Course description: This course provides a concise and accessible introduction to some basic concepts and techniques used when comparing information structures or solutions to optimization problems. It will introduce concepts like supermodularity and the single crossing property and show its relevance for guaranteeing that the solutions to an optimization problem are monotonically increasing (or decreasing) with respect to various parameters. Consideration will be given to both decision-making with and without uncertainty. Lehmann’s notion of informativeness and its role in valuing information structures will also be examined. Applications to games with strategic complementarities and to statistical decision theory will be discussed.

Style: This is a short course and not a seminar. While there will be some discussion of very new results, most of the time will be spent on known results in this area that I consider important. The emphasis is on providing an accessible introduction that will allow anyone interested in the techniques or its applications to do further reading and research. To give a flavor of how the arguments work, I shall at various points be proving results in some detail. In other words, I shall be sacrificing some coverage for the sake of providing greater depth and texture.

Readings:


ATHEY, S., P. MILGROM, AND J. ROBERTS (1998): Robust Comparative Statics. (Draft Chapters, available on Athey’s webpage.)


The closest thing to a textbook for the material covered in this course is the book by Topkis. The survey by Amir and the incomplete notes by Athey et al. are also relevant. The books by Gollier and Vives are not books on mathematical techniques as such, but they do make use of the concepts and techniques covered in the course. The book by Blackwell and Girsichik on statistical decision theory is very old, but it is beautifully written and provides an introduction to the topic that is congenial to economists. The readings most closely related to the lectures are indicated with asterisks*. 

2
Short Course on
Comparative Statics and Comparative Information

John Quah
One-dimensional comparative statics

Let \( X \subseteq \mathbb{R} \) and \( f, g : X \to \mathbb{R} \).

We are interested in comparing \( \text{argmax}_{x \in X} f(x) \) with \( \text{argmax}_{x \in X} g(x) \).

Standard approach:

Assume \( X \) is a compact interval and \( f \) and \( g \) are quasi-concave functions.

Let \( x^* \) be the unique maximizer of \( f \). Then \( f'(x^*) = 0 \). Show that \( g'(x^*) \geq 0 \). Then optimum has shifted to the right.
One-dimensional comparative statics

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This approach makes various assumptions, most notably the quasi-concavity of $f$ and $g$. 
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This approach makes various assumptions, most notably the quasi-concavity of $f$ and $g$. Not the most natural assumption; example:

let $x$ be output, $P$ the inverse demand function, and $c$ the marginal cost of producing good.

The profit function $\Pi(x) = xP(x) - cx$ is not naturally concave.
One-dimensional comparative statics

Assume that $f$ and $g$ are continuous functions and their domain $X$ is compact. Then $\text{argmax}_{x \in X} f(x)$ and $\text{argmax}_{x \in X} g(x)$ are nonempty. But these sets need not be singletons or intervals.

First question: how do we compare sets?

Definition: Let $S'$ and $S''$ be subsets of $R$. $S''$ dominates $S'$ in the strong set order ($S'' \geq S'$) if for any $x''$ in $S''$ and $x'$ in $S'$, we have $\max\{x'', x'\}$ in $S''$ and $\min\{x'', x'\}$ in $S'$.

Example: $\{3, 5, 6, 7\} \not\geq \{1, 4, 6\}$
One-dimensional comparative statics

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Example: $\{3, 5, 6, 7\} \not\geq \{1, 4, 6\}$ but $\{3, 4, 5, 6, 7\} \geq \{1, 3, 4, 5, 6\}$. 
One-dimensional comparative statics

Assume that $f$ and $g$ are continuous functions and their domain $X$ is compact. Then $\argmax_{x \in X} f(x)$ and $\argmax_{x \in X} g(x)$ are nonempty. But these sets need not be singletons or intervals.

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Example: $\{3, 5, 6, 7\} \not\geq\ {1, 4, 6}$ but $\{3, 4, 5, 6, 7\} \geq\ {1, 3, 4, 5, 6}$.

Note: if $S'' = \{x''\}$ and $S' = \{x'\}$, then $x'' \geq x'$.

More generally, largest element in $S''$ is larger than the largest element in $S'$; smallest element in $S''$ is larger than the smallest element in $S'$. 
One-dimensional comparative statics

Definition: $g$ dominates $f$ by the single crossing property ($g \succeq_{sc} f$) if for all $x''$ and $x'$ such that $x'' > x'$, the following holds:

$$f(x'') - f(x') \geq (>) 0 \implies g(x'') - g(x') \geq (>) 0.$$  \hspace{1cm} (1)

Let $S$ be a partially ordered set. A family of real-valued functions $\{f(\cdot, s)\}_{s \in S}$ is an SCP family if the functions are ordered by the single crossing property (SCP), i.e., whenever $s'' > s'$, we have $f(\cdot, s'') \succeq_{sc} f(\cdot, s')$. 
One-dimensional comparative statics

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Motivation for the term ‘single crossing’: for any $x'' > x'$, the function $\Delta : S \to R$ defined by

$$\Delta(s) = f(x'', s) - f(x', s)$$

crosses the horizontal axis at most once.
One-dimensional comparative statics

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Let $S$ be a partially ordered set.
A family of real-valued functions \{\{f(\cdot, s)\}_{s \in S}\} is an SCP family if the functions are ordered by the single crossing property (SCP), i.e., whenever $s'' > s'$, we have $f(\cdot, s'') \succeq_{sc} f(\cdot, s')$.

Motivation for the term ‘single crossing’: for any $x'' > x'$, the function $\Delta : S \to R$ defined by

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Single crossing is an ordinal property: if $g \succeq_{sc} f$, then $\phi \circ g \succeq_{sc} \psi \circ f$, where $\phi$ and $\psi$ are increasing functions mapping $R$ to $R$. 
One-dimensional comparative statics

In this case, \( f(\cdot, s'') \preceq_sc f(\cdot, s') \)
One-dimensional comparative statics

\[ f \]

\[ g \]

\[ x^* \]

\[ x^{**} \]

\[ g \]
One-dimensional comparative statics

In the top picture $f \succeq_{SC} g$, but not in the bottom picture.
One-dimensional comparative statics

Definition: $g$ dominates $f$ by the increasing differences ($g \succeq_{\text{IN}} f$) if for all $x''$ and $x'$ such that $x'' > x'$, the following holds:

$$g(x'') - g(x') \geq f(x'') - f(x')$$

(4)

Clearly, $g \succeq_{\text{IN}} f$ implies $g \succeq_{\text{SC}} f$.

Similarly, a family $\{f(\cdot, s)\}_{s \in S}$ satisfies increasing differences if the functions are ordered by increasing differences, i.e., whenever $s'' > s'$, we have $f(\cdot, s'') \succeq_{\text{IN}} f(\cdot, s')$.

