Serial Dictatorship with Infinitely Many Agents*

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Abstract. We extend an existing impossibility theorem to the environment where there are infinitely many agents. When the number of agents is infinite, it is impossible to identify a dictator whose preference dictates the outcome and it is proved that a set of decisive coalitions forms an ultrafilter (Fishburn, 1970; Kirman and Sondermann, 1972). Eraslan and McLennan (2004) show that a solution is a serial dictatorship if it satisfies unanimity, independence and stability. We extend the result of serial dictatorship by using hierarchical ultrafilters. An immediate consequence of this characterization is the existence of a series of individual dictators in the case of a finite number of agents and it gives a concise proof of an existing impossibility theorem with serial dictatorship. At the same time, the same characterization shows that effectively serial dictatorship persists also in the infinite case.

Key Words: Impossibility theorem, Social choice, Serial dictatorship, Ultrafilter.

JEL Classification Numbers: D71.

1 Introduction

In this paper, we establish a serial dictatorship with infinitely many agents and more than three alternatives by using a hierarchy of ultrafilters. In the literature of strategic candidacy, Dutta, Jackson, and Le Breton (2001) initiate the study of manipulation of voting procedures by a candidate who withdraws from the election. Eraslan and McLennan (2004) extend their framework by allowing: (a) the outcome of the procedure to be a set of candidates; (b) some or all of the agents to have weak preference orderings of the candidates. We extend the result of serial dictatorship in Eraslan and McLennan (2004) to the case where there are infinitely many agents. Our result is not confined to the number of voters being infinite. An immediate consequence of our main theorem is the existence of a series of individual dictators in the case of a finite number of agents. At the same time, the same characterization shows that effectively serial dictatorship persists also in the infinite case.

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When the number of agents is infinite, it is impossible to identify a dictator whose preference dictates the outcome and it is proved that there is an “invisible” dictator (Fishburn, 1970; Kirman and Sondermann, 1972). Our main theorem also extends the result of invisible dictators in a domain of weak preference profiles and formulates a serial dictatorship by using a hierarchy of ultrafilters.

Within the framework of social choice, recently Man and Takayama (2013) have proposed the independence and stability axioms together with unanimity and show that many well-known impossibility theorems including Arrow’s impossibility theorem (Arrow, 1959), Gibbert Satterthwaite theorem (?Satterthwaite, 1975) and the theorem of serial dictatorship (Eraslan and McLennan, 2004) follow their main theorem as corollaries when the number of voters is finite. We extend the analysis by allowing the number of agents to be infinite, and we show that a solution satisfying the three axioms is characterized by our generalized serial dictatorship. In this sense, we extend the previous research of dictatorship to a more general environment. It combines two strands of Kirman and Sondermann (1972) and Man and Takayama (2013) in the literature of social choice.

The organization of the paper is as follows. Section 2 presents the model. Section 3 defines a generalized serial dictatorship. Section 4 provides the main theorem and proves it.

2 The Model

Let $\mathcal{X}$ be the set of potential alternatives. We assume that $|\mathcal{X}| \geq 3$ and it is finite. Let $\mathcal{N}$ be the set of potential agents. We assume that $\mathcal{N}$ is infinite. Let $\mathcal{R}$ be the entire space of weak preferences over $\mathcal{X}$. Let $\succeq_i \in \mathcal{R}$ be agent $i$’s preference, and, for each $N \subset \mathcal{N}$, let $\succeq \in \mathcal{R}^N$ be a preference profile of all the agents in $N$. For each $x, y \in \mathcal{X}$, we say that $x \succ_i y$ if $x$ is strictly preferred to $y$, i.e., $x \succeq_i y$ but not $y \succeq_i x$, and say that $x \sim_i y$ if $x$ is indifferent to $y$, i.e., $x \succeq_i y$ and $y \succeq_i x$.

For an arbitrary set $Z$, let $\mathcal{P}(Z) \equiv 2^Z \setminus \{\emptyset\}$. An environment is a pair $(\mathcal{X}, \succeq) \in \mathcal{P}(\mathcal{X}) \times \mathcal{R}^N$. Given population $N$, a solution $\phi_N$ is a correspondence from environments to alternatives such that

$$\phi_N : \mathcal{P}(\mathcal{X}) \times \mathcal{R}^N \rightarrow \mathcal{X},$$

s.t. $\phi_N(\mathcal{X}, \succeq) \in \mathcal{P}(\mathcal{X})$.

Next we define three properties which we impose on $\phi_N$.

**Definition** (Unanimity). For each $X \in \mathcal{P}(\mathcal{X})$, $\succeq \in \mathcal{R}^N$, and $x \in X$, if, for each $i \in N$ and $y \in X$, $x \succeq_i y$, and there exists $j \in N$ s.t. $x \succ_j y$, then $\phi_N(X, \succeq) = \{x\}$.

We say that $\succeq$ and $\succeq' \in \mathcal{R}^N$ agree on $X$ if $\succeq$ and $\succeq'$ are the same on $X$. Then we say that $\succeq =_X \succeq'$.

**Definition** (Independence). For each $X \in \mathcal{P}(\mathcal{X})$ and each $\succeq, \succeq' \in \mathcal{R}^N$, if $\succeq =_X \succeq'$, then $\phi_N(X, \succeq) = \phi_N(X, \succeq')$.

**Definition** (Stability). For each $X, X' \in \mathcal{P}(\mathcal{X})$ and each $\succeq \in \mathcal{R}^N$, if $X' \subset X$ and $\phi_N(X, \succeq) \cap X' \neq \emptyset$, then $\phi_N(X', \succeq) = \phi_N(X, \succeq) \cap X'$.
3 A Generalized Serial Dictatorship

In this section, we define a “generalized serial dictatorship.” To do this, first we introduce our tool, an ultrafilter in our definition of the generalized serial dictatorship.

