}(q), \\ \widetilde{v}(\widetilde{r}(q)) = \delta'(\widetilde{\lambda}(q))y(\widetilde{r}(q), q), \\ \dfrac{d\widetilde{v}(\widetilde{r}(q))}{dp} = \delta'(\widetilde{\lambda}(q))\dfrac{\partial y(\widetilde{r}(q), q)}{\partial p}. \end{cases} \tag{17}$$

As explained in the introduction, the offering of output sharing contracts inevitably leads to an inefficient allocation. This is confirmed by comparing the set of conditions in Proposition 1 with those for the second best allocation, (6)-(12). In particular, the only difference in the boundary value problems is the condition (10) and (16). The condition (10) can be interpreted as the workers' incentive compatibility condition in price competition. Bilateral meeting implies $\delta' < \eta$. Under the *same* allocation, the condition (10) and (16) indicate that the information rent for workers grows at a slower rate when output shares are posted. This is an extension of the linkage principle in the security-bid auction literature (DeMarzo, Kremer, and Skrzypacz, 2005). The tricky part is the allocations in the two setting are always different. We will discuss the differences in the next section.

Nevertheless, a subset of the inequalities (13) in Remark 1 still applies to the equilibrium payoffs.

Lemma 1. *Consider any* $q \geq \underset{\sim}{q}$. *For all* $p \leq \widetilde{r}(q)$, *the equilibrium payoffs satisfy*

$$U(q) + \lambda V(p) \geq \delta(\lambda)y(p,q), \tag{18}$$

and equality holds if and only if $p = \widetilde{r}(q)$ *and* $\lambda = \widetilde{\lambda}(q)$*. Furthermore, there exist some types* $\widehat{q} > q$ *such that for any* $q^H \in (q, \widehat{q})$

$$U(q) < \delta(\widetilde{\lambda}(q^H))y(\widetilde{r}(q^H), q) - \widetilde{\lambda}(q^H)V(\widetilde{r}(q^H)) \tag{19}$$

$$< \delta(\widetilde{\lambda}(q))y(\widetilde{r}(q^H), q) - \widetilde{\lambda}(q)V(\widetilde{r}(q^H)). \tag{20}$$

The best way to appreciate Lemma 1 is to consider a thought experiment in which a single asset owner becomes able to identify the worker's type she pairs with. She may post a contract specifying the output share as well as the type of workers accepted. The resulting queue length provides the targeted type of workers their equilibrium payoff.[14] We are interested in which type of workers she may make a poaching offer.

We focus on an asset owner of $q \geq \underset{\sim}{q}$. The maximum payoff she can obtain by poaching workers of type p is given by

$$\widetilde{U}(p, q) = \max_{\lambda \geq 0} \{\delta(\lambda)y(p, q) - \lambda\widetilde{v}(p)\}.$$

The unique maximizer $\widetilde{\Lambda}(q, p)$ is determined by the FOC

$$\delta'(\widetilde{\Lambda}(p, q))y(p, q) = \widetilde{v}(p).$$

For $p \geq \underset{\sim}{p}$, the Hosios condition (9) implies that $\widetilde{\Lambda}(p, q)) > (<)\widetilde{\lambda}(\widetilde{\kappa}(p))$ if $q > (<)\widetilde{\kappa}(p)$. It is efficient to provide agents a higher matching probability whenever their potential partner's type is lower.

From the workers' IC condition (16) and the Hosios condition,

$$\begin{aligned}
&\frac{\partial}{\partial p}\widetilde{U}(p, q) = \delta(\widetilde{\Lambda}(p, q))\frac{\partial}{\partial p}y(p, q) - \widetilde{\Lambda}(p, q)\frac{\partial}{\partial p}\widetilde{v}(p) \\
&= \delta(\widetilde{\Lambda}(p, q))\frac{\partial}{\partial p}y(p, q) - \widetilde{\Lambda}(p, q)\delta'(\widetilde{\lambda}(\widetilde{\kappa}(p)))\frac{\partial y(p, \widetilde{\kappa}(p))}{\partial p} \\
&> \widetilde{\Lambda}(p, q)\left[\eta(\widetilde{\Lambda}(p, q))\frac{\partial}{\partial p}y(p, q) - \eta(\widetilde{\lambda}(\widetilde{\kappa}(p)))\frac{\partial y(p, \widetilde{\kappa}(p))}{\partial p}\right] \\
&= \widetilde{\Lambda}(p, q)\widetilde{v}(p)\left[\frac{\eta(\widetilde{\Lambda}(p, q))}{\delta'(\widetilde{\Lambda}(p, q))}\frac{\partial \ln y(p, q)}{\partial p} - \frac{\eta(\widetilde{\lambda}(\widetilde{\kappa}(p)))}{\delta'(\widetilde{\lambda}(\widetilde{\kappa}(p)))}\frac{\partial \ln y(p, \widetilde{\kappa}(p))}{\partial p}\right]
\end{aligned}$$

The strict inequality holds because, when supporting the same allocation, the information rent for workers grows at a faster rate under prices than under sharing contracts. Under Assumptions (Y) and (M), $\frac{\partial}{\partial p}\widetilde{U}(p, q) > 0$ for $p \leq \widetilde{r}(q)$, and hence the inequality (18) holds.

[14] This is in line with the definition of equilibrium queue length in an inactive market.

Now consider pairing up assets q with workers slightly above $\widetilde{r}(\widehat{q})$. As the Hosios condition holds for pair $(\widetilde{r}(\widehat{q}), \widehat{q})$, a change in the queue length has at most a second-order effect on the expected payoff of the asset owners, whereas an improvement in the workers results in a first-order increase in asset owners' expected payoff. This leads to the inequality (19). The inequality (20) then follows from the workers' IC condition.

Lemma 1 implies that asset owners never attempt to poach weaker workers in equilibrium. In contrast to Remark 1, poaching better workers is potentially profitable. An asset owner gains from poaching workers of some type $\widetilde{r}(q^H)$ if she accepts a queue length $\widetilde{\lambda}(q)$ or $\widetilde{\lambda}(q^H)$, and adjusts the output share to compensate for the lower asset quality.

Without the ability to make the contract term contingent on the worker's type, the asset owners will not succeed at poaching in equilibrium. In the setting here, they do not have any instrument to screen out better workers. When an owner of asset q posts a share, the workers of $\widetilde{r}(q)$ are willing to endure the longest queue for it, crowding out all other types of workers.

5 Equilibrium Distortion

Although I have characterized the second best allocation and equilibrium, they vary with the distribution of types. I now establish the form of distortion in equilibrium for any distribution of types.

Proposition 2. *Compared with the second best allocation, the equilibrium allocation has the following features:*

1. *The queue length for the best assets is greater,* $\lambda_{SB}(1) < \widetilde{\lambda}(1)$; *and*

2. *All participating asset owners pair up with worse partners,* $r_{SB}(q) > \widetilde{r}(q)$ *for* $q \in (\underline{\widetilde{q}}, 1)$; *and*

3. *Higher participation on the workers' side,* $\underline{p}_{SB} \geq \underline{\widetilde{p}}$, *but lower participation on the asset side,* $\underline{\widetilde{q}} \geq \underline{q}_{SB}$; *and*

4. *The threshold type on one side remains unchanged only if that side features full participation in Second Best allocation; that is, if* $\underline{p}_{SB} = \underline{\widetilde{p}}$ $(\underline{\widetilde{q}} = \underline{q}_{SB})$, *then* $\underline{p}_{SB} = \underline{\widetilde{p}} = 0$ $(\underline{\widetilde{q}} = \underline{q}_{SB} = 0)$.

Corollary 1. *Compared with price competition,*

1. *the best workers are strictly worse off, whereas the lowest types of the participating workers of the threshold type $\underline{p}_{SB}$ strictly benefit; and*

2. *the owners of the highest quality assets are strictly better off, whereas those of the threshold asset quality $\underline{\widetilde{q}}$ must be strictly worse off.*

To better understand the form of distortion, it is instructive to start with an equilibrium in price competition. Suppose we replace the posted prices with output shares $s_{SB}(q) = 1 - \frac{d \ln \delta(\lambda_{SB}(q))}{d \ln \lambda}$, keeping the same division of the output for every matched pair. This set of contracts is not incentive compatible for the workers. Compared with a fixed price, a fixed share of output costs more to the better workers but less to the low types. The workers will have a higher deviating payoff from the active markets for better assets. In particular, a worker of $r_{SB}(q')$ can always profit from searching for assets slightly above q'.

Nevertheless, such deviation need not be the most profitable. If λ_{SB} is decreasing at $q = q'$, the worker will prefer the market for assets of even higher quality, which features a lower queue length and a lower share s_{SB}.[15] Therefore, only the best assets always have a longer queue of workers.[16]

In response, their owners will post a greater share to partially offset the increase in the queue length. These asset owners decide to retain a longer queue than in the second best allocation because their private value of matching probability increases with their share of the output.[17] On the other side, the best workers will suffer from reductions in their output share and their matching probability.

Under assortative matching, the pool of workers available to the lower quality assets must deteriorate amid the increase in queue length for the better assets. So distortion in queue lengths necessarily leads to distortion in sorting pattern. The asset owners in the intermediate range face two counteracting forces. First, a sharing contract costs less for weaker workers, intensifying local competition between workers. To maintain sorting, the

[15] Under Assumptions (Y) and (M), $t_{SB}(q)$ must be monotonic in q while $s_{SB}(q)$ increases with $\lambda_{SB}(q)$, and may not be monotonic.

[16] Alternatively, one can view this result through the lens of incentive compatibility. Suppose we replace $t_{SB}(q)$ with the set of incentive compatible contracts $\widehat{s}(q)$. As $\frac{\partial}{\partial p} \ln v_{SB}(p) > \frac{\partial}{\partial p} \ln \widehat{V}(p)$, the best workers suffer the greatest percentage reduction in their share of the surplus upon matching. Under PAM, the owners of the best assets then gain the most from an improvement in their matching probability. They always induce a longer queue of workers at the expense of other asset owners.

[17] In a setting with a large but finite number of agents, the average type of workers will decrease with the queue length. The owners of the best assets will still decide to retain a longer queue in equilibrium. Otherwise, a single asset owner will deviate to post a more generous term, drawing workers from other owners of the best assets.

asset owner offer less generous terms. However, they are now left with weaker workers. This is how the distortion in sorting pattern feedbacks to the distortion in queue lengths. In comparison with price competition, these asset owners' gain from a match may go in either direction, so does the distortion in the queue lengths.

The relative strengths of the two forces depend on the distribution of types. Surprisingly, it turns out that all but the best assets must settle with weaker partners, regardless of the distribution of types. By continuity, this must happen to assets slightly below the highest quality. Suppose, to the contrary, that we move down from the top and find the owners of asset $\widehat{q} > \underline{\widetilde{q}}$ pairing with the same type $\widehat{p} = \widetilde{r}(\widehat{q}) = r_{SB}(\widehat{q})$ as in the second best allocation. As workers slightly better than $\widehat{p}$ now match with better assets, the queue for assets $\widehat{q}$ must be shorter than that in the second best allocation, $\lambda_{SB}(\widehat{q}) \geq \widetilde{\lambda}(\widehat{q})$, under PAM. Figure 5 depicts the situation.