Let $S$ be an open subset of $\mathbb{R}^l$ and $X$ an open interval. Then a sufficient (and necessary) condition for increasing differences is

$$\frac{\partial^2 f}{\partial x \partial s_i}(x, s) \geq 0$$

at every point $(x, s)$ and for all $i$. 

One-dimensional comparative statics

Theorem 1: (MS-94) Suppose $X \subseteq \mathbb{R}$ and $f, g : X \rightarrow \mathbb{R}$; then
\[
\arg\max_{x \in Y} g(x) \geq \arg\max_{x \in Y} f(x)
\]
for any $Y \subseteq X$ if and only if $g \succeq_{sc} f$. 
One-dimensional comparative statics

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$$\arg\max_{x \in Y} g(x) \geq \arg\max_{x \in Y} f(x)$$
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Proof: Assume $g \succeq_{sc} f$, $x'' \in \arg\max_{x \in Y} f(x)$, and
$x' \in \arg\max_{x \in Y} g(x)$. We have to show that
$$\max\{x', x''\} \in \arg\max_{x \in Y} g(x) \quad \text{and} \quad \min\{x', x''\} \in \arg\max_{x \in Y} f(x).$$
We need only consider the case where $x'' > x'$.
Theorem 1: (MS-94) Suppose $X \subseteq R$ and $f, g : X \rightarrow R$; then 
argmax_{x \in Y} g(x) \geq \argmax_{x \in Y} f(x)$ for any $Y \subseteq X$ if and only if $g \succeq_{SC} f$.

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We need only consider the case where $x'' > x'$.

Since $x''$ is in $\argmax_{x \in Y} f(x)$, we have $f(x'') \geq f(x')$ and since 
g \succeq_{SC} f, we also have $g(x'') \geq g(x')$; thus $x''$ is in $\argmax_{x \in Y} g(x)$.

Furthermore, $f(x'') = f(x')$ so that $x'$ is in $\argmax_{x \in Y} f(x)$. If not, 
f(x'') > f(x') which implies (by the fact that $g \succeq_{SC} f$) that 
g(x'') > g(x'), contradicting the assumption that $g$ is maximized at $x'$. 


One-dimensional comparative statics

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Since \( x'' \) is in \( \operatorname{argmax}_{x \in Y} f(x) \), we have \( f(x'') \geq f(x') \) and since \( g \succeq_{sc} f \), we also have \( g(x'') \geq g(x') \); thus \( x'' \) is in \( \operatorname{argmax}_{x \in Y} g(x) \).

Furthermore, \( f(x'') = f(x') \) so that \( x' \) is in \( \operatorname{argmax}_{x \in Y} f(x) \). If not, \( f(x'') > f(x') \) which implies (by the fact that \( g \succeq_{sc} f \)) that \( g(x'') > g(x') \), contradicting the assumption that \( g \) is maximized at \( x' \).

Necessity: clear that if SCP property is violated at \( x' \) and \( x'' \), then
\[
\operatorname{argmax}_{x \in \{x', x''\}} g(x) \not\geq \operatorname{argmax}_{x \in \{x', x''\}} f(x).
\]
One-dimensional comparative statics

Application: Recall $\Pi(x, -c) = xP(x) - cx$. Then $\{\Pi(\cdot, -c)\}_{c \in \mathbb{R}^-}$ obey increasing differences, since

$$\frac{\partial^2 \Pi}{\partial x \partial c} = -1.$$

By Theorem 1, $\operatorname{argmax}_{x \in X} \Pi(x, -c)$ is increasing in $-c$.

For example, $\operatorname{argmax}_{x \in X} \Pi(x, -3) \geq \operatorname{argmax}_{x \in X} \Pi(x, -5)$.

In other words, the profit-maximizing output decreases as the marginal cost of output increases.
One-dimensional comparative statics

**Application:** Bertrand Oligopoly with differentiated products, with

\[
\Pi_a(p_a, p_{-a}) = (p_a - c_a) D_a(p_a, p_{-a})
\]
\[
\ln \Pi_a(p_a, p_{-a}) = \ln(p_a - c_a) + \ln D_a(p_a, p_{-a})
\]

So \( \{\ln \Pi_a(\cdot, p_{-a})\}_{-a \in -A} \) has increasing differences if

\[
\frac{\partial^2}{\partial p_a \partial p_{-a}} [\ln D_a] \geq 0.
\]

This is equivalent to

\[
\frac{\partial}{\partial p_{-a}} \left[ -\frac{p_a}{D_a} \frac{\partial D_a}{\partial p_a} \right] \leq 0;
\]

i.e., firm \( a \)'s own-price elasticity of demand,

\[
\epsilon_a(p_a, p_{-a}) = -\frac{p_a}{D_a} \frac{\partial D_a}{\partial p_a} \text{ decreases with } p_{-a}.
\]

If this assumption holds, \( \arg \max_{p_a \in P} \Pi_a(p_a, p_{-a}) \) increases with \( p_{-a} \).
Let $X = \prod_{i=1}^{N} X_i$, where each $X_i$ is a compact interval.

**Theorem 2:** (Tarski) Suppose $\phi : X \rightarrow X$ is an increasing function. Then the set of fixed points $\{x \in X : \phi(x) = x\}$ is nonempty. In fact, $x^{**} = \sup\{x \in X : x \leq \phi(x)\}$ is a fixed point and is the largest fixed point, i.e., for any other fixed point $x^*$, we have $x^* \leq x^{**}$.

Note: $\phi$ need not be continuous.
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Note: $\phi$ need not be continuous.

**Theorem 3:** Suppose $\phi(\cdot, t) : X \rightarrow X$ is increasing in $(x, t)$. Then the largest fixed point of $\phi(\cdot, t)$ is increasing in $t$. 
Supermodular Games

Theorem 3: Suppose $\phi(\cdot, t) : X \to X$ is increasing in $(x, t)$. Then the largest fixed point of $f(\cdot, t)$ is increasing in $t$.

Proof: By Theorem 2, the largest fixed point

$$\bar{x}(t) = \sup\{x \in X : x \leq \phi(x, t)\}.$$ 

Suppose $t'' > t'$; since $\phi(x, t'') \geq \phi(x, t')$ for all $x$,

$$\{x \in X : x \leq \phi(x, t')\} \subseteq \{x \in X : x \leq \phi(x, t'')\}.$$

Therefore,

$$\sup\{x \in X : x \leq \phi(x, t')\} \leq \sup\{x \in X : x \leq \phi(x, t'')\}.$$ 

By Theorem 2 again, $\bar{x}(t') \leq \bar{x}(t'').$ 

QED
Supermodular Games

Bertrand Oligopoly: assume the set of firms is $A$; the typical firm $a$ chooses its price from the compact interval $P$ to maximize

$$\Pi_a(p_a, p_{-a}) = (p_a - c_a)D_a(p_a, p_{-a}).$$

Recall: if own-price elasticity is decreasing in $p_{-a}$ then the family $\{D_a(\cdot, p_{-a})\}_{p_{-a}}$ forms an SCP family.