**Definition (Ultrafilter).** A family $\mathcal{F} \subset 2^N$ is an ultrafilter of $N$ if

1. $\emptyset \notin \mathcal{F}$.
2. If $S \in \mathcal{F}$, and $S' \supset S$, then $S' \in \mathcal{F}$.
3. If $S, S' \in \mathcal{F}$, then $S \setminus S' \in \mathcal{F}$.
4. If $S \in N$, then either $S \in \mathcal{F}$ or $N \setminus S \in \mathcal{F}$.

An ultrafilter picks out one set from a finite partition. Kirman and Sondermann (1972) show that the following $U$ is an ultrafilter:

$$U \equiv \{ U \subset N \mid \forall x, y \in X, \forall \succ \in \mathcal{P}^N, x \succ_U y \land y \succ_{N \setminus U} x \Rightarrow x P(\succ) y \}.$$  

In our model that admits a full domain of preferences, we generalize the definition of $U$. To describe the idea intuitively, imagine the following “veto process.” We call a group of agents who hold a same preference a “coalition.” A coalition which belongs to the first ultrafilter with the ground set $N$ is asked to veto what they do not like and the next coalition that belongs to the second ultrafilter vetoes what the initial coalition does not veto. This process is repeated until all the voters are consulted. Then, we apply a tie-breaking rule to a set of survivors from this process to obtain the final winners. In this way, we can obtain a sequence of ultrafilters that each decisive coalition in each round belongs to. This veto process is the algorithm that the generalized serial dictatorship describes.

Here, we introduce a formal definition of the generalized serial dictatorship. To do so, we begin with defining a “coherent family of ultrafilters.” As we described for the veto process, this process depends on a preference profile. An ultrafilter indeed selects one set from partitions of the ground set at each round where each partition holds a same preference. Naturally we can imagine that consequence ultrafilters that “slightly” different preference profiles and in turn slightly different ground sets induce would hold some parallel relationship with each other. For example, in the veto process described above, if one person in the initial coalition under a preference profile $\succ$ has a same preference with the second coalition instead under a preference profile $\succ'$ with holding everything else the same between $\succ$ and $\succ'$, there may as well be some relationship between the two consecutive ultrafilters. The next definition requires what condition the relationship between the two consecutive ultrafilters must satisfy.

**Definition.** A coherent hierarchy of ultrafilters $\mathcal{U}$ associates each $N \in \mathcal{P}(N)$ to an ultrafilter over $N$, $\mathcal{U}_N$ such that for all $N, N' \in \mathcal{P}(N)$, if $N \subset N'$ and $N' \setminus N \not\in \mathcal{U}_N$, then $M \in \mathcal{U}_N$; if and only if $M \cap N \in \mathcal{U}_N$.

To grasp the concept, we provide a simple example of violation of coherence. Let $N = \{1, 2, 3, 4\}$ and $\mathcal{U}_N = \{U: 1 \in U\}$. Consider the following two cases:

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• \( U_1 = \{1, 4\} \) and \( U_{\{2,3\}} = \{U : 2 \in U\} \), and
• \( U'_1 = \{1\} \) and \( U_{\{2,3,4\}} = \{U : 3 \in U\} \).

The relationship between \( U_{\{2,3\}} \) and \( U_{\{2,3,4\}} \) does not satisfy coherence because \( \{3, 4\} \notin U_{\{2,3\}} \), although \( \{3, 4\} \cap \{2, 3\} \notin U_{\{2,3\}} \).

Let \( U_N \) be an ultrafilter over \( N \) and \( \mathcal{P} \) be a finite partition of \( N \). By a property of ultrafilters, there exists a unique set \( U \in \mathcal{P} \setminus U_N \) (that is, \( \mathcal{P} \setminus U_N = \{U\} \)). Let \( \succeq = (\succeq_1, \ldots, \succeq_i, \ldots) \in \mathcal{R}^N \). For each \( R \in \mathcal{R} \), let \( U(R) \equiv \{i \in N \mid \succeq_i = R\} \). Then, \( \mathcal{P} \equiv \{U(R) \subset N \mid R \in \mathcal{R}\} \) becomes a finite partition of \( N \), because \( X \) is finite. There exists a unique \( R^* \in \mathcal{R} \) such that \( U(R^*) \in U_N \). We denote such an \( R^* \) by \( \succeq_{U_N} \). For each \( R \in \mathcal{R} \), let \( Top(X, R) \equiv \{x \in X \mid \text{for each } y \in X, x \succeq y\} \).

**Definition.** A solution \( \phi_N \) is a generalized serial dictatorship if there exists a coherent hierarchy of ultrafilters \( U \) and a tie-breaking rule \( \rho \in \mathcal{R} \) s.t., for each \( (X, \succeq) \in \mathcal{P}(X) \times \mathcal{R}^N \), there exist \( K(X, \succeq) \in \mathbb{N}_+ \), a series \( \{T_k, C_k\}_{k=0}^{K(X, \succeq)} \) s.t.

1. \( T_0 = X \) and \( C_0 = N \),
2. For each \( k \in \{1, \cdots, K(X, \succeq)\} \), \( T_k = Top(T_{k-1}, \succeq_{U_{C_{k-1}}}) \),
3. For each \( k \in \{1, \cdots, K(X, \succeq)\} \), \( C_k = \{i \in N \mid Top(T_{k-1}, \succeq_i) \neq T_k\} \),
4. For each \( k \in \{1, \cdots, K(X, \succeq) - 1\} \), \( C_k \neq \emptyset \), and \( C_{K(X, \succeq)} = \emptyset \).
5. \( \phi(X, \succeq) = Top(T_{K(X, \succeq)}, \rho) \).

### 4 The Main Theorem

Our main result is;

**Theorem 1.** A solution \( \phi \) satisfies Unanimity, Independence, and Stability if and only if \( \phi \) is a generalized serial dictatorship.

**4.1 Proof of the sufficiency**

First we show the sufficiency part.