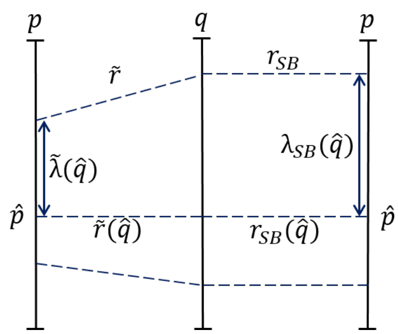

Figure 5: Implication of same matched pair of types on law of motion

Now consider the thought experiment where all workers above $\widehat{p}$ and assets above $\widehat{q}$ are removed. The utilitarian planner still finds the original second best allocation optimal for this truncated distribution of types. Otherwise, she would have improved upon it in the first place. The same set of equilibrium conditions in Proposition 1 applies to the truncated distribution of types. Even though workers of type $\widehat{p}$ now have no better assets to deviate to, owners of asset quality $\widehat{q}$ must continue to offer the same contract in order to deter deviation by workers slightly weaker than $\widehat{p}$. From the preceding discussion, we conclude that the queue for assets $\widehat{q}$ must be inefficiently short for this truncated distribution of types. This contradicts my first claim! As $\widehat{p}$ and $\widehat{q}$ are now the highest types, the owners of asset quality $\widehat{q}$ will induce an inefficiently long queue of workers, $\lambda_{SB}(\widehat{q}) < \widetilde{\lambda}(\widehat{q})$.

To understand the distortion in threshold types, let us return to our preceding discussion on the hypothetical set of contracts $s_{SB}(q)$. Suppose $\underline{p}_{SB} > 0$, workers of the threshold type

$\underline{p}_{SB}$ are indifferent about their outside option and entering the market $(s_{SB}(\underline{q}_{SB}), \underline{q}_{SB})$. Those of type below $\underline{p}_{SB}$ now pay less under the sharing contract $s_{SB}(\underline{q}_{SB})$. Some of them will be induced to participate. As such, participation on the workers' side must increase in equilibrium if $\underline{p}_{SB} > 0$.

We now turn to the threshold quality on the asset side. If no assets are assigned to their outside options in the second best allocation, $\underline{q}_{SB} = 0$, it is trivial that the threshold quality can only increase in equilibrium. On another hand, If no workers take their outside options in the second best allocation, it follows that $\underline{p}_{SB} = \underline{\widetilde{p}} = 0$. We have established that workers of the threshold type pair up with better asset in equilibrium, so threshold quality of asset must be inefficiently high.[18] It remains to be argued that some asset owners must be discouraged from participating in the case $\underline{p}_{SB} > 0$ and $\underline{q}_{SB} > 0$. The utilitarian planner will keep assigning agents into participation until the net expected surplus for the last pair of types declines to zero. Now compare this pair of threshold types to that in equilibrium. The threshold asset quality must be higher in equilibrium to compensate for the lower worker's type. Otherwise, one side will be better off taking the outside option.

With the distortion of the threshold types, the change in equilibrium payoffs, $\widetilde{v}(\underline{p}_{SB}) > v_{SB}(\underline{p}_{SB})$ and $\widetilde{u}(\underline{\widetilde{q}}) < u_{SB}(\underline{\widetilde{q}})$, can be deduced from the boundary conditions and the Hosios condition.

Note that the above arguments hinge on the Hosios condition and the property that better workers always pay more under the sharing contracts.[19] Nevertheless, we are still able to draw conclusions on how the matching pattern and participation margin are distorted.

Again, two counteracting forces affect the distortion of the queue length at the bottom. On the one hand, the owners of assets slightly above $\underline{\widetilde{q}}$ now benefit less from a higher matching probability as they face weaker partners. On the other hand, their cost of increasing their matching probability may also fall. This happens when $\widetilde{v}(\underline{p}_{SB}) > v_{SB}(\underline{p}_{SB}) > \widetilde{v}(\underline{\widetilde{p}})$. Therefore, the queue length at the lower end can be distorted in either direction. This is illustrated in the following example.

Example (A symmetric example)**.** *Suppose types on the two sides are both uniformly*

[18] A more involved argument is need to rule out the possibility that $\underline{p}_{SB} = \underline{\widetilde{p}} = 0$ and $\underline{q}_{SB} = \underline{\widetilde{q}} > 0$. In this case, the queue length for the pair of threshold types must also remain the same under the Hosios condition. As workers near the lowest type pay less under the sharing contract, asset owners slightly above the threshold quality face a longer queue than in price competition. This is exactly opposite to the case in Figure 5. These asset owners must pair up with better workers than in the second best allocation, contradicting my second claim.

[19] Formally, the expected payment received by the asset owner strictly increases with the worker's type for any contracts and asset quality.

distributed over $[0,1]$*. The outside options for the two sides yield the same payoff* $\underline{V} = \underline{U}$*. Consider a market with O-ring production technology* $y(p,q) = (\overline{y} - \underline{y})pq + \underline{y}$ *and random matching technology* $\delta(\lambda) = \frac{\lambda}{\lambda+1}$*. The Second Best allocation inherits the symmetry between both sides in the setup. It is given by* $r_{SB}(q) = q$*,* $\lambda_{SB}(q) = 1$*,* $v_{SB}(p) = \frac{1}{4}y(p,p)$ *and* $u_{SB}(q) = \frac{1}{4}y(q,q)$*.* $\underline{p}_{SB} = \underline{q}_{SB}$ *satisfy* $\frac{1}{4}[(\overline{y} - \underline{y})\underline{q}_{SB}^2 + \underline{y}] = \underline{U}$ *if* $\frac{1}{4}\underline{y} < \underline{U}$*. Otherwise,* $\underline{p}_{SB} = \underline{q}_{SB} = 0$*.*

I first consider the case $\frac{1}{4}\underline{y} < \underline{U}$*. The boundary conditions at the bottom immediately imply that* $\underline{\widetilde{p}\widetilde{q}} > 0$ *and*

$$\left(\frac{1}{\widetilde{\lambda}(\underline{\widetilde{q}})+1}\right)^2 [(\overline{y} - \underline{y})\underline{\widetilde{p}\widetilde{q}} + \underline{y}] = \underline{V} = \underline{U} = \left(\frac{\widetilde{\lambda}(\underline{\widetilde{q}})}{\widetilde{\lambda}(\underline{\widetilde{q}})+1}\right)^2 [(\overline{y} - \underline{y})\underline{\widetilde{p}\widetilde{q}} + \underline{y}].$$

It follows that $\widetilde{\lambda}(\underline{\widetilde{q}}) = 1$ *and* $\underline{\widetilde{p}\widetilde{q}} = \underline{p}_{SB}\underline{q}_{SB}$*. Proposition 2 implies that* $\underline{\widetilde{p}} < \underline{p}_{SB} = \underline{q}_{SB} < \underline{\widetilde{q}}$*. Proposition 1 states that* $\widetilde{\lambda}(q)$ *is strictly increasing. I conclude that the queue length in every active market, except the one for threshold quality, is inefficiently high,* $\widetilde{\lambda}(q) > 1$*.*[20]

For the case $\frac{1}{4}\underline{y} > \underline{U}$*, Proposition 2 states that* $0 = \underline{\widetilde{p}} = \underline{p}_{SB} = \underline{q}_{SB} < \underline{\widetilde{q}}$*. The boundary condition for* $\underline{\widetilde{q}} > 0$ *implies that*

$$\tfrac{1}{4}\underline{y} > \underline{U} = \left(\frac{\widetilde{\lambda}(\underline{\widetilde{q}})}{\widetilde{\lambda}(\underline{\widetilde{q}})+1}\right)^2 \underline{y},$$

and hence $\widetilde{\lambda}(\underline{\widetilde{q}}) < 1$*. There is a threshold asset quality* $\widehat{q}$ *such that all active markets for* $q > \widehat{q}$ *feature an inefficiently high queue length and the opposite occurs to the active markets for* $q < \widehat{q}$*.*

5.1 Reassignment by utilitarian planner

In the previous section, I relate the form of distortion to the incentives of individual agents. In this subsection, I approach the form of distortion from the utilitarian planner's perspective. Consider the following thought experiment. Participating agents are distributed across markets according to (K, L) in equilibrium, and have not formed matches. Suppose the utilitarian planner may reassign a randomly chosen agent to any market or her outside option.[21] Furthermore, an asset can be assigned to a market indexed by a different asset quality. I study how the utilitarian planner's reassignment decision varies with the agent's

[20] This conclusion also extends to the case $\frac{1}{4}\underline{y} = \underline{U}$.

[21] The reassignment of a single agent does not affect the expected payoff of any other agent. It does not matter whether the agents anticipate the reassignment beforehand.

type, and relate it to the Hosios condition and workers' IC condition. Again, I focus on "distribution-free" conclusion on the reassignment decision, and explain how it is consistent with the findings in Proposition 2.

As the reassigned agent will not get matched in an inactive market, the planner considers only the set of active markets, where each features a single pair of types $(\widetilde{r}(q), q)$. If an additional worker of type $\widehat{p}$ is assigned to an active market for assets of q', the aggregate surplus will change by

$$\eta(\widetilde{\lambda}(q'))y(\widehat{p}, q') + \widetilde{\lambda}(q')\eta'(\widetilde{\lambda}(q'))y(\widetilde{r}(q'), q').$$

The new worker will generate an increase in surplus by $\eta(\widetilde{\lambda}(q'))y(\widehat{p}, q')$ himself. At the same time, his participation will decrease the number of matches of the pair $(\widetilde{r}(q'), q')$, reducing the aggregate surplus by $-\widetilde{\lambda}(q')\eta'(\widetilde{\lambda}(q'))y(\widetilde{r}(q'), q')$.[22] If an asset of quality $\widehat{q}$ is assigned to an active market for assets of q', the aggregate surplus will change by

$$\delta(\widetilde{\lambda}(q'))y(\widetilde{r}(q'), \widehat{q}) - \widetilde{\lambda}(q')\delta'(\widetilde{\lambda}(q'))y(\widetilde{r}(q'), q').$$

The increase in surplus generated by the additional asset is $\delta(\widetilde{\lambda}(q'))y(\widetilde{r}(q'), \widehat{q})$, while the loss due to the reduction in matches of the pair $(\widetilde{r}(q'), q')$ is given by $\widetilde{\lambda}(q')\delta'(\widetilde{\lambda}(q'))y(\widetilde{r}(q'), q')$.

Conditional on the equilibrium allocation, the marginal contribution to aggregate surplus by an agent is the reduction in the aggregate surplus if she is removed from the population. For example, the marginal contribution by a worker of type $\widetilde{r}(q)$ is the change in aggregate surplus when a worker of the same type is added to the active market for assets q. Note that everyone receives her marginal contribution in the equilibrium here. This is because the Hosios condition holds in every active market.