Define $B_a(p_{-a}) = \arg\max_{p_a \in P} \Pi_a(p_a, p_{-a})$. This is the set of $a$’s best responses to the pricing of other firms. It is nonempty if $\Pi_a$ is continuous.
Supermodular Games

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This is the set of \( a \)'s best responses to the pricing of other firms. It is nonempty if \( \Pi_a \) is continuous.

Define \( \bar{B}_a(p_{-a}) = \max \left[ \arg\max_{p_a \in P} \Pi_a(p_a, p_{-a}) \right] \);

This is the \textit{largest} best response to the pricing of other firms. This is an increasing function of \( p_{-a} \) (by Theorem 1).
Supermodular Games

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Define $\bar{B}_a(p_{-a}) = \max \left[ \arg\max_{p_a \in P} \Pi_a(p_a, p_{-a}) \right]$; This is the *largest* best response to the pricing of other firms. This is an increasing function of $p_{-a}$ (by Theorem 1).

Define $\bar{P} = P \times P \times \ldots \times P$ and the map $\bar{B}: \bar{P} \to \bar{P}$ by

$$\bar{B}(p) = (\bar{B}_a(p_{-a}))_{a \in A}.$$

A fixed point of this map is a NE of the game.
Supermodular Games

By Theorem 1, $\bar{B}$ is an increasing function. By Tarski’s Fixed Point Theorem, a fixed point exists. Specifically, $p^* = \sup \{ p \in \bar{P} : p \leq \bar{B}(p) \}$ is a fixed point of the map $\bar{B}$ and thus a NE. In fact, this is the largest NE. Why?
Supermodular Games

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Let $\hat{p} = (p_a)_{a \in A}$ be any other NE. Then $\hat{p}_a \in B_a(\hat{p}_-a)$. So

$$\hat{p}_a \leq \bar{B}_a(\hat{p}_-a),$$

and we obtain $\hat{p} \leq \bar{B}(\hat{p})$. Thus, $\hat{p} \in \{p \in \bar{P} : p \leq \bar{B}(p)\}$, which implies $\hat{p} \leq p^* \equiv \sup\{p \in \bar{P} : p \leq \bar{B}(p)\}$. 

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Supermodular Games

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In this game, best response function is monotonic but not necessarily continuous. Existence guaranteed by appealing to Tarski’s fixed point theorem, rather than Kakutani’s. The game has a largest NE. We can do comparative statics exercises on the largest NE...
Supermodular Games

What happens to the largest NE when firm $\tilde{a}$ experiences an increase in marginal cost from $c_{\tilde{a}}$ to $c'_{\tilde{a}}$? Recall

$$
\Pi_{\tilde{a}}(p_{\tilde{a}}, p_{-\tilde{a}}, c_{\tilde{a}}) = (p_{\tilde{a}} - c_{\tilde{a}})D_{\tilde{a}}(p_{\tilde{a}}, p_{-\tilde{a}})
$$

$$
\ln \Pi_{\tilde{a}}(p_{\tilde{a}}, p_{-\tilde{a}}, c_{\tilde{a}}) = \ln(p_{\tilde{a}} - c_{\tilde{a}}) + \ln D_{\tilde{a}}(p_{\tilde{a}}, p_{-\tilde{a}})
$$

Observe that

$$
\frac{\partial}{\partial p_{\tilde{a}} \partial c_{\tilde{a}}} \left[ \ln \Pi_{\tilde{a}} \right] > 0.
$$

By Theorem 1, firm $a$’s best response increase with $c_{\tilde{a}}$ (for fixed $p_{-a}$). Formally,

$$
B_{\tilde{a}}(p_{-\tilde{a}}, c'_{\tilde{a}}) \geq B_{\tilde{a}}(p_{-\tilde{a}}, c_{\tilde{a}}).
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Supermodular Games

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B_{\tilde{a}}(p_{-\tilde{a}}, c'_{\tilde{a}}) \geq B_{\tilde{a}}(p_{-\tilde{a}}, c_{\tilde{a}}).
$$

This implies that $\bar{B}(p, c'_{\tilde{a}}) \geq \bar{B}(p, c_{\tilde{a}})$. So largest fixed point of $\bar{B}(\cdot, c'_{\tilde{a}})$ is larger than the largest fixed point of $\bar{B}(\cdot, c_{\tilde{a}})$ (by Theorem 3). In other words, if firm $\tilde{a}$’s marginal cost increases from $c_{\tilde{a}}$ to $c'_{\tilde{a}}$, the largest NE increases: every firm increases its price.
Decisions Under Uncertainty

Recall that a family \( \{f(\cdot, s)\}_{s \in S} \) is an SCP family (for \( S \subseteq R \)) if for any \( x'' > x' \), the function \( \Delta : S \to R \) defined by

\[
\Delta(s) = f(x'', s) - f(x', s)
\]

crosses the horizontal axis at most once.

Definition: A function \( \Delta : S \to R \) is said to have property \((\star)\) if, whenever \( s'' > s' \),

\[
\Delta(s') \geq (>) 0 \implies \Delta(s'') \geq (>) 0.
\]

Definition: Consider the family \( h(\cdot, t) : S \to R \) where \( S \subseteq R \). This family obeys log increasing differences if \( \{\ln h(\cdot, t)\}_{t \in T} \) obeys increasing differences. Equivalently, whenever \( s'' > s' \),

\[
\frac{h(s'', t)}{h(s', t)} \text{ increases with } t.
\]
Theorem 4: Suppose $S \subseteq R$, the function $\Delta : S \to R$ has property ($\star$), and $\{h(\cdot, t)\}_{t \in T}$ obeys log increasing differences. Then

$$\phi(t) = \int_{S} \Delta(s) h(s, t) ds$$

has property ($\star$).

Indeed, if $\Delta$ crosses the horizontal axis at $s = s_0$, then, for $t_2 > t_1$,

$$\phi(t_2) \geq \frac{h(s_0, t_2)}{h(s_0, t_1)} \phi(t_1).$$
Decisions Under Uncertainty

Proof: We split the integral \( \phi(t_2) = \int_S \Delta(s) h(s, t_2) ds \) into two:

\[
\phi(t_2) = \int_{-\infty}^{s_0} \Delta(s) h(s, t_1) \frac{h(s, t_2)}{h(s, t_1)} ds + \int_{s_0}^{\infty} \Delta(s) h(s, t_1) \frac{h(s, t_2)}{h(s, t_1)} ds.
\]

Between \([-\infty, s_0]\), \( \Delta(s) \leq 0 \), so the first term on the right is greater than

\[
\frac{h(s_0, t_2)}{h(s_0, t_1)} \int_{-\infty}^{s_0} \Delta(s) h(s, t_1) ds.
\]

Between \([s_0, \infty]\), \( \Delta(s) \geq 0 \), so the second term is greater than

\[
\frac{h(s_0, t_2)}{h(s_0, t_1)} \int_{s_0}^{\infty} \Delta(s) h(s, t_1) ds.
\]

Adding up the two lower bounds gives us

\[
\phi(t_2) \geq \frac{h(s_0, t_2)}{h(s_0, t_1)} \int_S \Delta(s) h(s, t_1) ds = \frac{h(s_0, t_2)}{h(s_0, t_1)} \phi(t_1).
\]

. QED
Decisions Under Uncertainty

Application: consider Bernoulli utility functions $v(\cdot, t)$ parameterized by $t$. Assume that the family $\{v'(\cdot, t)\}_{t \in T}$ obeys log increasing differences. This is equivalent to

$$- \frac{v''(z, t_2)}{v'(z, t_2)} \leq - \frac{v''(z, t_1)}{v'(z, t_1)}$$

whenever $t_2 > t_1$; i.e., type $t_2$ is less risk averse than type $t_1$.