**Proposition 1.** If \( \phi \) is a generalized serial dictatorship, then \( \phi \) satisfies Unanimity, Independence, and Stability.

We show it through the three lemmas properties below. In the proofs, we assume that a solution \( \phi \) is a generalized serial dictatorship associated with coherent hierarchy of ultrafilters \( U \).

**Lemma 1.** If a solution \( \phi \) is a generalized serial dictatorship, then \( \phi \) satisfies Unanimity.
Proof. Let $X \in \mathcal{P}(X)$ and $\succeq \in \mathcal{R}^N$. Let $x \in X$. Suppose that, for each $i \in N$ and each $y \in X$ with $y \neq x$, $x \succ_i y$ and there exists $i' \in N$ such that $x \succ_{i'} y$.

Then, by construction, for each $k \in \{1, \cdots, K(X, \succeq)\}$, we have that $x \in T_k$. On the other hand, for each $y \neq x$, let $i_y \in N$ be such that $x \succ_{i_y} y$. Then, there exists $k_y \in \{0, \cdots, K(X, \succeq)\}$ such that $x \in T_{k_y}$. It implies that $y \notin T_{k_y+1}$. Therefore $T_{K(X, \succeq)} = \{x\}$. Thus we have that $\phi(X, \succeq) = \text{Top}(T_{K(X, \succeq)}, \rho) = \{x\}$. □

Lemma 2. If a solution $\phi$ is a generalized serial dictatorship, then $\phi$ satisfies Independence.

Proof. Let $X \in \mathcal{P}(X)$ and $\succeq, \succeq' \in \mathcal{R}^N$ with $\succeq = x \succ_{i'} y$. Let $\{T_k, C_k\}_{k=0}^{K(X, \succeq)}$ and $\{T'_k, C'_k\}_{k=0}^{K(X, \succeq')}$ be the series of vetoes and remaining agents for $(X, \succeq)$ and $(X, \succeq')$ respectively. Let $\mathcal{U}_{C_k}$ and $\mathcal{U}_{C'_k}$ be the preferences picked out by ultrafilters $\mathcal{U}_{C_k}$ and $\mathcal{U}_{C'_k}$ respectively. We show that, for each $k \in \{0, \cdots, \min\{K(X, \succeq), K(X, \succeq')\}\}$, $T_k = T'_k$.

Note that $T_0 = T'_0 = X$ and $C_0 = C'_0 = N$. Therefore $\mathcal{U}_{C_0} = \mathcal{U}_{C_0'}$. By the definition of $\mathcal{U}_{C_0}$, we have that $\{i \in C_0 \mid \succeq_i = \succeq_i \mathcal{U}_{C_0}\} \in \mathcal{U}_{C_0}$. By the same logic, we have that $\{i \in C'_0 \mid \succeq'_i = \succeq_i \mathcal{U}_{C_0}\} \in \mathcal{U}_{C_0'}$. Since every two sets in an ultrafilter has a non-empty intersection, there exists $i \in N$ such that $\succeq_i = \succeq_i \mathcal{U}_{C_0}$ and $\succeq'_i = \succeq_i \mathcal{U}_{C_0'}$. By the assumption, $\succeq'_i = x \succ_{i'} y$. Therefore $\mathcal{U}_{C_0} = x \succ_{i'} y \mathcal{U}_{C_0'}$. It implies that $T_1 = \text{Top}(C_0, \succeq_i \mathcal{U}_{C_0}) = \text{Top}(C'_0, \succeq_i \mathcal{U}_{C_0'}) = T'_1$. Since, for each $i \in N$, $\succeq_i = x \succ_{i'} y$, we also have that $C_1 = \{i \in C_0 \mid \text{Top}(T_0, \succeq_i) \neq T_1\} = \{i \in C'_0 \mid \text{Top}(T'_0, \succeq_i) \neq T'_1\} = C'_1$.

Now that $T_1 = T'_1$ and $C_1 = C'_1$, we can repeat the same argument as above to obtain that $T_2 = T'_2$ and $C_2 = C'_2$. We can repeat this process until it ends. As a result, we have that $K(X, \succeq) = K(X, \succeq')$ and $T_{K(X, \succeq)} = T_{K(X, \succeq')}$. It implies that $\phi(X, \succeq) = \phi(X, \succeq')$. □

Lemma 3. If a solution $\phi$ is a generalized serial dictatorship, then $\phi$ satisfies Stability.

Proof. Let $X \subset X' \subset X$ and $\succeq \in \mathcal{R}^N$. Let $\{T_k, C_k\}_{k=1}^{K(X, \succeq)}$ and $\{T'_l, C'_l\}_{l=1}^{K(X', \succeq)}$ be the series of top sets and agents waiting for vetoes in the definition of the generalized serial dictatorship for $(X, \succeq)$ and $(X', \succeq)$ respectively.

Suppose that $\phi_N(X', \succeq) \cap X \neq \emptyset$. We show that $T_{K(X, \succeq)} = T_{K(X', \succeq)} \cap X$ by mathematical induction. For $k = 0$, $T_0 = X$, $T'_0 = X'$ and $C_0 = C'_0 = N$. Thus $\succeq_i \mathcal{U}_{C_0} = \succeq_i \mathcal{U}_{C'_0}$.

Then $T_1 = \text{Top}(T_0, \succeq_i \mathcal{U}_{C_0})$ and $T'_1 = \text{Top}(T'_0, \succeq_i \mathcal{U}_{C'_0})$, and we have that $T_1 = T'_1 \cap X$. Let $U_1 = \{i \in C_0 \mid \text{Top}(T_0, \succeq_i) = T_1\}$ and $U'_1 = \{i \in C'_0 \mid \text{Top}(T'_0, \succeq_i) = T'_1\}$. Since $T'_1 \cap X \neq \emptyset$, for each $j \in U'_1$, we have that

$$\text{Top}(T_0, \succeq_i) = \text{Top}(T'_0 \cap X, \succeq_i) = \text{Top}(T'_0, \succeq_i) \cap X = \text{Top}(T'_0, \succeq_i \mathcal{U}_{C'_0}) \cap X = \text{Top}(T_0, \succeq_i \mathcal{U}_{C_0}).$$

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It implies that $U'_1 \subset U_1$. Since $C_1 = N \setminus U_1$ and $C'_1 = N \setminus U'_1$, we have that $C_1 \subset C'_1$. Note that, for each $j \in C'_1 \setminus C_1$, $j \notin U_1 \setminus U'_1$. Therefore $Top(T_0, \succsim_j) = T_1$.