We first consider a case where the chosen agent is a worker of type $\widehat{p} < \underline{\widetilde{p}}$, who contributes $\underline{V}$ to the aggregate surplus. If he is reassigned to an active market for assets of q', the aggregate surplus will change by $\eta(\widetilde{\lambda}(q'))y(\widehat{p}, q') + \widetilde{\lambda}(q')\eta'(\widetilde{\lambda}(q'))y(\widetilde{r}(q'), q')$. The planner never reassigns a worker of type $\widehat{p} < \underline{\widetilde{p}}$ because

$$\begin{aligned} \underline{V} &> \delta'(\widetilde{\lambda}(q'))y(\widehat{p}, q') = \eta(\widetilde{\lambda}(q'))y(\widehat{p}, q') + \widetilde{\lambda}(q')\eta'(\widetilde{\lambda}(q'))y(\widehat{p}, q') \\ &> \eta(\widetilde{\lambda}(q'))y(\widehat{p}, q') + \widetilde{\lambda}(q')\eta'(\widetilde{\lambda}(q'))y(\widetilde{r}(q'), q'). \end{aligned}$$

The first inequality comes from incentive compatibility. The second inequality holds because the reassigned worker will displace better workers of type $\widetilde{r}(q')$.

[22]Recall that $M(l, k)$ is the aggregate matching function. Then, $\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left[M(l, \frac{l}{l+\epsilon}k) - M(l, k)\right] = \frac{l}{k}\eta'(\frac{l}{k})$. The remaining expressions are derived in a similar manner.

Now consider a worker of type $\widehat{p} = \widetilde{r}(\widehat{q}) > \underline{\widetilde{p}}$. The planner will not reassign him to the outside option because his marginal contribution $V(\widehat{p}) > \underline{V}$. The worker will not be assigned to match with better assets either. For $q^H > \widehat{q}$, the change in aggregate surplus is given by

$$\begin{aligned} & \eta(\widetilde{\lambda}(q^H))y(\widehat{p}, q^H) + \widetilde{\lambda}(q^H)\eta'(\widetilde{\lambda}(q^H))y(\widetilde{r}(q^H), q^H) \\ < \;& \delta'(\widetilde{\lambda}(q^H))y(\widehat{p}, q^H) < \delta'(\widetilde{\lambda}(\widehat{q}))y(\widehat{p}, \widehat{q}) = V(\widehat{p}). \end{aligned}$$

Again, the first inequality is due to the displacement of better workers, while the second inequality is due to the workers' IC condition. In fact, the planner will gain from reassigning the worker to some inferior assets q^L. The net change in aggregate surplus can be rewritten as

$$\begin{aligned} & \eta(\widetilde{\lambda}(q^L))y(\widehat{p}, q^L) + \widetilde{\lambda}(q^L)\eta'(\widetilde{\lambda}(q^L))y(\widetilde{r}(q^L), q^L) - \delta'(\widetilde{\lambda}(\widehat{q}))y(\widehat{p}, \widehat{q}) \\ = \;& \eta(\widetilde{\lambda}(q^L))[y(\widehat{p}, q^L) - y(\widetilde{r}(q^L), q^L)] - [\delta'(\widetilde{\lambda}(\widehat{q}))y(\widehat{p}, \widehat{q}) - \delta'(\widetilde{\lambda}(q^L))y(\widetilde{r}(q^L), q^L)]. \end{aligned}$$

As $\eta(\widetilde{\lambda}(\widehat{q}) > \delta'(\widetilde{\lambda}(\widehat{q}))$, the above expression is positive when q^L is sufficiently close to $\widehat{q}$. The root cause is that incentive compatibility requires a slower growth of information rent for the workers under sharing contracts than under prices.

The above arguments also imply that the planner cannot gain from reassigning a worker of the threshold type $\underline{\widetilde{p}}$, be it $\underline{\widetilde{p}} > 0$ or $V(\underline{\widetilde{p}}) > \underline{V}$.

We now carry out the same exercise for the asset side, with markets indexed by asset quality. First, the utilitarian planner will reassign an asset of $\widehat{q} \geq \underline{\widetilde{q}}$ to better workers. If the asset remains in the active market for quality $\widehat{q}$, the marginal contribution will be $U(\widehat{q}) \geq \underline{U}$. Consider asset qualities $\widehat{q} > q^L \geq \underline{\widetilde{q}}$. Rearranging the inequality (18) with $(p, q, \lambda) = (\widetilde{r}(q^L), \widehat{q}, \widetilde{\lambda}(q^L))$,

$$\begin{aligned} U(\widehat{q}) &= [\delta(\widetilde{\lambda}(\widehat{q})) - \widetilde{\lambda}(\widehat{q})\delta'(\widetilde{\lambda}(\widehat{q}))]y(\widetilde{r}(\widehat{q}), \widehat{q}) \\ &> \delta(\widetilde{\lambda}(q^L))y(\widetilde{r}(q^L), \widehat{q}) - \widetilde{\lambda}(q^L)\delta'(\widetilde{\lambda}(q^L))y(\widetilde{r}(q^L), q^L). \end{aligned}$$

As such, the aggregate surplus declines if an asset is reassigned to weaker workers. Lemma 1 states that there are some types of assets q^H slightly better than $\widehat{q}$ satisfying

$$\begin{aligned} U(\widehat{q}) &= \delta(\widetilde{\lambda}(\widehat{q}))y(\widetilde{r}(\widehat{q}), \widehat{q}) - \widetilde{\lambda}(\widehat{q})V(\widetilde{r}(\widehat{q})) \\ &< \delta(\widetilde{\lambda}(q^H))y(\widetilde{r}(q^H), \widehat{q}) - \widetilde{\lambda}(q^H)V(\widetilde{r}(q^H)). \end{aligned}$$

I conclude that the aggregate surplus will increase after reassigning the asset to slightly better workers.

By continuity, the above argument also implies that the utilitarian planner will reassign an asset of quality slightly below $\underline{\widetilde{q}}$

Like Proposition 2, the conclusion on the reassignment decision applies to all distributions of types. Let us compare the two results. The equilibrium features excessive entry of workers and inefficiently low participation on the asset side. This is consistent with the utilitarian planner's decision to leave a worker of type below $\underline{\widetilde{p}}$ to his outside option, but reassign some assets of quality slightly below $\underline{\widetilde{q}}$ to match with workers. The planner can increase the aggregate surplus by reassigning a worker of type above $\underline{p}_{SB}$ to assets of lower quality or an asset above $\underline{\widetilde{q}}$ to better workers. This observation is consistent with two features of the distortion in equilibrium. First, the participating workers pair up with better assets than in the second best allocation. Second, the equilibrium queue length is inefficiently high for the best assets, as the planner reassigns the best workers but not the best assets. In comparison, it is less obvious that the planner will reassign a worker of $p \in (\underline{\widetilde{p}}, \underline{p}_{SB})$ to assets of lower quality q^L instead of his outside option. Insufficient participation on the asset side is the underlying reason. The quality of assets to which the worker is reassigned is still above the efficient threshold $\underline{q}_{SB}$. As such, this reassignment increases the aggregate surplus by

$$\begin{aligned}&\eta(\widetilde{\lambda}(q^L))y(\widehat{p}, q^L) + \widetilde{\lambda}(q^L)\eta'(\widetilde{\lambda}(q^L))y(\widetilde{r}(q^L), q^L)\\ >&\delta'(\widetilde{\lambda}(q^L))y(\widetilde{r}(q^L), q^L) > \underline{V}.\end{aligned}$$

5.2 Policy intervention

In Section 3, I describe how the utilitarian planner may induce the second best allocation by directly imposing the share schedule $\widehat{s}(q)$ for $q \geq \underline{q}_{SB}$. However, the share schedule $\widehat{s}(q)$ provides the asset side a payoff above its shadow value. In decentralized markets, the asset owners will demand a lower share to improve their matching probability. A Pigouvian intervention can correct the asset side's incentive. Let $\tau(q)$ denote the tax payment levied on an owner of asset $q \geq \underline{q}_{SB}$ when she pairs up with a worker. $\tau(q)$ is determined by the condition

$$\delta'(\lambda_{SB}(q))[y(r_{SB}(q), q) - \tau(q)] = \widehat{V}(r_{SB}(q)).$$

With $\tau(q)$ in place, there is an equilibrium in which asset owners above the threshold quality will post $\widehat{s}(q)$, and the second best allocation will be decentralized. [23]When an

[23]Strictly speaking, we have to formally define the expected payoffs and an equilibrium in a setting with the intervention. I omit it here as the argument is very straightforward.

owner of asset $q \geq \underline{q}_{SB}$ posts a different share, she will still match with a worker of $r_{SB}(q)$. Suppose she induces a queue length slightly above $\lambda_{SB}(q)$, she must receive $\widehat{V}(r_{SB}(q))$ less to compensate the workers. This amount equals to $\delta'(\lambda_{SB}(q))[y(r_{SB}(q), q) - \tau(q)]$, the associated increase in expected payoff net of tax. With the usual assumptions on matching technology, the asset owner will not gain from distorting the queue length. Asset owners below the threshold type will also refrain from participating, as the owners of the threshold quality assets are earning their outside option.[24]

As the goal is to reduce asset owners' gain from a match, $\tau(q)$ can also be implemented as the subsidy paid to a participating asset owner if she is left unmatched, or the combination of both to achieve budget balance. It is stressed that the design of Pigouvian intervention requires knowledge of the entire distribution of types, contrasting with the distribution-free results.

6 Discussion

6.1 Source of sorting inefficiency

The recipe for inefficiency here contains three ingredients: output sharing contracts, private types, and search friction. The equilibrium allocation would be (constrained) efficient if we remove any of them from the model framework here.

Eeckhout and Kircher (2010) show that second best allocations can always be decentralized in price competition. If workers' types are contractible, a menu of output shares may function as a posted price, as its term can be made contingent on types to implement the required transfer. Therefore, the second best allocation can be decentralized. In such equilibrium, condition (13) in Remark 1 must hold, so that an asset owner cannot gain from posting a menu of output shares specifying different expected payments for different types.[25]

Now consider an environment where the parties face no search friction. In the first best allocation, the matching is perfectly assortative with a unit queue length for every matched pair. The first best allocation always prevails if the asset side post prices. When we replace the posted prices with the output shares keeping the same divisions of the outputs, the

[24] The exact argument mirrors that in Section 4.

[25] The root cause is the bilateral meeting assumption. Specifically, an asset owner cannot meet with multiple applicants and pick the most productive one ex post, as in Shi (2002); Shimer (2005). Consequently, the asset owner does not benefit from inducing different queue lengths for different types.

workers will again deviate to better assets, resulting in a longer queue for the best assets. Without search friction, the owners of the best assets will increase their posted shares until the queue length restores to unity. The asset owners still have a greater share of the surplus upon matching, and hence a higher marginal value of matching probability. However, they cannot improve their matching probability by distorting the queue length. Their decisions in turn leave the same pool of workers to owners of the second highest quality assets. Inductively, the equilibrium allocation remains first best amid higher equilibrium payoffs for all participating asset owners.

6.2 Efficiency benchmark

The utilitarian planner's program has been widely adopted as an efficiency benchmark in the sorting and competitive search literature. A principal reason is that its solution often coincides with the set of equilibrium allocations (e.g., Shapley and Shubik, 1971; Chiappori, McCann, and Nesheim, 2010; Moen, 1997). In the current framework, this is indeed the case when asset owners post prices or workers' types are public. My results can also be viewed in terms of how sorting is affected when feasible contracts change from prices to output shares or the types on one side are no longer observable.