Agent has wealth $w$ and chooses between a safe and risky asset to maximize expected utility

$$V(x, t) = \int v((w - x)r + xs, t) \lambda(s) \, ds$$

$-r$ and $s$ are payoffs of safe and risky assets, latter with density $\lambda$; $x \in [0, w]$ is the investment in the risky asset.
Decisions Under Uncertainty

Recall $V(x, t) = \int v((w - x)r + xs, t) \lambda(s) \, ds$. Differentiating w.r.t $x$,

$$V'(x, t) = \int_{s \in S} (s - r)\lambda(s) v'((w - x)r + xs, t) \, ds.$$ 

By Theorem 4 and the fact that $\{v'(\cdot, t)\}_{t \in T}$ obeys log increasing differences, for any $t_2 > t_1$,

$$V'(x, t_2) \geq \frac{v'(wr, t_2)}{v'(wr, t_1)} V'(x, t_1).$$
Decisions Under Uncertainty

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$$V'(x, t_2) \geq \frac{v'(wr, t_2)}{v'(wr, t_1)} V'(x, t_1).$$

So whenever $x'' > x'$,

$$V(x'', t_2) - V(x', t_2) \geq \frac{v'(wr, t_2)}{v'(wr, t_1)} [V(x'', t_1) - V(x', t_1)].$$

Clearly, the family $\{V(\cdot, t)\}_{t \in T}$ obeys SCP.
Decisions Under Uncertainty

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\[
V(x'', t_2) - V(x', t_2) \geq \frac{v'(wr, t_2)}{v'(wr, t_1)} [V(x'', t_1) - V(x', t_1)]
\]

Clearly, the family \( \{V(\cdot, t)\}_{t \in T} \) obeys SCP.

By Theorem 1, argmax\( _{x \in [0, w]} V(x, t) \) is increasing in \( t \),
i.e., the less risk averse agent invests more in the risky asset.
Definition: A family of density functions \( \{ \lambda(\cdot, t) \}_{t \in T} \) (defined on \( S \subseteq R \)) that obeys log increasing differences is said to respect the \textit{monotone likelihood ratio order} (MLR).
Decisions Under Uncertainty

Definition: A family of density functions \( \{\lambda(\cdot, t)\}_{t \in T} \) (defined on \( S \subseteq R \)) that obeys log increasing differences is said to respect the monotone likelihood ratio order (MLR).

Fact: If \( \{\lambda(\cdot, t)\}_{t \in T} \) is MLR-ordered then it is ordered by first order stochastic dominance (FOSD), i.e., whenever \( t_2 > t_1 \), \( \lambda(\cdot, t_2) \) first order stochastically dominates \( \lambda(\cdot, t_1) \).
Decisions Under Uncertainty

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Theorem 5: Suppose \( \{ f(\cdot, s) \}_{s \in S} \) (for \( S \subseteq R \)) is an SCP family. Then the family \( \{ F(\cdot, t) \}_{t \in T} \) (for \( T \subseteq R \)) given by

\[
F(x, t) = \int_{s \in S} f(x, s) \lambda(s, t) \, ds
\]

is also an SCP family if \( \{ \lambda(\cdot, t) \}_{t \in T} \) is MLR-ordered.
Theorem 5: Suppose \( \{f(\cdot, s)\}_{s \in S} \) (for \( S \subseteq R \)) is an SCP family. Then the family \( \{F(\cdot, t)\}_{t \in T} \) given by

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\]

is also an SCP family if \( \{\lambda(\cdot, t)\}_{t \in T} \) is MLR-ordered.

Proof: The function \( \Delta(s) = f(x'', s) - f(x', s) \) has property (\( \star \)). By Theorem 4, the function \( \phi \) given by

\[
\phi(t) = F(x'', t) - F(x', t) = \int_{s \in S} [f(x'', s) - f(x', s)] \lambda(s, t) \, ds
\]

also obeys property (\( \star \)). In other words, \( \{F(\cdot, t)\}_{t \in T} \) is an SCP family.

QED
Decisions Under Uncertainty

Theorem 6: Suppose $\{f(\cdot, s)\}_{s \in S}$ (for $S \subseteq R$) satisfies increasing differences. Then the family $\{F(\cdot, t)\}_{t \in T}$ given by

$$F(x, t) = \int_{s \in S} f(x, s) \lambda(s, t) \, ds$$

also satisfies increasing differences if $\{\lambda(\cdot, t)\}_{t \in T}$ is FOSD-ordered.
Decisions Under Uncertainty

Theorem 6: Suppose \( \{f(\cdot, s)\}_{s \in S} \) (for \( S \subseteq \mathbb{R} \)) satisfies increasing differences. Then the family \( \{F(\cdot, t)\}_{t \in T} \) given by

\[
F(x, t) = \int_{s \in S} f(x, s) \lambda(s, t) \, ds
\]

also satisfies increasing differences if \( \{\lambda(\cdot, t)\}_{t \in T} \) is FOSD-ordered.

Proof: Since \( \{f(\cdot, s)\}_{s \in S} \) satisfies increasing differences, for \( x'' > x' \), the function \( \Delta(s) = f(x'', s) - f(x', s) \) is increasing. By a standard result (Rothschild-Stiglitz),

\[
\int_{s \in S} \Delta(s) f(s, t_2) \, ds \geq \int_{s \in S} \Delta(s) f(s, t_1) \, ds \quad \text{if} \ t_2 > t_1
\]

Re-writing this expression, we obtain

\[
V(x'', t_2) - V(x', t_2) \geq V(x'', t_1) - V(x', t_1),
\]

as required by increasing differences. QED
Decisions Under Uncertainty

Application: The family \( \{ f(\cdot, s) \}_{s \in S} \) given by

\[
    f(x, s) = v((w - x)r + xs)
\]

is an SCP family. By Theorem 5, the family \( \{ F(\cdot, t) \}_{t \in T} \) given by

\[
    F(x, t) = \int v((w - x)r + xs) \lambda(s, t) \, ds
\]

is an SCP family provided \( \{ \lambda(\cdot, t) \}_{t \in T} \) is MLR-ordered.