Let $k_2$ be the first number such that $T_{k_2} \subseteq T_1$, and $l_2$ be the first number such that $T'_{l_2} \cap X \subseteq T_1$. We show that $T_{k_2} = T'_{l_2} \cap X$.

Let $N_{k_2-1} = \{ i \in C_1 \mid \succsim_{U_{C_{k_2-1}}} = \succsim_i \}$ and $N'_{l_2-1} = \{ i \in C'_1 \mid \succsim_{U_{C'_{l_2-1}}} = \succsim_i \}$. Since only agents whose top set in $T_1$ equals to $T_1$ have left for both $T_{k_2}$ and $T'_{l_2}$, we have that, for each $i \in C_{k_2-1} \Delta C'_{l_2-1}$, $Top(T_1, \succsim_i) = T_1$. Therefore we have that $N_{k_2-1} \subset C_{k_2-1} \cap C'_{l_2-1}$ and $N'_{l_2-1} \subset C_{k_2-1} \cap C'_{l_2-1}$.

Since $N_{k_2-1} \in U_{C_{k_2-1}}$, we have that $C_{k_2-1} \setminus C'_{l_2-1} \notin U_{C_{k_2-1}}$. By the definition of coherent hierarchy of ultrafilters, $N_{k_2-1} \in U_{C_{k_2-1} \cap C'_{l_2-1}}$. Since $N'_{l_2-1} \in U_{C'_{l_2-1}}$, we have that $C'_{l_2-1} \setminus C_{k_2-1} \notin U_{C'_{l_2-1}}$. Therefore, by the definition of coherent hierarchy of ultrafilters, $N_{k_2-1} \in U_{C'_{l_2-1}}$. Thus $N_{k_2-1} \cap N'_{l_2-1} \neq \emptyset$. It means that $\succsim_{C_{k_2-1}} = \succsim_{C'_{l_2-1}}$. Since $T_1 = T'_1 \cap X$ and $T'_{l_2} \cap X \neq \emptyset$, we have that $T_{k_2} = T'_{l_2} \cap X$.

We can do the same process until it ends. □

4.2 Proof of the necessity

Next we show the necessity part of the theorem.

**Proposition 2.** If a solution $\phi$ satisfies Unanimity, Independence, and Stability, then $\phi$ is a generalized serial dictatorship.

4.2.1 Social preference

First we have to show that a solution satisfying the three properties is equivalent to find a social welfare function.

**Lemma 4.** If a solution $\phi_N$ satisfies Unanimity, Independence and Stability, then, for each $\succsim \in \mathcal{R}^N$, there exists $R \in \mathcal{R}$ such that, for each $X \subset \mathcal{X}$ with $|X| \geq 2$, $\phi_N(X, \succsim) = Top(X, R)$.

**Proof.** Let $\succsim \in \mathcal{R}^N$. We define a binary relationship $R$ on $X$ so that $x \mathrel{R} y$ if $x \in \phi_N(\{x, y\}; \succsim)$. Then, since $\phi_N(\{x, y\}; \succsim) \neq \emptyset$, $R$ is complete.

Next we show transitivity. Suppose that $x, y, z \in \mathcal{X}$ satisfy that $x \mathrel{R} y$ and $y \mathrel{R} z$. Then $x \in \phi_N(\{x, y\}; \succsim)$ and $y \in \phi_N(\{y, z\}; \succsim)$. If $x \notin \phi_N(\{x, z\}; \succsim)$, then $x \notin \phi_N(\{x, y, z\}; \succsim)$ by Stability. It means that $y \notin \phi_N(\{x, y, z\}; \succsim)$ by Stability. This leads to that $\phi_N(\{x, y, z\}; \succsim) = \{z\}$. But, by Stability, this implies that $y \notin \phi_N(\{y, z\}; \succsim)$. It is a contradiction. Therefore $x \in \phi_N(\{x, z\}; \succsim)$, i.e., $x \mathrel{R} z$. Now $R$ is a complete transitive relationship, that is, a weak preference.

Finally we show that $\phi_N$ is the top set of $R$. Let $X \subset \mathcal{X}$ with $|X| \geq 2$. First, suppose that there exists $x \in \phi_N(X, \succsim)$ but $x \notin Top(X, R)$. Then there exists $y \in X$ such that $y \in \phi_N(\{x, y\}; \succsim)$ but $x \notin \phi_N(\{x, y\}; \succsim)$. However, since $\{x, y\} \subset X$, $x \in \phi_N(\{x, y\}; \succsim)$ by Stability. It is a contradiction. Thus $\phi_N(X, \succsim) \subset Top(X, R)$. Next suppose that $x \in Top(X, R)$. Then, for each $y \in X$, $x \mathrel{R} y$. It means that, for each $y \in X$, $x \in \phi_N(\{x, y\}; \succsim)$. By Stability, it implies that $x \in \phi_N(X, \succsim)$. As a result, $Top(X, R) \subset \phi_N(X, \succsim)$. □
For each $\succeq \in \mathcal{R}^N$, we denote its associated social preference as $R(\succeq)$. We say that $x P(\succeq) y$ if $x$ is strictly preferred to $y$ under $R(\succeq)$, and say that $x I(\succeq) y$ if $x$ is indifferent to $y$. Since $R(\cdot)$ becomes a social welfare function, we can apply Arrow’s impossibility theorem, especially the one extended to infinitely many agents by Kirman and Sondermann (1972).