6.3 Sharing contracts

A notable feature of my setting is that contract choice affects only the division, but not the size, of the expected output. Consequently, the sum of the values of the matching probabilities for any pair is given by their output.[26] For any pair of types, utility can be transferred between two sides perfectly by adjusting the contract term, but imperfectly by changing the queue length. This setup yields a unique second best allocation and allows us to focus on the distortion of extensive margins, that is, queue lengths, matching pattern, and participation thresholds.

The introduction of general contracts along with incentive provision or risk sharing into matching gives rise to two known complications even in frictionless settings with public types.

First, Legros and Newman (2007) point out that utility is no longer perfectly transferable. When adjusting the term of a contract to transfer utility between two sides, the

[26]This is a cardinal property, and stronger than the ordinal notion of transferable utility in frictionless matching models.

conversion rate typically depends on types. In particular, Serfes (2005); Legros and Newman (2007); Chiappori and Reny (2016) demonstrate that the complementarity between type and transferability can lead to negative assortative matching.

Second, the form of the contracts can affect the type dependence of the matching surplus. Kaya and Vereshchagina (2014) illustrated this point. The cost of incentive provision in their setting can be either supermodular or submodular in type, depending on ownership structure, and hence the optimal contracts.

The above channels hinge on the model specification in the post-matching stage. A fruitful avenue for future research is determining how these channels in particular applications interact with search friction and private types when forming matches.

6.4 Hosios condition

In a market where both sides are homogeneous, the Hosios condition ensures that an agent's private benefit from participation is the same as her marginal contribution to the aggregate surplus, provided that the queue length is second best. With free entry on one side, the equilibrium queue length must indeed be second best.

With heterogeneity on both sides, the matching pattern and queue lengths for different types of assets are interlinked. The interpretation of the Hosios condition merits some elaboration. In my setting, the Hosios condition for the active market $(q, \widetilde{s}(q))$ is equivalent to the following: the equilibrium payoff for a worker of type $\widetilde{r}(q)$ is given by his marginal contribution to the aggregate surplus, and the same for an asset of quality q.

However, the utilitarian planner is better off reassigning the participating agents, except the best assets, to other active markets. I define the social value of an agent *given the equilibrium allocation* as the maximum increase in aggregate surplus the planner may achieve from assigning a new agent of the same type. In Section 5.1, I illustrate the role of the Hosios condition by studying the systematic departure of social value from the marginal contribution of agents given the equilibrium allocation.

For other forms of securities or contingent contracts, the preference over the matching probability and the term of the contract differ across workers' types. In equilibrium, the matched pairs do not fully separate into a continuum of markets where the Hosios condition is satisfied. In such a case, an asset owner will profit from distorting the queue length to screen out better workers. Under the Hosios condition, an incremental distortion of the queue length leads only to a second-order loss, while an improvement in the partner's type yields a first-order gain. As a result, the channel of sorting inefficiency will confound with the distortions associated with the screening by the asset owners and the well-known search externalities, including the thick market, congestion, and compositional effects.[27] I do not have any distribution-free results in such a setting. Studying the interaction between the above forces and the resulting distortion is left to future research. The results in this paper will serve as a useful benchmark.

6.5 Distribution-free results

In this highly stylized framework, a simple contract determines only the division of output. In applications, considerations of incentive provision, screening, or risking sharing certainly influence the choice of contracts offered. The distribution-free results here should by no means be interpreted as robust predictions when the asset side switches from offering prices to sharing contracts or the types on one side become private.

For sorting with search friction, distortions in queue lengths and the sorting pattern are intertwined. The central challenge is that the relative strengths of various channels and their interactions depend on the entire distribution of types. The contribution of this paper, in my view, is to identify and single out a new channel of inefficiency. The distribution-free features of the distortion then manifest that the economic forces underlying the sorting inefficiency are always at play.

[27] In general, asset owners may screen workers using two margins: term of contract and queue length. Unlike the standard screening models, queue lengths and the composition of workers in the active markets are jointly determined in equilibrium and must be consistent with the distribution of types.

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A Appendix

A.1 Proof of Remark 1

Eeckhout and Kircher (2010) show that the boundary value problem for the second best allocation admits a solution.[28] Fix $(\underline{p}_{SB}, \underline{q}_{SB}, r_{SB}, \lambda_{SB}, v_{SB}, u_{SB})$, I first show that it satisfies the inequality (13), and use the inequality to establish uniqueness of the second best allocation. Define

$$\widehat{U}(p,q) = \max_{\lambda \geq 0} \{\delta(\lambda) y(p,q) - \lambda v_{SB}(p)\}.$$

As $\delta(\lambda)$ is strictly concave, the unique maximizer, denoted by $\widehat{\Lambda}(p,q)$, is determined by the FOC,

$$\delta'(\widehat{\Lambda}(p,q)) y(p,q) = v_{SB}(p).$$

I first consider $p \geq \underline{p}_{SB}$. From the condition (9),

$$v_{SB}(p) = \delta'(\lambda_{SB}(\kappa_{SB}(p))) y(p, \kappa_{SB}(p)).$$

Therefore, $\widehat{\Lambda}(p,q) > (<) \lambda_{SB}(\kappa_{SB}(p))$ if $q > (<) \kappa_{SB}(p)$. By envelope theorem,

$$\begin{aligned}
\frac{\partial}{\partial p}\widehat{U}(p,q) &= \delta(\widehat{\Lambda}(p,q)) \frac{\partial}{\partial p} y(p,q) - \widehat{\Lambda}(p,q) \frac{\partial}{\partial p} v_{SB}(p) \\
&= \delta(\widehat{\Lambda}(p,q)) \frac{\partial}{\partial p} y(p,q) - \widehat{\Lambda}(p,q) \eta(\lambda_{SB}(\kappa_{SB}(p))) \frac{\partial}{\partial p} y(p, \kappa_{SB}(p)) \\
&= \widehat{\Lambda}(p,q) v_{SB}(p) \left[\frac{\eta(\widehat{\Lambda}(p,q))}{\delta'(\widehat{\Lambda}(p,q))} \frac{\partial \ln y(p,q)}{\partial p} - \frac{\eta(\lambda_{SB}(\kappa_{SB}(p)))}{\delta'(\lambda_{SB}(\kappa_{SB}(p)))} \frac{\partial \ln y(p, \kappa_{SB}(p))}{\partial p} \right]
\end{aligned}$$

The second inequality obtained by substituting the condition (10). Under Assumptions (Y) and (M), $\frac{\partial}{\partial p}\widehat{U}(p,q) > (<) 0$ if $q > (<) \kappa_{SB}(p)$. For $p < \underline{p}_{SB}$, $v_{SB}(p) = \underline{V} = v_{SB}(\underline{p}_{SB})$, so $\widehat{U}(p,q) < \max_{\lambda \geq 0}\{\delta(\lambda) y(\underline{p}_{SB}, q) - \lambda v_{SB}(\underline{p}_{SB})\} = \widehat{U}(\underline{p}_{SB}, q)$.

Together, for $q \geq \underline{q}_{SB}$ and any $p \in [0,1]$

$$\begin{aligned}
u_{SB}(q) &= \delta(\lambda_{SB}(q)) y(r_{SB}(q), q) - \lambda_{SB}(q) v_{SB}(r_{SB}(q)) \\
&= \widehat{U}(r_{SB}(q), q) = \max_{\lambda \geq 0}\{\delta(\lambda) y(p,q) - \lambda v_{SB}(p)\}
\end{aligned}$$

For $q < \underline{q}_{SB}$, the boundary condition requires that

$$u_{SB}(q) = \underline{U} = \max_{\lambda \geq 0}\{\delta(\lambda) y(p, \underline{q}_{SB}) - \lambda v_{SB}(p)\} > \max_{\lambda \geq 0}\{\delta(\lambda) y(p,q) - \lambda v_{SB}(p)\}.$$

[28] Although Eeckhout and Kircher (2010) assume the values of outside options to be zero, their proof is readily extended to the case of positive outside options.

This establishes the inequality (13).

Let $supp(L)$ denote the support of measure L. The aggregate surplus for (K, L) is given by

$$\begin{aligned}
& \int_{supp(L)} \eta(\frac{dL_{qs}}{dK}) y(p,q) dL + [F(1) - L_p(1)]\underline{V} + [G(1) - K_q(1)]\underline{U} \\
\leq & \int_{supp(L)} \frac{dK}{dL_{qs}} u_{SB}(q) + v_{SB}(p) dL + [F(1) - L_p(1)]\underline{V} + [G(1) - K_q(1)]\underline{U} \\
\leq & \int u_{SB}(q) dG(q) + \int v_{SB}(p) dF(p) \\
= & \int_{\underline{q}}^{1} \delta(\lambda_{SB}(q)) y(r_{SB}(q), q) dG(q) + F(\underline{p})\underline{V} + G(\underline{q})\underline{U}
\end{aligned}$$

The first inequality is due to the inequality (13) and the second stems from the boundary conditions for the second best allocation. The above inequality holds with equality if and only if (K, L) features PAM with $(\underline{p}_{SB}, \underline{q}_{SB}, \kappa_{SB})$ and $\frac{dL_{qs}}{dK} = \lambda_{SB}(q)$ almost everywhere in the support of L.

A.2 Proof of Proposition 1

In this proof, I first show the properties listed in Proposition 1. I then proceed to show that the boundary value problem in system (17) admits a unique solution. Note that $(\widetilde{r}, \widetilde{\lambda}, \widetilde{v})$ in any solution must be continuously differentiable and strictly increasing. In particular, $\frac{d \ln \widetilde{v}(\widetilde{r}(q))}{dp} = \frac{\partial \ln y(\widetilde{r}(q), q)}{\partial p}$, so the gain from matching with a better asset must be offset by a reduction in $\delta'(\widetilde{\lambda}(q))$. Hence, $\widetilde{\lambda}$ is strictly increasing.

Characterization of the equilibria

Any equilibrium satisfies the properties in Proposition 1.

"Only if"

Fix an equilibrium (K, L). The assumption in (1) ensures that the set of active markets is non-empty. Suppose not, consider the inactive market $(1, s')$ where s' and λ' satisfy $\underline{V} = \delta'(\lambda') y(1, 1) = \eta(\lambda')(1 - s') y(1, 1)$. As $\eta(\lambda')(1 - s') y(p, 1) < \underline{V}$ if $p < 1$, only the best workers will be attracted to this inactive market, and the resulting queue length is λ'. The deviating payoff for an owner of asset quality q is $[\delta(\lambda') - \delta'(\lambda')\lambda'] y(1, 1) > \underline{U}$! As participation is costly, every active market must have a positive finite queue length, and $s \in (0, 1)$. Furthermore, participation on the workers' side must be monotonic.