By Theorem 1, \( \arg\max_{x \in [0, w]} F(x, t) \) increases with \( t \).

Interpretation: an MLR shift in the payoff distribution \( \lambda(\cdot, t) \) means that higher payoffs for the risky asset are more likely, so investment in the risky asset increases.
Short Course on
Comparative Statics and Comparative Information

Part II

John Quah

john.quah@economics.ox.ac.uk
One-dimensional comparative statics

Recall....

Definition: Let $S'$ and $S''$ be subsets of $R$. $S''$ dominates $S'$ in the strong set order ($S'' \geq S'$) if for any for $x''$ in $S''$ and $x'$ in $S'$, we have $\max\{x'', x'\}$ in $S''$ and $\min\{x'', x'\}$ in $S'$.

Example: $\{3, 5, 6, 7\} \not\geq \{1, 4, 6\}$ but $\{3, 4, 5, 6, 7\} \geq \{1, 3, 4, 5, 6\}$.

Note: if $S'' = \{x''\}$ and $S' = \{x'\}$, then $x'' \geq x'$.
One-dimensional comparative statics

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Definition: $g$ dominates $f$ by the single crossing property ($g \succeq_{sc} f$) if for all $x''$ and $x'$ such that $x'' > x'$, the following holds:

$$f(x'') - f(x') \geq (>) 0 \iff g(x'') - g(x') \geq (>) 0.$$  \hspace{1cm} (2)
One-dimensional comparative statics

Recall....

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Definition: $g$ dominates $f$ by the single crossing property ($g \succeq_{sc} f$) if for all $x''$ and $x'$ such that $x'' > x'$, the following holds:

$$f(x'') - f(x') \geq (> 0) \iff g(x'') - g(x') \geq (> 0). \quad (3)$$

Theorem 1: (MS-94) Suppose $X \subseteq R$ and $f, g : X \rightarrow R$; then $\arg\max_{x \in Y} g(x) \geq \arg\max_{x \in Y} f(x)$ for any $Y \subseteq X$ if and only if $g \succeq_{sc} f$. 

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Multidimensional comparative statics

Definitions:

We endow $\mathbb{R}^l$ with the product order: $x' \geq x$ if $x'_i \geq x_i$ for all $i$.

An element $y$ in $\mathbb{R}^l$ is an upper bound of $S \subset \mathbb{R}^l$ if $y \geq x$ for all $x$ in $S$.

It is the least upper bound (or supremum) if it is an upper bound and for any other other upper bound $y'$, we have $y' \geq y$. 
Multidimensional comparative statics

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An element $z$ in $\mathbb{R}^l$ is a lower bound of $S \subset \mathbb{R}^l$ if $z \leq x$ for all $x$ in $S$. It is the greatest lower bound (or infimum) if it is a lower bound and for any other lower bound $z'$, we have $z' \leq z$. 
Multidimensional comparative statics

Definitions:

We endow $R^l$ with the **product order**: $x' \geq x$ if $x'_i \geq x_i$ for all $i$.

An element $y$ in $R^l$ is an **upper bound** of $S \subset R^l$ if $y \geq x$ for all $x$ in $S$.

It is the least upper bound (or **supremum**) if it is an upper bound and for any other other upper bound $y'$, we have $y' \geq y$.

An element $z$ in $R^l$ is a **lower bound** of $S \subset R^l$ if $z \leq x$ for all $x$ in $S$. It is the greatest lower bound (or **infimum**) if it is a lower bound and for any other other lower bound $z'$, we have $z' \leq z$.

Note:

(1) $\sup S$ is *unique* if it exists. (2) $\sup S$ need not be in $S$. 
Multidimensional comparative statics

We denote the supremum of $x'$ and $x''$ by $x' \lor x''$ and their infimum by $x' \land x''$. Easy to check that

$$x' \lor x'' = (\max\{x'_1, x''_1, \max\{x'_2, x''_2\}, \ldots, \max\{x'_l, x''_l\}\}) \text{ and }$$

$$x' \land x'' = (\min\{x'_1, x''_1, \min\{x'_2, x''_2\}, \ldots, \min\{x'_l, x''_l\}\}).$$

A set $S$ is a sublattice of $R^l$ if for any $x'$ and $x''$ in $S$, $x' \lor x''$ and $x' \land x''$ are both in $S$. 

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Multidimensional comparative statics

More definitions:

Let $S'$ and $S''$ be subsets of $\mathbb{R}^l$. $S''$ dominates $S'$ in the strong set order ($S'' \succeq S'$) if for any for $x''$ in $S''$ and $x'$ in $S'$, we have $x' \lor x''$ in $S''$ and $x' \land x''$ in $S'$. 
Multidimensional comparative statics

More definitions:

Let $S'$ and $S''$ be subsets of $R^l$. $S''$ dominates $S'$ in the strong set order ($S'' \geq S'$) if for any for $x''$ in $S''$ and $x'$ in $S'$, we have $x' \lor x''$ in $S''$ and $x' \land x''$ in $S'$.

Note:

(1) If $S'' = \{x''\}$ and $S' = \{x'\}$, then $S'' \geq S'$ if and only if $x'' \geq x'$.

(2) Suppose $S'$ and $S''$ both contain their suprema; then $S'' \geq S'$ implies that $\sup S'' \geq \sup S'$.

Proof of (2): Let $x''$ and $x'$ be suprema of $S''$ and $S'$ respectively. Then $x'' \lor x'$ is in $S''$. Since $x''$ is $\sup S''$, we have $x'' \geq x'' \lor x'$, which implies $s'' \geq x'$.

QED
Multidimensional comparative statics

Let $X$ be a sublattice of $\mathbb{R}^l$ and $f : X \rightarrow \mathbb{R}$. The function $f$ is supermodular (SPM) if

$$f(x' \lor x'') - f(x'') \geq f(x') - f(x' \land x'').$$
Multidimensional comparative statics

Let $X$ be a sublattice of $R^l$ and $f : X \rightarrow R$. The function $f$ is supermodular (SPM) if

$$f(x' \lor x'') - f(x'') \geq f(x') - f(x' \land x'').$$

The function $f$ is quasisupermodular (QSM) if

$$f(x') - f(x' \land x'') \geq (>)0 \implies f(x' \lor x'') - f(x'') \geq (>)0.$$

Note: If $X \subset R$, then any function $f$ is SPM.
Multidimensional comparative statics

**Theorem 2.1:** Let $X$ be a sublattice of $R^l$ and suppose that $f : X \rightarrow R$ is a QSM function. Then $\text{argmax}_{x \in X} f(x)$ is a sublattice.