4.2.2 Dictatorial coalitions

Since Kirman-Sondermann only deals with the strict preferences, we have to extend their result to the full domain of preferences. Let $U \subset N$ and $x, y \in \mathcal{X}$. We denote $x \succeq_U y$ if, for each $i \in U$, $x \succeq_i y$. Throughout the argument below, let $U' \equiv N \setminus U$, and, for each $U \subset U'$, let $U'' \equiv U' \setminus \hat{U}$.

We consider three families of groups of agents:

The first family is\footnote{We abuse notation $\mathcal{U}_N$ here. The $\mathcal{U}_N$ below is not related to a coherent hierarchy of ultrafilters $\mathcal{U}$.}

$$U_N \equiv \{ U \subset N \mid \forall x, y \in \mathcal{X}, \forall \succeq \in \mathcal{R}^N, x \succ_U y \land y \succeq_U x \Rightarrow x P(\succeq) y \}.$$ 

Note that each member of $U_N$ is a “dictatorial coalition” in the sense that as long as they agree to strictly prefer some alternatives to the others, those unfavoured alternatives are never chosen. Let the other two families be

$$U'_N \equiv \{ U \subset N \mid \exists x, y \in \mathcal{X} \text{ with } x \neq y, \forall \succeq \in \mathcal{R}^N, x \succ_U y \land y \succeq_U x \Rightarrow x P(\succeq) y \},$$
$$U''_N \equiv \{ U \subset N \mid \exists x, y \in \mathcal{X} \text{ with } x \neq y, \forall \hat{U} \subset U', \exists \succeq \in \mathcal{R}^N, x \succ_U y \land y \sim_{\hat{U}} x \land y \succ_{U''} x \land x P(\succeq) y \}.$$ 

We want to show that $U_N$ is an ultrafilter on $N$. We prove it by showing that (1) $U_N = U''_N$ and (2) $U'_N$ is an ultrafilter.

By definition, it is clear that $U_N \subset U'_N \subset U''_N$. The following two lemmas show that the inverse inclusions also hold.

\textbf{Lemma 5.} $U''_N \subset U'_N$.

\textbf{Proof.} Let $U \in U''_N$ and $U' \equiv N \setminus U$. Let $\succeq \in \mathcal{R}^N$. Suppose that $x \succ_U y$ and $y \succeq_U x$. Now let $\hat{U} \equiv \{ i \in U' \mid x \sim_i y \}$, and let $U'' \equiv U' \setminus \hat{U}$. Then, since $U \in U''_N$, there exists $x, y \in \mathcal{X}$ and $\succeq^* \in \mathcal{R}^N$ such that $x \succ_U^* y$, $y \sim_{\hat{U}}^* x$, $y \succ_{U''}^* x$, and $x P(\succeq^*) y$. It is clear that $\succeq^* = \{ x, y \} \succeq$. Thus, by Independence, $\phi_N(\{ x, y \}, \succeq^*) = \phi_N(\{ x, y \}, \succeq^*)$. Since $x P(\succeq^*) y$, $\phi_N(\{ x, y \}, \succeq^*) = \phi_N(\{ x, y \}, \succeq^*) = \{ x \}$. It means that $x P(\succeq) y$. Thus $U \in U'_N$. \hspace{1cm} \Box

\textbf{Lemma 6.} $U'_N \subset U_N$.
Proof. Let \( U \in \mathcal{U}_N \), and let \( U' \equiv N \setminus U \). Then there exists \( x, y \in \mathcal{X} \) with \( x \neq y \) such that

\[
\forall \zeta \in \mathcal{R}^N, \quad x \succ_U y \land y \succ_{U'} x \Rightarrow x \in P(\zeta) y.
\]

Let \( z \in \mathcal{X} \setminus \{x, y\} \) and \( \zeta \in \mathcal{R}^N \). Suppose that \( z \succ_U y \) and \( y \succ_{U'} z \). Now we want to show that \( z \in P(\zeta) y \).

Let \( \zeta' \in \mathcal{R}^N \) such that \( z \succ_U' x \prec_U y, y \succ_{U'} z \), and, for each \( i \in U' \), \( \zeta_i = (y, z) \succ_{U'} \). Then, by Unanimity, \( \phi_N(\{x, z\}, \zeta') = \{z\} \). It means that \( z \in P(\zeta') x \). Now \( x \succ_U y \) and \( y \succ_U' x \). Since \( U \in \mathcal{U}_N \), we have \( x \in P(\zeta') y \). Since \( P(\zeta') \in \mathcal{R}^N \), we have \( z \in P(\zeta') y \) by transitivity. Since \( \zeta = (y, z) \succ_{U'} \), we have that \( \phi_N(\{y, z\}, \zeta) = \phi_N(\{y, z\}, \zeta') \). Therefore \( z \in P(\zeta) y \).

Let \( w \in \mathcal{X} \setminus \{x, y, z\} \). Now, for \( y, z \), it holds that, for each \( \zeta \in \mathcal{R}^N \), if \( z \succ_U y \) and \( y \succ_{U'} z \), then \( z \in P(\zeta) y \). Thus, by applying the same argument as above, we have that, for each \( \zeta \in \mathcal{R}^N \), if \( z \succ_U w \) and \( w \succ_{U'} z \), then \( z \in P(\zeta) w \). It implies that \( U \in \mathcal{U}_N \). \( \Box \)

Corollary 1. \( \mathcal{U}_N = \mathcal{U}'_N = \mathcal{U}''_N \)

Finally we show that \( \mathcal{U}_N \) is an ultrafilter.

Proposition 3. \( \mathcal{U}_N \) is an ultrafilter.