Step 1: All equilibria feature PAM

Suppose not, then there must exist $\{(q^H, s_1), (q^L, s_0)\} \in \Psi$, where $q^H > q^L$, and $p^H > p^L$ where p^L and p^H are in the support of $R(q^H, s_1)$ and $R(q^L, s_0)$ respectively. Workers' optimality condition is met only if

$$\begin{aligned} \eta(\Lambda(q^L, s_0))(1-s_0)y(p^H, q^L) &\geq \eta(\Lambda(q^H, s_1))(1-s_1)y(p^H, q^H), \text{ and} \\ \eta(\Lambda(q^H, s_1))(1-s_1)y(p^L, q^H) &\geq \eta(\Lambda(q^L, s_0))(1-s_0)y(p^L, q^L), \end{aligned}$$

This implies $y(p^H, q^L)y(p^L, q^H) \geq y(p^H, q^H)y(p^L, q^L)$, which contradicts strict log-SPM of $y(p, q)$!

As the equilibrium allocation features PAM, the threshold types $(\underline{p}, \underline{q})$ and $\kappa(p)$ are well defined. κ is strictly increasing because the distribution of types is atomless and every active market must have a positive finite queue length.

Step 2: $U(q)$ is strictly increasing for $q \geq \underline{q}$

Suppose $(q^L, s_0) \in \Psi$ and $q^L = \kappa(p^L)$. Fix $\widehat{q} > q^L$. Consider the inactive market $(\widehat{q}, \widehat{s})$, where $\widehat{s}$ is given by

$$(1-\widehat{s})y(p^L, \widehat{q}) = (1-s_0)y(p^L, q^L).$$

$\Lambda(\widehat{q}, \widehat{s}) \geq \Lambda(q^L, s_0)$ because $V(p^L) \geq \eta(\Lambda(\widehat{q}, \widehat{s}))(1-\widehat{s})y(p^L, \widehat{q})$. For all $p < p^L$,

$$V(p) \geq \eta(\Lambda(q^L, s_0))(1-s_0)y(p, q^L) > \eta(\Lambda(\widehat{q}, \widehat{s}))(1-\widehat{s})y(p, \widehat{q}).$$

The strict inequality follows from the log-SPM of $y(p, q)$. It follows that the support of $R(\widehat{q}, \widehat{s})$ contains no type below p^L. An owner of asset $\widehat{q}$ can ensure herself a payoff of

$$\delta(\Lambda(\widehat{q}, \widehat{s}))\widehat{s} \int y(p, \widehat{q})dR(\widehat{q}, \widehat{s}) > U(q^L).$$

Therefore, the participation of the asset side is monotonic, and the function $r(q)$ is well defined. By definition,

$$V(p) = \eta(\Lambda(q, s))(1-s)y(p, q) \text{ if } q = \kappa(p) \text{ and } (q, s) \in \Psi.$$

Step 3: For any $p \in [0, 1]$ and $(q, s) \in \Psi$,

$$V(p) > \eta(\Lambda(q, s))(1-s)y(p, q) \textit{ if } q \neq \kappa(p).$$

Consider two active markets (q^H, s_1) and (q^L, s_0), where $q^H > q^L$. Suppose a worker of type $r(q^H) > 0$ is indifferent between these two markets. His decision is optimal only if

$$\begin{aligned} \eta(\Lambda(q^L, s_0))(1-s_0)y(r(q^H), q^L) &= \eta(\Lambda(q^H, s_1))(1-s_1)y(r(q^H), q^H) \\ &\geq \eta(\Lambda(q, s'))(1-s')y(r(q^H), q) \end{aligned}$$

for all $(q, s') \in \Psi$ where $q \in (q^L, q^H)$. Strict log-SPM of $y(p,q)$ implies that for any $p < r(q^H)$,

$$\frac{\eta(\Lambda(q^L, s_0))(1-s_0)}{\eta(\Lambda(q^H, s_1))(1-s_1)} = \frac{y(r(q^H), q^H)}{y(r(q^H), q^L)} > \frac{y(p, q^H)}{y(p, q^L)}$$

and

$$\frac{\eta(\Lambda(q^L, s_0))(1-s_0)}{\eta(\Lambda(q', s'))(1-s')} \geq \frac{y(r(q^H), q')}{y(r(q^H), q^L)} > \frac{y(p, q')}{y(p, q^L)},$$

for all $(q', s') \in \Psi$ where $q' \in (q^L, q^H)$. PAM then implies that $\Lambda(q', s') = 0$ if $(q', s') \in \Psi$ and $q' \in (q^L, q^H)$. Hence, $U(q') = \underline{U} \leq U(q^L)$, contradicting my previous claim.

Suppose $1 > q^L = k(p^L)$ and $(q^L, s_0) \in \Psi$. A symmetric argument rules out the case that a worker of type p^L is indifferent between (q^L, s_0) and another active market (q^H, s_1) where $q^H > q^L$.

Step 4:Characterize active marketsΨ.

Lemma 2. *Suppose* $(q, s') \in \Psi$ *and for any* $p \in [0,1]$,

$$V(p) \geq \eta(\Lambda(q, s'))(1-s')y(p,q),$$

with equality if and only if $q = \kappa(p)$. *Then, for any* $s \in [0,1)$, $R(q,s)$ *is degenerate at* $r(q)$ *if* $\Lambda(q,s) > 0$. *Furthermore, an owner of asset quality* q *has no profitable deviations if and only if* $\Lambda(q, s')$ *satisfies*

$$\delta'(\Lambda(q, s')) = \eta(\Lambda(q, s'))(1-s'). \tag{21}$$

Proof. For $s \in [0,1)$ and $p \neq r(q)$,

$$\frac{V(p)}{V(r(q))} > \frac{y(p,q)}{y(r(q),q)} = \frac{\eta(\Lambda(q,s))(1-s)y(p,q)}{\eta(\Lambda(q,s))(1-s)y(r(q),q)}.$$

Suppose $\Lambda(q,s) > 0$. Then, $V(p) = \eta(\Lambda(q,s))(1-s)y(p,q)$ if and only if $p = r(q)$, and hence $R(q,s)$ is degenerate at $r(q)$. In this case, $\Lambda(q,s)$ is determined by

$$V(r(q)) = \eta(\Lambda(q,s'))(1-s')y(r(q),q) = \eta(\Lambda(q,s))(1-s)y(r(q),q).$$

An asset owner has no profitable deviations if and only if

$$U(q) \geq \delta(\Lambda(q,s))sy(r(q),q),$$

with equality at $s = s'$. This can be further simplified as

$$\Lambda(q,s') \in \arg\max_{\lambda \in [0,\infty]} \delta(\lambda) - \lambda\eta(\Lambda(q,s'))(1-s').$$

As $\delta(\lambda)$ is strictly concave and $\Lambda(q,s') \in (0,\infty)$, the above holds if and only if the equality

(21) holds. $\square$

Step 5: Establish the boundary value problem (17).
There exists a function $\widetilde{\lambda} : [0, 1] \to (0, \infty)$ *such that*

$$\Psi = \Big\{(q, s) : q \in [\underline{q}, 1], s = 1 - \frac{d \ln \delta(\widetilde{\lambda}(q))}{d \ln \lambda}\Big\},$$

and $\Lambda(q, s) = \widetilde{\lambda}(q)$ *for* $(q, s) \in \Psi$. $(\widetilde{\lambda}, V, r)$ *is continuously differentiable and satisfies the differential equation system in (17) along with* $\underline{q}$.

Recall that participation is monotonic on both sides, so there is at least one active market (q', s') for any $q' \in [\underline{q}, 1]$. Consider workers of $p \geq \underline{p}$. Substituting the equality (21), the expression of their equilibrium payoff can be expressed as

$$V(r(q')) = \delta'(\Lambda(q', s'))y(r(q'), q'),$$

and incentive compatibility requires

$$V(r(q')) = \max_{(q,s)\in\Psi} \{\delta'(\Lambda(q, s))y(r(q'), q)\}.$$

The envelope theorem implies that V is continuously differentiable (C^1). As $\delta'(\lambda)$ and $y(p, q)$ are C^1, there must exist a C^1 function $\lambda : [0, 1] \to (0, \infty)$ such that

$$\frac{dV(r(q))}{dp} = \delta'(\lambda(q))\frac{\partial y(r(q), q)}{\partial p}.$$

This also establishes that for each asset quality $q \geq \underline{q}$, there is exactly one active market (q', s') with $s' = 1 - \frac{d \ln \delta(\lambda(q'))}{d \ln \lambda}$. Furthermore, $\Lambda(q', s') = \lambda(q')$. Given full participation for $p \in [\underline{p}, 1]$ and $q \in [\underline{q}, 1]$, λ and r must satisfy the law of motion (8).

The boundary conditions for the threshold types remain to be shown. Suppose $\underline{p} > 0$. As $y(p, q)$ is strictly increasing in p, the workers $p < \underline{p}$ all take their outside option only if $V(\underline{p}) = \underline{V}$. Now suppose $\underline{q} > 0$, $(\underline{q}, \underline{s}) \in \Psi$ and $U(\underline{q}) > \underline{U}$. If $\underline{p} = 0$, an owner of q' slightly below $\underline{q}$ can secure a payoff $\delta(\lambda(\underline{q}))s'y(0, q') > \underline{U}$ by posting a share s' satisfying $\eta(\lambda(\underline{q}))(1-s')y(0, q') = V(0)$. I turn to the case $\underline{p} > 0$ so that $V(\underline{p}) = \eta(\lambda(\underline{q}))(1-\underline{s})y(\underline{p}, \underline{q}) = \underline{V}$. By continuity, for q' slightly below $\underline{q}$, there must exist $s' < \underline{s}$ and $p' < \underline{p}$ satisfying both

$$\delta(\lambda(\underline{q}))s'y(p', q') > \underline{U}$$
$$\eta(\lambda(\underline{q}))(1 - s')y(p', q') = \underline{V}$$

It follows that $\Lambda(q', s') \geq \lambda(\underline{q})$, and $\eta(\Lambda(q', s'))(1 - s')y(p, q') < \underline{V}$ for any $p < p'$. Hence, the expected payoff for an asset owner to participate in (q', s') must be above $\underline{U}$, rendering the deviation profitable. Therefore, $U(\underline{q}) = \underline{U}$ if $\underline{q} > 0$.

The preceding analysis verifies all properties in Proposition *1.*

"If"

Fix a solution $(\underline{\widetilde{p}}, \underline{\widetilde{q}}, \widetilde{r}, \widetilde{\lambda}, \widetilde{v})$ to the boundary value problem in system (17). A unique candidate equilibrium $(\widetilde{K}, \widetilde{L})$ can be recovered, satisfying the properties listed in Proposition *1.* Let $\widetilde{\kappa}$ denote the inverse of $\widetilde{r}$. Define $\widetilde{s}(q) = 1 - \frac{d \ln \delta(\widetilde{\lambda}(q))}{d \ln \lambda}$. As $\widetilde{\lambda}(q)$ is continuous and strictly increasing, $\widetilde{s}(q)$ is also continuous and increasing in q under Assumption (M). $\widetilde{K}(q', s') = 0$ if $q' \leq \underline{\widetilde{q}}$ or $s' \leq \widetilde{s}(\underline{\widetilde{q}})$. Otherwise, $\widetilde{K}(q', s') = G(\sup\{q \leq q' : \widetilde{s}(q) \leq s'\}) - G(\underline{\widetilde{q}})$. $\widetilde{L}(p', q', s') = F(\sup\{p \leq p' : \widetilde{\kappa}(p) \leq q', \widetilde{s}(\widetilde{\kappa}(p)) \leq s'\}) - F(\underline{\widetilde{p}})$ if $p > \underline{\widetilde{p}}$, $q' > \underline{\widetilde{q}}$ and $s' > \widetilde{s}(\underline{\widetilde{q}})$. Otherwise, $\widetilde{L}(p', q', s') = 0$.