**Proof:** Suppose $x'$ and $x''$ are both in $\text{argmax}_{x \in X} f(x)$. Since $X$ is a sublattice, $x' \lor x''$ and $x' \land x''$ are both in $X$. Note that $f(x') \geq f(x' \land x'')$ since $x' \in \text{argmax}_{x \in X} f(x)$. The QSM property guarantees that $f(x' \lor x'') \geq f(x'')$, which implies that $x' \lor x'' \in \text{argmax}_{x \in X} f(x)$. Etc. QED
Multidimensional comparative statics

Definition: \( g \) dominates \( f \) by the **single crossing property** \( (g \succeq_{sc} f) \) if for all \( x'' \) and \( x' \) such that \( x'' > x' \), the following holds:

\[
f(x'') - f(x') \geq (>) 0 \implies g(x'') - g(x') \geq (>) 0. \tag{4}
\]

Definition: \( g \) dominates \( f \) by the **increasing differences** \( (g \succeq_{in} f) \) if for all \( x'' \) and \( x' \) such that \( x'' > x' \), the following holds:

\[
g(x'') - g(x') \geq f(x'') - f(x'). \tag{5}
\]
Multidimensional comparative statics

A family \( \{ f(\cdot, s) \}_{s \in S} \) satisfies SCP if the functions are ordered by SCP, i.e., whenever \( s'' > s' \), we have \( f(\cdot, s'') \succeq_{SC} f(\cdot, s') \). Clearly, \( g \succeq_{IN} f \) implies \( g \succeq_{SC} f \).

Similarly, a family \( \{ f(\cdot, s) \}_{s \in S} \) satisfies increasing differences if the functions are ordered by increasing differences, i.e., whenever \( s'' > s' \), we have \( f(\cdot, s'') \succeq_{IN} f(\cdot, s') \).
Multidimensional comparative statics

A family \( \{ f(\cdot, s) \}_{s \in S} \) satisfies SCP if the functions are ordered by SCP, i.e., whenever \( s'' > s' \), we have \( f(\cdot, s'') \succeq_{SC} f(\cdot, s') \). Clearly, \( g \succeq_{IN} f \) implies \( g \succeq_{SC} f \).

Similarly, a family \( \{ f(\cdot, s) \}_{s \in S} \) satisfies increasing differences if the functions are ordered by increasing differences, i.e., whenever \( s'' > s' \), we have \( f(\cdot, s'') \succeq_{IN} f(\cdot, s') \).

Fact: Let \( S \) be an open subset of \( \mathbb{R}^l \) and \( X \) an open interval. Then a sufficient (and necessary) condition for increasing differences is

\[
\frac{\partial^2 f}{\partial x_i \partial s_j}(x, s) \geq 0
\]

at every point \((x, s)\) and for all \(i\) and \(j\).
Theorem 2.1: (MS-94) Suppose $X \subseteq R$ is a sublattice and $f, g : X \to R$ two QSM functions. Then
\[
\arg\max_{x \in Y} g(x) \geq \arg\max_{x \in Y} f(x)
\]
for every sublattice $Y$ contained in $X$ if and only if $g \preceq_{sc} f$.

Proof: Let $x' \in \arg\max_{x \in Y} f(x)$ and $x'' \in \arg\max_{x \in Y} g(x)$. Then
\[
f(x') \geq f(x' \land x'') .
\]
By QSM of $f$, $f(x' \lor x'') \geq f(x'')$. Since $g \preceq_{sc} f$, we obtain $g(x' \lor x'') \geq g(x'')$, which implies that $x' \lor x''$ is in
\[
\arg\max_{x \in Y} g(x).
\]
Etc. QED
Multidimensional comparative statics

Application:

Let $x$ denote the vector of inputs (drawn from $X = R^l_+$), $p$ the vector of input prices, and $R$ the revenue function mapping input vector $x$ to revenue (in $R$). The firm’s profit is

$$\Pi(x; p) = R(x) - p \cdot x.$$ 

Note that

$$\frac{\partial^2 \Pi}{\partial x_i \partial p_j}(x; p) = -1.$$ 

So there is decreasing differences.

If $R$ is supermodular, then $\Pi$ is supermodular.

Conclusion:

By Theorem 2.1, $\arg\max_{x \in X} \Pi(x; p') \geq \arg\max_{x \in X} \Pi(x; p'')$ if $p' < p''$. 
One-dimensional comparative statics - again!

Recall that $g \succeq_{sc} f$ if, for $x^{**} > x^*$, we have

$$f(x^{**}) - f(x^*) \geq (>) 0 \Rightarrow g(x^{**}) - g(x^*) \geq (>) 0.$$ 

But look at these pictures
One-dimensional comparative statics - again!

Recall that $g \succeq_{sc} f$ if, for $x^{**} > x^*$, we have

$$f(x^{**}) - f(x^*) \geq (>) 0 \Rightarrow g(x^{**}) - g(x^*) \geq (>) 0.$$ 

But look at these pictures

$g \succeq_{sc} f$ in the upper diagram but not in the lower. This is unsatisfactory since in both cases, the optimum has increased.
The interval dominance order

Motivation for the interval dominance order: to develop an ordering for functions that captures both situations.

Let $X \subseteq \mathbb{R}$ and $f, g : X \to \mathbb{R}$.

Recall: $g \succeq_{sc} f$ if, for any pair $x^{**} > x^*$, we have

$$f(x^{**}) - f(x^*) \geq (>) 0 \Rightarrow g(x^{**}) - g(x^*) \geq (>) 0.$$
The interval dominance order

Motivation for the interval dominance order: to develop an ordering for functions that captures both situations.

Let $X \subseteq R$ and $f, g : X \rightarrow R$.

Recall: $g \succeq_{sc} f$ if, for any pair $x^{**} > x^*$, we have

$$f(x^{**}) - f(x^*) \geq (>) 0 \Rightarrow g(x^{**}) - g(x^*) \geq (>) 0.$$

$g$ dominates $f$ by the interval dominance order ($g \succeq_{I} f$) if for any pair $x^{**} > x^*$ satisfying
The interval dominance order

Motivation for the interval dominance order: to develop an ordering for functions that captures both situations.

Let \( X \subseteq \mathbb{R} \) and \( f, g : X \rightarrow \mathbb{R} \).