Proof. By Corollary 1, it is enough to show that \( \mathcal{U}''_N \) is an ultrafilter. First, we show that \( \emptyset \notin \mathcal{U}''_N \). If not so, by taking \( U = \emptyset \) and \( \bar{U} \neq N \), there exist \( x, y \in \mathcal{X} \) and \( \zeta \in \mathcal{R}^N \) such that \( y \sim_{\bar{U}} x, y \succ_{N \setminus \bar{U}} x \), and \( x \in P(\zeta) y \). It contradicts Unanimity.

Next we show that \( \mathcal{U}''_N \) is closed under finite intersection. Let \( W_1, W_2 \in \mathcal{U}''_N \). We separate \( N \) into four disjoint subsets:

\[
\begin{align*}
V_1 & \equiv W_1 \cap W_2, \\
V_2 & \equiv W_1 \setminus V_1, \\
V_3 & \equiv W_2 \setminus V_1, \\
V_4 & \equiv N \setminus (W_1 \cup W_2).
\end{align*}
\]

Let \( \{a, b, c\} \subset \mathcal{X} \) and \( \bar{V}_1 \subset N \setminus V_1 \). Then we can find \( \zeta^* \in \mathcal{R}^N \) such that

\[
\begin{align*}
c & \succ_{\bar{V}_1} a \succ_{V_1} b, \ a \succ_{V_2} b \succ_{V_2} c, \\
b & \succ_{V_3} c \succ_{V_4} a, \ b \succ_{V_4} a \succ_{V_4} c, \\
b & \sim_{V_1} c, \text{ and } b \succ_{N \setminus (V \cup \bar{V}_1)} c.
\end{align*}
\]

Note that \( a \succ_{W_1} b \), and \( b \succ_{N \setminus W_1} a \). By Corollary 1, we have \( W_1 \in \mathcal{U}''_N = \mathcal{U}_N \). Therefore \( a \in P(\zeta^*) b \). By the same way, \( W_2 \in \mathcal{U}_N, c \succ_{W_2} a, \) and \( a \succ_{N \setminus W_2} c \). It implies that \( c \in P(\zeta^*) a \). By transitivity, we have \( c \in P(\zeta^*) b \). Now we also have that \( c \succ_{V_1} b, \ b \sim_{V_1} c, \text{ and } b \succ_{N \setminus (V \cup \bar{V}_1)} c \). Thus, by the definition of \( \mathcal{U}''_N \), \( V_1 = W_1 \cap W_2 \in \mathcal{U}''_N \).
Next we show that, for each $V \subset N$, $V \in \mathcal{U}_N^\prime$ or $N \setminus V \in \mathcal{U}_N^\prime$. If $V \in \mathcal{U}_N^\prime$, then it is immediately satisfied. So we suppose that $V \notin \mathcal{U}_N^\prime$. Then, for each $\{a, b, c\} \subset \mathcal{X}$, there exists $\hat{V} \subset N \setminus V$ such that, for each $\preceq \in \mathcal{R}^N$, if $b \succ_V a$, $b \sim_{\hat{V}} a$, and $a \succ_{N \setminus (V \cup \hat{V})} b$, then $a \overset{\text{R}}{\succ} b$. Now let $\hat{V} \subset V$. Then we can find $\preceq \in \mathcal{R}^N$ such that $b \overset{\text{V}}{\succeq} a$, $c \overset{\text{V}}{\succeq} a$, $a \overset{\text{V}}{\sim} b$, and $a \overset{\text{V}}{\succeq} c$. Since $\hat{V}$ can be an arbitrary subset of $V$, by the definition of $\mathcal{U}_N^\prime$, we have that $N \setminus V \in \mathcal{U}_N^\prime$.

Finally we show that, for each $U \in \mathcal{U}_N^\prime$, if $W \supset U$, then $W \in \mathcal{U}_N^\prime$. Suppose not. Then there exist $W, U \subset N$ with $W \supset U$ such that $U \in \mathcal{U}_N^\prime$ but $W \notin \mathcal{U}_N^\prime$. Then, by the above argument, $N \setminus W \in \mathcal{U}_N^\prime$. Since $\mathcal{U}_N^\prime$ is closed under intersection, we have that $U \cap (N \setminus W) \in \mathcal{U}_N^\prime$. However $U \cap (N \setminus W) = \emptyset$. It is contradiction. □

4.2.3 Necessary condition

By Lemma 4 and Proposition 3, we show the necessity part of our theorem. We separate the proof into two lemmas.

Lemma 7. If a solution $\phi_N$ satisfies Unanimity, Independence, and Stability, then there exists a coherent hierarchy of ultrafilters $\mathcal{U}$ such that, for each $(X, \preceq) \in \mathcal{P}(\mathcal{X}) \times \mathcal{R}^N$, there exist $K(X, \preceq) \in \mathbb{N}_+$ and a series $\{T_k, C_k\}_{k=0}^\infty$ satisfying 1-4 in the definition of the generalised serial dictatorship.

Proof. Suppose that $\phi_N$ satisfies Unanimity, Independence, and Stability. First we define a coherent hierarchy of ultrafilters $\mathcal{U}$ as follows.

For each $N \in \mathcal{P}(\mathbb{N})$ and $\preceq \in \mathcal{R}^N$, let $\preceq_N \in \mathcal{R}^N$ be such that

$$\text{for each } i \in N, \quad \preceq_i \equiv \gamma_i,$$

$$\text{for each } i \in N \setminus N, \text{ for each } x, y \in \mathcal{X}, \quad x \sim_i y.$$

Let $\psi_N$ be a solution such that, for each $(X, \preceq) \in \mathcal{P}(\mathcal{X}) \times \mathcal{R}^N$, $\psi_N(X, \preceq) \equiv \phi_X(X, \preceq_N)$. We show that $\psi_N$ satisfies Unanimity, Independence, and Stability.