I first verify that workers' optimality condition. $V(p) = \widetilde{v}(p) > \underline{V}$ for $p > \underline{\widetilde{p}}$. Combining the conditions (9) and (15),

$$\frac{d \ln \widetilde{v}(p)}{dp} = \frac{\partial \ln y(p, \widetilde{\kappa}(p))}{\partial p}, p \geq \underline{\widetilde{p}}.$$

Consider any $p_0 \geq \underline{\widetilde{p}}$ and $p_1 \neq p_0$,

$$\begin{aligned}
& \ln V(p_1) - \ln \delta'(\widetilde{\kappa}(p_0)) y(p_1, \widetilde{\kappa}(p_0)) \\
= \; & [\ln V(p_1) - \ln \widetilde{v}(p_0)] - [\ln y(p_1, \widetilde{\kappa}(p_0)) - \ln y(p_0, \widetilde{\kappa}(p_0)] \\
= \; & \int_{p_0}^{p_1} \frac{d \ln V(p)}{dp} - \frac{\partial \ln y(p, \widetilde{\kappa}(p_0))}{\partial p} dp \\
\geq \; & \int_{p_0}^{\max\{p_1, \underline{\widetilde{p}}\}} \frac{\partial \ln y(p, \widetilde{\kappa}(p))}{\partial p} - \frac{\partial \ln y(p, \widetilde{\kappa}(p_0))}{\partial p} dp > 0.
\end{aligned}$$

The last strict inequality is due to the strict log-SPM of $y(p, q)$ and $\widetilde{\kappa}$ is strictly increasing. Recall that $\eta(\Lambda(q, s))(1 - s) = \delta'(\widetilde{\lambda}(q))$ holds for any active market (q, s). Therefore, a worker of $p = \widetilde{r}(q)$ receives his highest payoff only at $(q, s) \in \Psi$, and the outside option is optimal for the workers of $p < \underline{\widetilde{p}}$.

I now turn to the asset side. For $q \geq \underline{\widetilde{q}}$,

$$U(q) = [\delta(\widetilde{\lambda}(q)) - \widetilde{\lambda}(q) \delta'(\widetilde{\lambda}(q))] y(\widetilde{r}(q), q) \geq \underline{U}.$$

Together with Lemma 2, the decision is optimal for owners of $q \geq \underline{\widetilde{q}}$.

Now suppose $\underline{\widetilde{q}} > 0$. Consider an inactive market (q^L, s') where $q^L < \underline{\widetilde{q}}$. For any $p^H > \underline{\widetilde{p}}$, $R(p^H | q^L, s') = 0$ because

$$\frac{V(p^H)}{V(\underline{\widetilde{p}})} > \frac{\delta'(\widetilde{\lambda}(\underline{\widetilde{q}})) y(p^H, \underline{\widetilde{q}})}{\delta'(\widetilde{\lambda}(\underline{\widetilde{q}})) y(\underline{\widetilde{p}}, \underline{\widetilde{q}})} > \frac{(1 - s') y(p^H, q^L)}{(1 - s') y(\underline{\widetilde{p}}, q^L)}.$$

Hence, $R(q^L, s')$ is degenerate at some $p^L \leq \underset{\sim}{p}$. The case $\Lambda(q^L, s') = 0$ is trivial. For the case $\Lambda(q^L, s') > 0$, $\Lambda(q^L, s')$ satisfies $\eta(\Lambda(q^L, s'))(1 - s')y(p^L, q^L) = V(p^L)$. If deviating to market (q^L, s'), an asset owner will receive

$$\begin{aligned} \delta(\Lambda(q^L, s'))s'y(p^L, q^L) &= \delta(\Lambda(q^L, s'))y(p^L, q^L) - \Lambda(q^L, s')V(p^L) \\ &\leq \max_{\lambda}[\delta(\lambda)y(p^L, q^L) - \lambda V(p^L)] \\ &= \max_{\lambda}[\delta(\lambda)y(p^L, q^L) - \lambda V(\underset{\sim}{p})] < U(\underset{\sim}{q}) = \underline{U}. \end{aligned}$$

The second equality holds because of the boundary condition $\underset{\sim}{p}(V(\underset{\sim}{p}) - \underline{V}) = 0$. As such, it is never optimal for an owner of $q < \underset{\sim}{q}$ to participate.

Analysis of the boundary value problem

By differentiating the Hosios condition w.r.t. q and subtracting it from the expression $\frac{d \ln \tilde{v}(\tilde{r}(q))}{dp}$, I obtain

$$\frac{d \ln \delta'(\widetilde{\lambda}(q))}{dq} = -\frac{\partial \ln y(\tilde{r}(q), q)}{\partial q}$$

There exists a unique pair of $\underline{\lambda}$ and $\overline{\lambda}$ satisfying $[\delta(\underline{\lambda}) - \delta'(\underline{\lambda})\underline{\lambda}]y(1,1) = \underline{U}$ and $\delta'(\overline{\lambda})y(1,1) = \underline{V}$, respectively. Note that $\overline{\lambda} > \underline{\lambda}$. Consider the following initial value problem (IPV-$\lambda(1)$):

$$\begin{aligned} r'(q) &= \frac{g(q)}{f(r(q))}\lambda(q), \\ \frac{d \ln \delta'(\lambda(q))}{dq} &= -\frac{\partial \ln y(r(q), q)}{\partial q}, \end{aligned}$$

where the initial values are given by $r(1) = 1$ and $\lambda(1) = \lambda^1 \in [\underline{\lambda}, \overline{\lambda}]$. As the differential equation system is locally Lipschitz, Picard's existence theorem ensures that IPV-$\lambda(1)$ (in the downward direction) admits a unique solution $\{r(q; \lambda^1), \lambda(q; \lambda^1)\}$ over the interval $[\underline{q}(\lambda^1), 1]$, where $\underline{q}(\lambda^1)$ is the first level of q where either of the following cases occurs:

$$0 = r(q; \lambda^1)[\delta'(\lambda(q; \lambda^1))y(r(q; \lambda^1), q) - \underline{V}], \text{ or} \tag{22}$$

$$0 = q[(\delta(\lambda(q; \lambda^1)) - \delta'(\lambda(q; \lambda^1))\lambda(q; \lambda^1))y(r(q; \lambda^1), q) - \underline{U}]. \tag{23}$$

Furthermore, $\underline{q}(\lambda^1)$ and $\underline{p}(\lambda^1) := r(\underline{q}(\lambda^1); \lambda^1)$ are continuous in λ^1.

For $p \in [\underline{p}(\lambda^1), 1], \kappa(p; \lambda^1)$ denote the inverse of $r(q; \lambda^1)$. Define

$$\begin{aligned} v(p; \lambda^1) &= \delta'(\lambda(\kappa(p; \lambda^1); \lambda^1))y(p, \kappa(p; \lambda^1)), p \in [\underline{p}(\lambda^1), 1], \text{ and} \\ u(q; \lambda^1) &= [\delta(\lambda(q; \lambda^1)) - \delta'(\lambda(q; \lambda^1))\lambda(q; \lambda^1)]y(r(q; \lambda^1), q), q \in [\underline{q}(\lambda^1), 1]. \end{aligned}$$

Note that for all $q > \underline{q}(\lambda^1)$, $\lambda(q; \lambda^1)$, $r(q; \lambda^1)$, $r(q; \lambda^1)[v(r(q; \lambda^1); \lambda^1) - \underline{V}]$ and $q[u(q; \lambda^1) -$

$\underline{U}]$ are positive and strictly increasing in q.[29]

Existence of a solution

The boundary value problem has a solution if there exists some λ^1 such that the solution to the IPV-$\lambda(1)$ with $\lambda(1) = \lambda^1$ satisfies both condition (22) and (23) at $q = \underline{q}(\lambda^1)$. By construction, condition (22) holds at $q = \underline{q}(\overline{\lambda})$ and condition (23) holds at $q = \underline{q}(\underline{\lambda})$. Consider

$$\widehat{\lambda} = \inf\{\lambda' \geq \underline{\lambda} : [v(\underline{p}(\lambda^1); \lambda^1) - \underline{V}]\underline{p}(\lambda^1) = 0, \forall \lambda^1 \geq \lambda'\}.$$

By continuity, $[v(\underline{p}(\widehat{\lambda}); \widehat{\lambda}) - \underline{V}]\underline{p}(\widehat{\lambda}) = 0$. If $\widehat{\lambda} = \underline{\lambda}$. Then, I have argued that condition (23) also holds at $q = \underline{q}(\widehat{\lambda})$. Suppose $\widehat{\lambda} > \underline{\lambda}$, the construction of $\widehat{\lambda}$ ensures that there is a convergent sequence $\{\lambda^1_n\}$ with limit $\widehat{\lambda}$ such that $\lambda^1_n < \widehat{\lambda}$ and only condition (23) holds at $q = \underline{q}(\lambda^1_n)$ for $\lambda^1 = \lambda^1_n$. By continuity, $\underline{q}(\widehat{\lambda})[u(\underline{q}(\widehat{\lambda}); \widehat{\lambda}) - \underline{U}] = 0$ must hold as well. Therefore, the solution to the IPV-$\lambda(1)$ with $\lambda(1) = \widehat{\lambda}$ solves the boundary value problem.

Uniqueness of the solution

Claim: Suppose $\lambda^H > \lambda^L$. For $p \in [\max\{\underline{p}(\lambda^H), \underline{p}(\lambda^L)\}, 1)$, $\kappa(p; \lambda^H) > \kappa(p; \lambda^L)$, $v(p; \lambda^H) < v(p; \lambda^L)$ and $\lambda(\kappa(p; \lambda^H); \lambda^H) > \lambda(\kappa(p; \lambda^L); \lambda^L)$.

Proof. As $r'(1; \lambda^H) > r'(1; \lambda^L)$, $\kappa(p; \lambda^H) > \kappa(p; \lambda^L)$, $\lambda(\kappa(p; \lambda^H); \lambda^H) > \lambda(\kappa(p; \lambda^L); \lambda^L)$ and $v(p; \lambda^H) < v(p; \lambda^L)$ must hold in some neighborhood of $p = 1$.