Recall: \( g \succeq_{sc} f \) if, for any pair \( x^{**} > x^* \), we have

\[
f(x^{**}) - f(x^*) \geq (>) 0 \Rightarrow g(x^{**}) - g(x^*) \geq (>) 0.
\]

\( g \) dominates \( f \) by the interval dominance order \( (g \succeq_I f) \) if for any pair \( x^{**} > x^* \) satisfying \( f(x^{**}) \geq f(x) \) for \( x \) in the interval \( [x^*, x^{**}] = \{ x \in X : x^* \leq x \leq x^{**} \} \), the following holds:

\[
f(x^{**}) - f(x^*) \geq (>) 0 \Rightarrow g(x^{**}) - g(x^*) \geq (>) 0.
\]
The interval dominance order

\[ g \succeq f \text{ if for any pair } x^{**} > x^* \text{ satisfying } f(x^{**}) \geq f(x) \text{ for } x \text{ in the interval } [x^*, x^{**}] = \{ x \in X : x^* \leq x \leq x^{**} \}, \text{ the following holds: } \]

\[ f(x^{**}) - f(x^*) \geq (>) 0 \implies g(x^{**}) - g(x^*) \geq (>) 0. \]  

\( (\star) \)

Notice that \((\star)\) need not be applied to \(x^{**}\) and \(x^*\).
The interval dominance order

A subset $I$ of $X$ is called an interval of $X$ if for any $x'$ and $x''$ in $I$ and $x$ in $X$ such that $x' < x < x''$, we have $x$ in $I$.

Example: If $X = \{1, 2, 3, 5, 6\}$, then $\{1, 2\}$ and $\{3, 5, 6\}$ are intervals of $X$ but $\{3, 6\}$ is not an interval of $X$. 
The interval dominance order

A subset $I$ of $X$ is called an interval of $X$ if for any $x'$ and $x''$ in $I$ and $x$ in $X$ such that $x' < x < x''$, we have $x$ in $I$.

Example: If $X = \{1, 2, 3, 5, 6\}$, then $\{1, 2\}$ and $\{3, 5, 6\}$ are intervals of $X$ but $\{3, 6\}$ is not an interval of $X$.

**Theorem 2.2:** Suppose that $f$ and $g$ are real-valued functions defined on $X \subset \mathbb{R}$. Then $g \succeq_I f$ if and only if
\[
\arg\max_{x \in I} g(x) \geq \arg\max_{x \in I} f(x)
\]
for any interval $I$ of $X$.

Compare this with
The interval dominance order

A subset $I$ of $X$ is called an interval of $X$ if for any $x'$ and $x''$ in $I$ and $x$ in $X$ such that $x' < x < x''$, we have $x$ in $I$.

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**Theorem 2.2:** Suppose that $f$ and $g$ are real-valued functions defined on $X \subset R$. Then $g \succeq_I f$ if and only if

$$\arg \max_{x \in I} g(x) \geq \arg \max_{x \in I} f(x)$$

for any interval $I$ of $X$.

Compare this with

**Theorem:** Suppose that $f$ and $g$ are real-valued functions defined on $X \subset R$. Then $g \succeq_{sc} f$ if and only if

$$\arg \max_{x \in S} g(x) \geq \arg \max_{x \in S} f(x)$$

for any subset $S$ of $X$. 
IDO with Uncertainty

Basic example of IDO family: for every $s$, $u(\cdot, s)$ is quasiconcave (in $x$) with $\arg\max u(x, s'') \geq \arg\max u(x, s')$ for $s'' > s'$.

This family of quasiconcave functions with increasing peaks *need not* be an SCP family.
At each state $s$ there is an optimal action, which increases with $s$. 
IDO with Uncertainty

Basic example of IDO family: for every \( s \), \( u(\cdot, s) \) is quasiconcave (in \( x \)) with argmax \( u(x, s'') \geq \text{argmax} u(x, s') \) for \( s'' > s' \).

This family of quasiconcave functions with increasing peaks need not be an SCP family. At each state \( s \) there is an optimal action, which increases with \( s \). Agent chooses \( x \) under uncertainty, i.e., before \( s \) is realized. He maximizes \( U(x) = \int_S u(x, s)\lambda(s)ds \), where \( \lambda : S \to R \) is the density function.
IDO with Uncertainty

We expect the optimal choice of $x$ to increase when higher states are more likely.

Recall: a family of density functions $\{\lambda(\cdot, t)\}_{t \in T}$ (defined on $S \subseteq R$) respects the monotone likelihood ratio order (MLR) if for $t'' > t'$,

$$\frac{\lambda(s, t'')}{\lambda(s, t')}$$

increases with $s$. 
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$$\frac{\lambda(s, t'')}{\lambda(s, t')} \text{ increases with } s.$$

**Theorem 2.2:** (QS-07) Suppose $\{u(\cdot, s)\}_{s \in S}$ forms an IDO family and $\{\lambda(\cdot, t)\}_{t \in T}$ is ordered by MLR. Then $\{U(\cdot, t)\}_{t \in T}$, defined by

$$U(x; t) = \int_S u(x, s)\lambda(s, t)ds$$

forms an an IDO family. Consequently, $\arg\max_{x \in X} U(x, t)$ is increasing in $t$. 
IDO with Uncertainty

Example: a Firm decides on the time $x$ to launch a new product. The more time the Firm gives itself, the more it can improve the quality of the product and its manufacturing process. But there is a Rival who may launch a similar product at time $s$. If the Firm is anticipated, it suffers a discrete fall in profit. Graphically, the Firm’s family of profit functions $\{\pi(\cdot, s)\}_{s \in S}$ looks like
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The firm chooses $x$ to maximize $F(x, \lambda) = \int_{s \in S} \pi(x, s) \lambda(s) ds$.

Note that $\{\pi(\cdot, s)\}_{s \in S}$ is an IDO, but not necessarily SCP family.

By the Theorem 2.2, firm's optimal launch time increases if $\lambda$ experiences MLR shift.
Statistical Decision Theory

Agent chooses action after observing a signal $z$; $z$ is drawn from an interval $Z (Z \subset \mathbb{R})$, but before realization of state. Agent’s decision rule is a map from $Z$ to $X \subset \mathbb{R}$. The distribution over signals at a state $s \in S$ is $H(z|s)$ (with density $h(z|s)$).

We refer to the family $\{H(\cdot|s)\}_{s \in S}$ as the information structure $H$.

Expected utility of rule $\phi$ in state $s$ is $\int_{Z} u(\phi(z), s) dH(z|s)$. 

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How should a decision maker choose amongst decision rules?

The Bayesian decision maker will have a prior $P$ on the states of the world $S$: the value to him of decision rule $\phi$ is

$$\int_S \left[ \int_Z u(\phi(z), s)dH(z|s) \right] dP(s).$$
Theorem 2.3: (QS-07) Suppose $\{u(\cdot, s)\}_{s \in S}$ is an IDO family and $\{h(\cdot|s)\}_{s \in S}$ is MLR-ordered. Then there exists an increasing decision rule that maximizes the Bayesian’s ex ante utility.
Statistical Decision Theory

Theorem 2.3: (QS-07) Suppose \( \{u(\cdot, s)\}_{s \in S} \) is an IDO family and \( \{h(\cdot|s)\}_{s \in S} \) is MLR-ordered. Then there exists an increasing decision rule that maximizes the Bayesian’s ex ante utility.