First we show that $\psi_N$ satisfies Unanimity. Let $x \in X$ and $\preceq \in \mathcal{R}^N$. Suppose that, for each $i \in N$ and $y \in X$, $x \sim_i y$, and there exists $i^* \in N$ such that $x \succ_{i^*} y$. Then, by construction, we have that, for each $i \in N$ and $y \in X$, $x \sim_i y$. Since $i^* \in N$, we have that $\preceq_{i^*} = \preceq_{i^*}$. So $x \succ_{i^*} y$. By Unanimity of $\phi_N$, $\phi_N(X, \preceq_N) = \{x\}$. It implies that $\psi_N(X, \preceq) = \{x\}$.

Next we show that $\psi_N$ satisfies Independence. Let $X \subset \mathcal{X}$, and $\preceq, \tilde{\preceq} \in \mathcal{R}^N$ be such that $\tilde{\preceq} = X \sim_i \tilde{\preceq}$. Then, for each $i \in N$, we have that $\preceq_N = X \sim_i \tilde{\preceq}_N$. For each $i \in N \setminus N$, $i$ is indifferent among $\mathcal{X}$ in the both cases. Therefore $\preceq_N = X \sim_i \tilde{\preceq}_N$. Thus we have that $\psi_N(X, \preceq) = \phi_X(X, \preceq_N) = \phi_N(X, \tilde{\preceq}_N) = \psi_N(X, \tilde{\preceq})$. The second equality is due to Independence of $\phi_N$. 

9
Finally we show Stability. Let $Y \subset X \subset \mathcal{X}$ with $Y \neq \emptyset$. Let $\succeq \in R^N$. Suppose that $\psi_N(X, \succeq) \cap Y \neq \emptyset$. Then $\phi_N(X, \succeq^N) \cap Y \neq \emptyset$. Since $\phi_N$ satisfies Stability, $\phi_N(Y, \succeq^N) = \phi_N(X, \succeq^N) \cap Y$. Therefore $\psi_N(Y, \succeq^N) = \psi_N(X, \succeq) \cap Y$.

Now we know that, for each $N \in \mathcal{P}(N)$, $\psi_N$ satisfies Unanimity, Independence, and Stability. By the above lemma, there exists an dictatorial ultrafilter over $N$. We denote the ultrafilter as $\mathcal{U}_N$. Let $\mathcal{U}$ be the mapping which maps each $N$ to $\mathcal{U}_N$. We show that $\mathcal{U}$ is a coherent hierarchy of ultrafilters.

Let $N, N' \in \mathcal{P}(N)$ with $N \subset N'$. For each $\succeq \in R^N$, let $\succeq^N$ be such that (1) for each $i \in N$, $\succeq_i = \succeq_i^N$, and (2) for each $i \in N \setminus N$ and $x, y \in \mathcal{X}$, $x \sim^N y$. Then we have that, for each $(X, \succeq) \in \mathcal{P}(X) \times R^N$, $\psi_N(X, \succeq) = \psi_N(X, \succeq^N)$. Suppose that, for $U \in \mathcal{U}_N$, it holds that $U \subset N$. By construction, for each $\succeq \in R^N$ and $x, y \in \mathcal{X}$, if $x \sim_U y$, $\psi_N(\{x, y\}, \succeq^N) = \{x\}$. This implies that, for each $\succeq \in R^N$ and $x, y \in \mathcal{X}$, if $x \sim_U y$, $\psi_N(\{x, y\}, \succeq) = \{x\}$. Thus $U \subset \mathcal{U}_N$.

Now suppose that $N \setminus N \notin \mathcal{U}_N$. Since $\mathcal{U}_N$ is an ultrafilter, $N \in \mathcal{U}_N$. If $M' \in \mathcal{U}_N$, then $N \cap M' \in \mathcal{U}_N$. The above argument implies that $N \cap M' \in \mathcal{U}_N$. If $N \cap M' \in \mathcal{U}_N$, we want to show that $N \cap M' \in \mathcal{U}_N$. Suppose not. Since $N \in \mathcal{U}_N$, we have that $N \setminus M' \in \mathcal{U}_N$. Since $N \setminus M' \in \mathcal{U}_N$, we have that $N \setminus M' \in \mathcal{U}_N$. However we assume that $N \cap M' \in \mathcal{U}_N$. It is a contradiction. Thus $N \cap M' \in \mathcal{U}_N$. Since $N \cap M' \in \mathcal{U}_N$, we have that $M' \in \mathcal{U}_N$.

The rest of the proof is to show that $\phi_N$ follows the process of the generalized serial dictatorship. Let $(X, \succeq) \in \mathcal{P}(X) \times R^N$. Let $T_0 \equiv X$ and $C_0 \equiv N$. Since ultrafilter $\mathcal{U}_0$ pick up only one set from the equivalent classes with respect to their preferences, there exists an unique $\succeq^* \in \mathcal{R}$ such that $\{i \in N \mid \succeq_i = \succeq^*\} \in \mathcal{U}_0$. We denote this $\succeq^*$ as $\succeq_{\mathcal{U}_0}$. Then we have that $\phi_N(X, \succeq) \subset T_1 \equiv \text{Top}(T_0, \succeq_{\mathcal{U}_0})$.

Next let $C_1 \equiv \{i \in N \mid \text{Top}(T_0, \succeq_i) \neq \{i\}\}$. Let $\succeq_{C_1} \in \mathcal{R}^{C_1}$ be such that, for each $i \in C_1$, $\succeq_{C_1} = \succeq_i$. Then, by construction, $\psi_{C_1}(T_1, \succeq_{C_1}) = \phi_N(T_1, \succeq_i)$. By Stability, we have that $\psi_{C_1}(T_1, \succeq_{C_1}) = \phi_N(X, \succeq_i)$. By the same argument as above, we can define $\succeq_{\mathcal{U}_{C_1}}$ as before. And we have that $\psi_{C_1}(T_1, \succeq_{C_1}) \subset T_2 \equiv \text{Top}(T_1, \succeq_{\mathcal{U}_{C_1}})$. Since the equivalent classes with respect to the agents’ preference is only finite, this process ends at a finite $K$ step. Thus we can construct $\{T_k, C_k\}_{k=1}^K$ such that

$$T_0 = 0,$$
$$C_0 = X,$$
$$\text{for each } k \in \{1, \ldots, K\}, \quad T_k = \text{Top}(T_{k-1}, \succeq_{\mathcal{U}_{C_{k-1}}}) \neq \emptyset,$$
$$C_k = \{i \in C_{k-1} \mid \text{Top}(T_{k-1}, \succeq_i) \neq T_k\},$$
$$C_K = \emptyset,$$
$$\phi_N(X, \succeq) \subset T_K. \quad \square$$

Next we show the existence of a tie-breaking rule;

**Lemma 8.** There exists $\rho \in \mathcal{R}$, such that, for each $X \in \mathcal{P}(X)$ and $\succeq \in R^N$, $\phi_N(X, \succeq) = \text{Top}(T_K(X, \succeq), \rho)$.