Consider the case that $\kappa(.; \lambda^H)$ and $\kappa(.; \lambda^L)$ intersects somewhere in $[\max\{\underline{p}(\lambda^H), \underline{p}(\lambda^L)\}, 1)$. $p_\kappa = \max\{p < 1 : \kappa(p; \lambda^H) = \kappa(p; \lambda^L)\}$ is then well defined and, by construction, $\kappa(p; \lambda^H) > \kappa(p; \lambda^L)$ for all $p \in (p_\kappa, 1)$. I now argue that $v(.; \lambda^H)$ and $v(.; \lambda^L)$ must intersect somewhere in $[p_\kappa, 1)$. Suppose not, for all $p \in (p_\kappa, 1]$,

$$\begin{aligned} \delta'(\lambda(\kappa(p; \lambda^H); \lambda^H))y(p, \kappa(p; \lambda^H)) = v(p; \lambda^H) \\ < v(p; \lambda^L) = \delta'(\lambda(\kappa(p; \lambda^L); \lambda^L))y(p, \kappa(p; \lambda^L)), \end{aligned}$$

and hence $\lambda(\kappa(p; \lambda^H); \lambda^H) > \lambda(\kappa(p; \lambda^L); \lambda^L)$. This contradicts the law of motion,

$$\begin{aligned} 0 > & \int_{p_\kappa}^1 \frac{1}{\lambda(\kappa(p; \lambda^H); \lambda^H)} - \frac{1}{\lambda(\kappa(p; \lambda^L); \lambda^L)} dF \\ = & [G(1) - G(\kappa(p_\kappa; \lambda^H))] - [G(1) - G(\kappa(p_\kappa; \lambda^L))] = 0! \end{aligned}$$

Consider the case where $v(.; \lambda^H)$ and $v(.; \lambda^L)$ intersects somewhere in $[\max\{\underline{p}(\lambda^H), \underline{p}(\lambda^L)\}, 1)$.

[29] $v(r(q; \lambda^1); \lambda^1)$ is strictly increasing in q because $\frac{\partial \ln \delta'(\lambda(q;\lambda^1))}{\partial q} + \frac{\partial \ln y(r(q;\lambda^1), q)}{\partial q} = 0$.

Define $p_v = \max\{p < 1 : v(p;\lambda^H) = v(p;\lambda^L)\}$. As $v(p;\lambda^H) < v(p;\lambda^L)$ for $p > p_v$,

$$\frac{\partial \ln y(p_v,\kappa(p_v;\lambda^H))}{\partial p} = \frac{d\ln v(p_v;\lambda^H)}{dp} \leq \frac{d\ln v(p_v;\lambda^L)}{dp} = \frac{\partial \ln y(p_v,\kappa(p_v;\lambda^L))}{\partial p}.$$

Assumption (Y) implies that $\kappa(p_v;\lambda^H) \leq \kappa(p_v;\lambda^L)$. So $\kappa(.;\lambda^H)$ and $\kappa(.;\lambda^L)$ intersects somewhere in $[p_v, 1]$.

Together, it must be that $\kappa(p;\lambda^H) > \kappa(p;\lambda^L)$ and $v(p;\lambda^H) < v(p;\lambda^L)$ throughout $[\max\{\underline{p}(\lambda^H), \underline{p}(\lambda^L)\}, 1)$. Otherwise, p_v and p_κ are well defined, satisfying $p_v > p_\kappa$ and $p_\kappa \geq p_v$.

$\lambda(\kappa(p;\lambda^H);\lambda^H) > \lambda(\kappa(p;\lambda^L);\lambda^L)$ then follows from the definition of $v(p;\lambda^H)$. □

There is a unique initial value of $\lambda(1)$ for which the solution to the IPV-$\lambda(1)$ satisfies both conditions (22) and (23) at $q = \underline{q}(\lambda^1)$.

Suppose not, the solutions to the IPV-$\lambda(1)$ with $\lambda(1) = \lambda^H$ and $\lambda(1) = \lambda^L$ satisfy both conditions (22) and (23) at $q = \underline{q}(\lambda^1)$, where $\lambda^H > \lambda^L$.

Consider the case where $\underline{p}(\lambda^L) > \underline{p}(\lambda^H)$. As $\underline{p}(\lambda^L) > 0$, $\underline{V} = v(\underline{p}(\lambda^L);\lambda^L) > v(\underline{p}(\lambda^L);\lambda^H)$. Condition (22) cannot be met at $\underline{p}(\lambda^H)$!

Now consider the case where $\underline{p}(\lambda^L) \leq \underline{p}(\lambda^H)$, then $\underline{q}(\lambda^H) = \kappa(\underline{p}(\lambda^H);\lambda^H) > \kappa(\underline{p}(\lambda^H);\lambda^L) \geq \underline{q}(\lambda^L)$ and $\lambda(\underline{q}(\lambda^H);\lambda^H) > \lambda(\kappa(\underline{p}(\lambda^H);\lambda^L);\lambda^L)$. This is impossible because $\underline{q}(\lambda^H) > 0$ implies

$$\underline{U} = u(\underline{q}(\lambda^H);\lambda^H) > u(\kappa(\underline{p}(\lambda^H);\lambda^L);\lambda^L)$$

Condition (23) cannot be met at $\underline{q}(\lambda^L)$!

A.3 Proof of Proposition 2

Denote the equilibrium payoff for asset owners by

$$\widetilde{u}(q) = [\delta(\widetilde{\lambda}(q)) - \delta'(\widetilde{\lambda}(q))\widetilde{\lambda}(q)]y(\widetilde{r}(q), q), q \geq \underline{\widetilde{q}}.$$

The equilibrium and second best allocations (or the equilibrium allocation in price competition) can be respectively recovered from $(\underline{\widetilde{p}}, \underline{\widetilde{q}}, \widetilde{r}, \widetilde{\lambda}, \widetilde{v}, \widetilde{u})$ and $(r_{SB}, \lambda_{SB}, \underline{p}_{SB}, \underline{q}_{SB}, v_{SB}, u_{SB})$, which both satisfy the following conditions.

Boundary conditions:

$$r(\underline{q}) = \underline{p}, r(1) = 1, \underline{q}[u(\underline{q}) - \underline{U}] = \underline{p}(v(p) - \underline{V}) = 0.$$

Law of motion:

$$r'(q) = \frac{g(q)}{f(r(q))}\lambda(q).$$

Hosios condition:

$$v(r(q)) = \delta'(\lambda(q))y(r(q), q).$$

The only difference is in the workers' IC conditions, which are given by

$$\begin{aligned}\frac{d\ln \widetilde{v}(\widetilde{r}(q))}{dp} &= \frac{\partial \ln y(\widetilde{r}(q), q)}{\partial p},\\ \frac{d\ln v_{SB}(r_{SB}(q))}{dp} &= \frac{\eta(\lambda_{SB}(q))}{\delta'(\lambda_{SB}(q))}\frac{\partial \ln y(r_{SB}(q), q)}{\partial p}.\end{aligned}$$

The listed set of conditions defines two boundary value problems, and we are going to compare their solutions.

Step 1: For any $\widehat{p} > \max\{\underline{p}_{SB}, \underline{\widetilde{p}}\}$ *and* $\widehat{q} > 0$ *satisfying* $\widehat{q} = \widetilde{\kappa}(\widehat{p}) = \kappa_{SB}(\widehat{p})$, *then* $\lambda_{SB}(\widehat{q}) < \widetilde{\lambda}(\widehat{q})$.

Suppose to the contrary that $\lambda_{SB}(\widehat{q}) \geq \widetilde{\lambda}(\widehat{q})$. Then, $v_{SB}(\widehat{p}) \leq \widetilde{v}(\widehat{p})$ and $\frac{d\ln \widetilde{v}(\widehat{p})}{dp} < \frac{d\ln v_{SB}(\widehat{p})}{dp}$. There must exist some $\epsilon > 0$ such that for all $p \in (\widehat{p} - \epsilon, \widehat{p})$, $v_{SB}(p) < \widetilde{v}(p)$ and $\kappa_{SB}(p) > \widetilde{\kappa}(p)$. [30]

Consider the case where $\widetilde{\kappa}$ and κ_{SB} intersect in $[\max\{\underline{p}_{SB}, \underline{\widetilde{p}}\}, \widehat{p})$. p_κ denotes the first intersection point of $\widetilde{\kappa}$ and κ_{SB} in $[\max\{\underline{p}_{SB}, \underline{\widetilde{p}}\}, \widehat{p})$, so that $\kappa_{SB}(p) > \widetilde{\kappa}(p)$ for all $p \in (p_\kappa, \widehat{p})$. Then, v_{SB} and $\widetilde{v}$ must intersect somewhere in between p_κ and $\widehat{p}$. Otherwise, for all $p \in (p_\kappa, \widehat{p})$, the Hosios condition implies $\delta'(\widetilde{\lambda}(\widetilde{\kappa}(p)))y(p, \widetilde{\kappa}(p)) > \delta'(\lambda_{SB}(\kappa_{SB}(p)))y(p_v, \kappa_{SB}(p))$, and hence $\widetilde{\lambda}(\widetilde{\kappa}(p)) < \lambda_{SB}(\kappa_{SB}(p))$. This contradicts the law of motion,

$$0 < \int_{p_\kappa}^{\widehat{p}} \frac{1}{\widetilde{\lambda}(\widetilde{\kappa}(p))} - \frac{1}{\lambda_{SB}(\kappa_{SB}(p))} dF = [G(\widehat{q}) - G(\widetilde{\kappa}(p_\kappa))] - [G(\widehat{q}) - G(\kappa_{SB}(p_\kappa))] = 0!$$

Consider the case where v_{SB} and $\widetilde{v}$ intersect in $[\max\{\underline{p}_{SB}, \underline{\widetilde{p}}\}, \widehat{p})$. Let p_v be the first intersection point of v_{SB} and $\widetilde{v}$ in $[\max\{\underline{p}_{SB}, \underline{\widetilde{p}}\}, \widehat{p})$, so that $v_{SB}(p) < \widetilde{v}(p)$ for all $p \in (p_v, \widehat{p})$. Then, $\widetilde{\kappa}$ and κ_{SB} must intersect at some point between p_v and $\widehat{p}$. Suppose not, the continuity of $\widetilde{\kappa}$ and κ_{SB} imply that $\kappa_{SB}(p) > \widetilde{\kappa}(p)$ for $p \in (p_v, \widehat{p})$. The Hosios condition again requires $\widetilde{\lambda}(\widetilde{\kappa}(p)) < \lambda_{SB}(\kappa_{SB}(p))$ for all $p \in (p_\kappa, \widehat{p})$. Under Assumptions (Y) and (M),

[30] For the case where $\lambda_{SB}(\widehat{q}) = \widetilde{\lambda}(\widehat{q})$, one can show $\frac{\partial \ln \delta'(\lambda_{SB}(\widehat{q}))}{\partial q} > \frac{\partial \ln \delta'(\widetilde{\lambda}(\widehat{q}))}{\partial q}$ by differentiating the Hosios condition $v(r(q)) = \delta'(\lambda(q))y(r(q), q)$ w.r.t. q and combining it with $\frac{d\ln v(r(q))}{dp}$.

for all $p \in (p_\kappa, \widehat{p})$,

$$\begin{aligned}\frac{d \ln v_{SB}(p)}{dp} &= \frac{\eta(\lambda_{SB}(\kappa_{SB}(p)))}{\delta'(\lambda_{SB}(\kappa_{SB}(p)))} \frac{\partial \ln y(p, \kappa_{SB}(p))}{\partial p} \\ &> \frac{\eta(\widetilde{\lambda}(\widetilde{\kappa}(p)))}{\delta'(\widetilde{\lambda}(\widetilde{\kappa}(p)))} \frac{\partial \ln y(p, \widetilde{\kappa}(p))}{\partial p} > \frac{\partial \ln \widetilde{v}(p)}{\partial p}.\end{aligned}$$

Hence, v_{SB} and $\widetilde{v}$ cannot intersect at p_v.