Proof: We denote that posterior distribution (over states) conditional on \( z \) by \( \tilde{H}_P(\cdot|z) \). The ex ante utility of rule \( \phi \) is

\[
\int_S \int_Z u(\phi(z), s) dH(z|s) dP(s) = \int_{z \in Z} \left[ \int_{s \in S} u(\phi(z), s) d\tilde{H}_P(s|z) \right] d\nu_H
\]

where \( \nu_H \) is the marginal distribution of \( z \). It follows that an optimal decision rule can be found by choosing, for each signal \( z \), an action that maximizes \( \arg\max_{x \in X} \int_{s \in S} u(x, s) d\tilde{H}_P(s, z) \).
Statistical Decision Theory

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where $\nu_H$ is the marginal distribution of $z$. It follows that an optimal decision rule can be found by choosing, for each signal $z$, an action that maximizes $\arg\max_{x \in X} \int_{s \in S} u(x, s) \tilde{H}_P(s, z)$. We know that $\{\tilde{h}_P(z|z)\}_{z \in Z}$ is MLR-ordered because $\{h(s|z)\}_{s \in S}$ is MLR-ordered. By Theorem 2.2, for $z'' > z'$, we have

$$\arg\max_{x \in X} \int_{s \in S} u(x, s) \tilde{h}_P(s, z'') ds \geq \arg\max_{x \in X} \int_{s \in S} u(x, s) \tilde{h}_P(s, z') ds.$$

For each $z$, choose $\phi^*(z) = \max \left[ \arg\max_{x \in X} \int_{s \in S} u(x, s) \tilde{h}_P(s, z) ds \right]$. Then $\phi^*$ is optimal and increasing in $z$. QED
Lehmann informativeness

Assume prior is $P$ and consider information structure $H$. If $\phi_H$ is the optimal decision rule, then agent’s ex ante utility is

$$\mathcal{V}(H, P) = \int_S \int_Z u(\phi_H(z), s) dH(z|s) dP(s)$$

$$= \int_{z \in Z} \left[ \int_{s \in S} u(\phi_H(z), s) d\tilde{H}_P(s|z) \right] d\nu_H$$

where $\nu_H$ is the marginal distribution of $z$.

We wish to compare $H$ with another information structure $G = \{G(\cdot|s)\}_{s \in S}$. The ex ante utility of $G$ is

$$\mathcal{V}(G, P) = \int_S \int_Z u(\phi_G(z), s) dG(z|s) dP(s)$$

where $\phi_G$ is the optimal decision rule for $G$. 
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When is $V(H, P) \geq V(G, P)$ for all $P$?
Lehmann informativeness

Given information structures $H \equiv \{H(\cdot|s)\}_{s \in S}$ and $G \equiv \{G(\cdot|s)\}_{s \in S}$, define $T(z,s)$ by $H(T(z,s)|s) = G(z|s)$.

Definition: $H$ is more informative than $G$ (in the sense of Lehmann) if $T(z, \cdot)$ is increasing in $s$. 

Lehmann informativeness

Theorem 2.4: (QS-07) Suppose that
(i) information structure $H$ is more informative than $G$
(ii) $\{u(\cdot, s)\}_{s \in S}$ is an IDO family
(iii) $G = \{g(\cdot|s)\}_{s \in S}$ is MLR-ordered.
Theorem 2.4: (QS-07) Suppose that
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Then $\mathcal{V}(H, P) \geq \mathcal{V}(G, P)$. 

Lehmann informativeness
Other decision criteria...

The risk of a decision rule $\phi : Z \rightarrow X$ is given by

$$\left[ \int_Z u(\phi(z), s) dH(z|s) \right]_{s \in S}. $$

The Bayesian (with prior $P$) chooses the rule that maximizes

$$\int_S \left[ \int_Z u(\phi(z), s) dH(z|s) \right] dP(s).$$

The minimax criterion chooses the rule that maximizes

$$\min_{s \in S} \int_Z u(\phi(z), s) dH(z|s).$$
Complete Class Theorem

Definition: A rule $\phi$ is better than $\psi$ if

$$\left[ \int_{z \in Z} u(\phi(z), s) h(z \mid s) \, dz \right]_{s \in S} \geq \left[ \int_{z \in Z} u(\psi(z), s) h(z \mid s) \, dz \right]_{s \in S}$$
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Geometrically…. when $S$ is finite, we can think of

$$\left[ \int_{Z} u(\phi(z), s) dH(z | s) \right]_{s \in S}$$

as a point in $R^{|S|}$. Each rule is associated with a point in $R^{|S|}$.

![Diagram showing the comparison of two functions with respect to a measure $H$ over the set $S$.]
Complete Class Theorem

Definition: A subset of decision rules $C$ form an essentially complete class if for every rule $\psi$ there is a rule $\phi$ in $C$ such that $\phi$ is better than $\psi$. 
Complete Class Theorem

Definition: A subset of decision rules \( C \) form an **essentially complete class** if for every rule \( \psi \) there is a rule \( \phi \) in \( C \) such that \( \phi \) is better than \( \psi \).

**Theorem 2.5:** (QS-07) Let \( H \) be an MLR-ordered information structure and \( \{ u(\cdot, s) \}_{s \in S} \) an IDO family. Then the increasing decision rules form an essentially complete class.
Complete Class Theorem

Definition: A subset of decision rules $C$ form an essentially complete class if for every rule $\psi$ there is a rule $\phi$ in $C$ such that $\phi$ is better than $\psi$.

Theorem 2.5: (QS-07) Let $H$ be an MLR-ordered information structure and $\{u(\cdot, s)\}_{s \in S}$ an IDO family. Then the increasing decision rules form an essentially complete class.

Notice that Theorem 2.5 implies Theorem 2.3: let $\psi$ be any rule maximizing ex ante utility for a Bayesian; by Theorem 2.5 there is an increasing rule $\phi$ that is better than $\psi$, so $\phi$ must also maximize the Bayesian’s ex ante utility.
Complete Class Theorem

Application (adapted from Manski):

Medical Treatment A is the status quo with known recovery probability of $\bar{p}^A$.

Treatment B is the new treatment with unknown recovery probability of $p^B$, taking values in set $S$. 
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Fact: The distribution of $z$ given $p^B$ is binomial and

$$\{h(\cdot|p^B)\}_{p^B \in S}$$

is MLR-ordered.
Statistical Decision Theory

Assume that cost of treating fraction $x$ of the population with B (and the rest with A) is $C(x)$.

Normalize utility of cure at 1 and that of no cure at 0. Planner’s utility if fraction $x$ of the population receives B (and the rest A) is

$$u(x, p^B) = (1 - x) \bar{p}^A + xp^B - C(x).$$

Notice that $\{u(\cdot, p^B)\}_{p^B \in S}$ is an IDO family.
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(7)

Notice that $\{u(\cdot, p^B)\}_{p^B \in S}$ is an IDO family.

Conclusion:

By Theorem 2.5, planner can confine herself to rules where $x$ increases with $z$. 