**Proof.** Let $\rho$ be a relationship on $\mathcal{X}$ such that, for each $x, y \in \mathcal{X}$, $x \rho y$ if there exist $\succeq \in R^N$ such that $\{x, y\} \subset T_K(X, \succeq)$ and $x \in \phi_N(X, \succeq)$. First we show that $\rho$ is a weak preference.
Let $x, y \in \mathcal{X}$. Then we can find $\tilde{\sim} \in \mathcal{R}^N$ such that, for each $i \in N$ and $z \in \mathcal{X}\{x, y\}$, $x \tilde{\sim}_iy$ and $x \preceq_i z$. Then $T_K(x, \tilde{\sim}) = \{x, y\}$. Since $\phi_N(\mathcal{X}, \tilde{\sim}) \subset T_K(x, \tilde{\sim})$ and $\phi_N(\mathcal{X}, \tilde{\sim}) \neq \emptyset$, we have $x \in \phi_N(\mathcal{X}, \tilde{\sim})$ or $y \in \phi_N(\mathcal{X}, \tilde{\sim})$, that is, $x \rho y$ or $y \rho x$. Therefore $\rho$ is complete.

Next we show transitivity. Let $x, y, z \in \mathcal{X}$ be such that $x \rho y$ and $y \rho z$. Then there exists $\sim \in \mathcal{R}^N$ such that $\{x, y\} \subset T_K(x, \sim)$ and $x \in \phi_N(\mathcal{X}, \sim)$. By the same way, there also exists $\sim' \in \mathcal{R}^N$ such that $\{y, z\} \subset T_K(x, \sim')$ and $y \in \phi_N(\mathcal{X}, \sim')$. Then, by Stability, $x \in \phi_N(\{x, y\}, \sim)$ and $y \in \phi_N(\{y, z\}, \sim')$. Now let $\sim^* \in \mathcal{R}^N$ be such that, for each $i \in N$ and each $w \in \mathcal{X}\{x, y, z\}$, $x \sim^* y \sim^* z$ and $z \sim^* w$. Then $T_K(x, \sim^*) = \{x, y, z\}$. It means that $\phi_N(\mathcal{X}, \sim^*) \subset \{x, y, z\}$. By Stability, we have that $\phi_N(\mathcal{X}, \sim^*) = \phi_N(\{x, y, z\}, \sim^*)$. Thus it is enough to show that $x \in \phi_N(\{x, y, z\}, \sim^*)$. Suppose that $x \not\in \phi_N(\{x, y, z\}, \sim^*)$. Since $\sim^* = \{y, z\} \sim^*$, we have that $y \in \phi_N(\{x, y, z\}, \sim^*) = \phi_N(\{y, z\}, \sim^*)$. Thus we have that $\phi_N(\{x, y, z\}, \sim^*) = \phi_N(\{x, y\}, \sim^*)$. Since $x \rho y$, we have that $x \in \phi_N(\{x, y\}, \sim^*) = \phi_N(\{x, y\}, \sim^*)$. It is a contradiction.

Now that we showed that $\rho$ is a weak preference over $\mathcal{X}$, we finally show that, for each $X \in \mathcal{P}(\mathcal{X})$ and $\sim \in \mathcal{R}^N$, $\phi_N(\mathcal{X}, \sim) = \text{Top}(T_K(\mathcal{X}, \sim), \rho)$. First we show that, for each $X \in \mathcal{P}(\mathcal{X})$ and each $\sim \in \mathcal{R}^N$, $\phi_N(\mathcal{X}, \sim) \subset \text{Top}(T_K(\mathcal{X}, \sim), \rho)$. Let $x \in \phi_N(\mathcal{X}, \sim)$ and $y \in \text{Top}(T_K(\mathcal{X}, \sim), \rho)$. Then $\{x, y\} \subset T_K(\mathcal{X}, \sim)$ and, by Stability, $x \in \phi_N(\{x, y\}, \sim)$. This implies that $x \rho y$. Since $y \in \text{Top}(T_K(\mathcal{X}, \sim), \rho)$, we have $x \in \text{Top}(T_K(\mathcal{X}, \sim), \rho)$.

Next we show that, for each $X \in \mathcal{P}(\mathcal{X})$ and each $\sim \in \mathcal{R}^N$, $\text{Top}(T_K(\mathcal{X}, \sim), \rho) \subset \phi_N(\mathcal{X}, \sim)$. Let $x \in \text{Top}(T_K(\mathcal{X}, \sim), \rho)$ and $y \in \phi_N(\mathcal{X}, \sim)$. Since $x \rho y$, there exists $\sim' \in \mathcal{R}^N$ such that $\{x, y\} \subset T_K(\mathcal{X}, \sim')$ and $x \in \phi_N(\mathcal{X}, \sim')$. Since $\{x, y\} \subset T_K(\mathcal{X}, \sim)$, we have that $\sim = (x, y) \sim'$. Therefore $x \in \phi_N(\{x, y\}, \sim') = \phi_N(\{x, y\}, \sim)$. By Stability, we have that $x \in \phi_N(\mathcal{X}, \sim)$. □

References