It follows that $v_{SB}(p) < \widetilde{v}(p)$ and $\kappa_{SB}(p) > \widetilde{\kappa}(p)$ throughout $[\max\{\underline{p}_{SB}, \widetilde{\underline{p}}\}, \widehat{p})$. Otherwise, p_v and p_κ will co-exist, satisfying $\widehat{p} > p_\kappa > p_v$ and $\widehat{p} > p_v > p_\kappa$! Again, $\widetilde{\lambda}(\widetilde{\kappa}(p)) < \lambda_{SB}(\kappa_{SB}(p))$ throughout $[\max\{\underline{p}_{SB}, \widetilde{\underline{p}}\}, \widehat{p})$ because of the Hosios condition.

These conclusions are inconsistent with the boundary conditions. Suppose $\widetilde{\underline{p}} > \underline{p}_{SB}$; then, the boundary condition for $\widetilde{\underline{p}} > 0$ requires $\underline{V} = \widetilde{v}(\widetilde{\underline{p}}) > v_{SB}(\widetilde{\underline{p}})$! Suppose $\widetilde{\underline{p}} \leq \underline{p}_{SB}$, then $\underline{q}_{SB} = \kappa_{SB}(\underline{p}_{SB}) > \widetilde{\kappa}(\underline{p}_{SB}) \geq \widetilde{\underline{q}}$ and $\lambda_{SB}(\underline{q}_{SB}) > \widetilde{\lambda}(\widetilde{\kappa}(\underline{p}_{SB}))$. The boundary condition for $\underline{q}_{SB} > 0$ then requires $\underline{U} = u_{SB}(\underline{q}_{SB}) > \widetilde{u}(\widetilde{\kappa}(\underline{p}_{SB})) \geq \widetilde{u}(\widetilde{\underline{q}})$!

Corollary: $\lambda_{SB}(1) < \widetilde{\lambda}(1)$, $v_{SB}(1) > \widetilde{v}(1)$ *and* $u_{SB}(1) < \widetilde{u}(1)$.

Step 2: $\widetilde{\kappa}(p) > \kappa_{SB}(p)$ *for any* $p \in (\max\{\underline{p}_{SB}, \widetilde{\underline{p}}\}, 1)$.

As $\lambda_{SB}(1) < \widetilde{\lambda}(1)$, the law of motion implies $r'_{SB}(1) < \widetilde{r}'(1)$, and hence $\widetilde{\kappa}(p) > \kappa_{SB}(p)$ for sufficient large p. Suppose $\widetilde{\kappa}(.)$ and $\kappa_{SB}(.)$ intersects somewhere in $(\max\{\underline{p}_{SB}, \widetilde{\underline{p}}\}, 1)$. Consider the first intersection point $\widehat{p} = \max\{p \in (0,1) : \widetilde{\kappa}(p) = \kappa_{SB}(p)\}$. Let $\widehat{q} = \widetilde{\kappa}(\widehat{p}) = \kappa_{SB}(\widehat{p})$. The previous claim states that $\lambda_{SB}(\widehat{q}) < \widetilde{\lambda}(\widehat{q})$. However, $r_{SB}(q) > \widetilde{r}(q)$ for $q > \widehat{q}$ by construction. From the law of motion, $r'_{SB}(\widehat{q}) \geq \widetilde{r}'(\widehat{q})$ only if $\lambda_{SB}(\widehat{q}) \geq \widetilde{\lambda}(\widehat{q})$!

Step 3: $\underline{p}_{SB} \geq \widetilde{\underline{p}}$ *and* $\widetilde{\underline{q}} \geq \underline{q}_{SB}$.

First, suppose to the contrary that $\widetilde{\underline{p}} > \underline{p}_{SB} \geq 0$. Then $\widetilde{\underline{q}} = \widetilde{\kappa}(\widetilde{\underline{p}}) \geq \kappa_{SB}(\widetilde{\underline{p}}) > \kappa_{SB}(\underline{p}_{SB}) = \underline{q}_{SB}$. The boundary conditions must be violated if $\widetilde{\underline{p}} > \underline{p}_{SB}$ and $\widetilde{\underline{q}} > \underline{q}_{SB}$.

$$\begin{aligned}\underline{U} &= \widetilde{u}(\widetilde{\underline{q}}) + \widetilde{\lambda}(\widetilde{\underline{q}})[\widetilde{v}(\widetilde{\underline{p}}) - \underline{V}] \\ &= \max_{\lambda \geq 0}[\delta(\lambda) y(\widetilde{\underline{p}}, \widetilde{\underline{q}}) - \lambda \underline{V}] > \max_{\lambda \geq 0}[\delta(\lambda) y(\underline{p}_{SB}, \underline{q}_{SB}) - \lambda v_{SB}(\underline{p}_{SB})] \qquad (24) \\ &= u_{SB}(\underline{q}_{SB})!\end{aligned}$$

The first equality is due to the boundary conditions for $\widetilde{\underline{q}} > 0$ and $\widetilde{\underline{p}} > 0$ while the second and last equalities derive from their FOCs and the Hosios condition, respectively. Interchanging the role of $(\underline{p}_{SB}, \underline{q}_{SB})$ and $(\widetilde{\underline{p}}, \widetilde{\underline{q}})$ in the inequality (24), the case $\underline{p}_{SB} > \widetilde{\underline{p}}$ and $\underline{q}_{SB} > \widetilde{\underline{q}}$ is also ruled out. A continuity argument rules out the case $\underline{p}_{SB} = \widetilde{\underline{p}}$ and $\underline{q}_{SB} > \widetilde{\underline{q}}$. For any p' slightly above $\widetilde{\underline{p}}$, $\widetilde{\kappa}(p') > \kappa_{SB}(p') > \underline{q}_{SB}$ and $\widetilde{\kappa}(\widetilde{\underline{p}}) = \widetilde{\underline{q}} < \underline{q}_{SB}$. $\widetilde{\kappa}$ must be discontinuous at $\widetilde{\underline{p}}$! Therefore, I establish that $\underline{p}_{SB} \geq \widetilde{\underline{p}}$ and $\widetilde{\underline{q}} \geq \underline{q}_{SB}$.

Step 4: $\underline{p}_{SB} = \underline{\widetilde{p}}$ only if $\underline{p}_{SB} = \underline{\widetilde{p}} = 0$ and $\widetilde{v}(0) > v_{SB}(0)$. $\underline{\widetilde{q}} = \underline{q}_{SB}$ only if $\underline{\widetilde{q}} = \underline{q}_{SB} = 0$ and $u_{SB}(0) > \widetilde{u}(0)$.

First, consider the case where $\underline{p}_{SB} = \underline{\widetilde{p}}$ and $\underline{\widetilde{q}} = \underline{q}_{SB}$. Differentiating the Hosios condition $\ln v(r(q)) = \ln \delta'(\lambda(q)) + \ln y(r(q), q)$ and subtracting it with the respective expressions of $\frac{d \ln v_{SB}(r_{SB}(q))}{dp}$ and $\frac{d \ln \widetilde{v}(\widetilde{r}(q))}{dp}$, I obtain $\frac{d \ln \delta'(\lambda_{SB}(\underline{\widetilde{q}}))}{dq} > \frac{d \ln \delta'(\widetilde{\lambda}(\underline{\widetilde{q}}))}{dq}$. It follows that $\widetilde{\lambda}(\underline{\widetilde{q}}) < \lambda_{SB}(\underline{\widetilde{q}})$. Otherwise, $\widetilde{\lambda}(q') > \lambda_{SB}(q')$ for q' slightly above $\underline{\widetilde{q}}$. The law of motion in turn implies that $\widetilde{r}(q') > r_{SB}(q')$, contradicting the previous claim in Step 2. We can immediately rule out the cases where $\underline{p}_{SB} = \underline{\widetilde{p}} > 0$ or $\underline{\widetilde{q}} = \underline{q}_{SB} > 0$. The boundary conditions and Hosios condition in such cases require $\widetilde{\lambda}(\underline{\widetilde{q}}) = \lambda_{SB}(\underline{\widetilde{q}})$! The remaining possibility is that $\underline{p}_{SB} = \underline{\widetilde{p}} = \underline{\widetilde{q}} = \underline{q}_{SB} = 0$. $\widetilde{v}(0) > v_{SB}(0)$ and $u_{SB}(0) > \widetilde{u}(0)$ because $\widetilde{\lambda}(0) < \lambda_{SB}(0)$.

Consider the case where $\underline{p}_{SB} = \underline{\widetilde{p}}$ and $\underline{\widetilde{q}} > \underline{q}_{SB}$. From the boundary conditions and Hosios condition,

$$\begin{aligned} \max_{\lambda \geq 0}[\delta(\lambda) y(\underline{p}_{SB}, \underline{q}_{SB}) - \lambda v_{SB}(\underline{p}_{SB})] &= u_{SB}(\underline{q}_{SB}) \\ &\geq \underline{U} = \widetilde{u}(\underline{\widetilde{q}}) = \max_{\lambda \geq 0}[\delta(\lambda) y(\underline{p}_{SB}, \underline{\widetilde{q}}) - \lambda \widetilde{v}(\underline{p}_{SB})]. \end{aligned}$$

This immediately implies that $\widetilde{v}(\underline{p}_{SB}) > v_{SB}(\underline{p}_{SB}) \geq \underline{V}$, and hence $\underline{p}_{SB} = \underline{\widetilde{p}} = 0$. The case $\underline{p}_{SB} > \underline{\widetilde{p}}$ and $\underline{\widetilde{q}} = \underline{q}_{SB}$ follows from a symmetric argument.

Corollary: $\widetilde{v}(\underline{p}_{SB}) > v_{SB}(\underline{p}_{SB})$ and $u_{SB}(\underline{\widetilde{q}}) > \widetilde{u}(\underline{\widetilde{q}})$

The previous claim establishes the cases where $\underline{p}_{SB} = \underline{\widetilde{p}}$ or $\underline{\widetilde{q}} = \underline{q}_{SB}$. Suppose $\underline{p}_{SB} > \underline{\widetilde{p}}$. The boundary condition immediately implies $\widetilde{v}(\underline{p}_{SB}) > v_{SB}(\underline{p}_{SB}) = \underline{V}$. Similarly, $u_{SB}(\underline{\widetilde{q}}) > \widetilde{u}(\underline{\widetilde{q}}) = \underline{U}$ if $\underline{\widetilde{q}} > \underline{q}_{SB}$.

Note that the above arguments require only the workers' IC condition to satisfy $\frac{d\widetilde{v}(\widetilde{r}(q))}{dp} < \eta(\widetilde{\lambda}(q)) \frac{\partial y(\widetilde{r}(q), q)}{\partial p}$ for $\widetilde{v}(p) \geq \underline{V}$.