Strategy-proofness and Efficiency with Non-quasi-linear Preferences: A Characterization of Minimum Price Walrasian Rule*

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Abstract

We consider the problems of allocating objects to a group of agents and how much agents should pay. Each agent receives at most one object and has non-quasi-linear preferences. Non-quasi-linear preferences describe environments where payments influence agents' abilities to utilize objects or derive benefits from them. The “minimum price Walrasian (MPW) rule” is the rule that assigns a minimum price Walrasian equilibrium allocation to each preference profile. We establish that the MPW rule is the unique rule satisfying strategy-proofness, efficiency, individual rationality, and no subsidy for losers. Since the outcome of the MPW rule coincides with that of the simultaneous ascending (SA) auction, our result supports SA auctions adopted by many governments.

Keywords: minimum price Walrasian equilibrium, simultaneous ascending auction, strategy-proofness, efficiency, heterogeneous objects, non-quasi-linear preferences

JEL Classification Numbers: D44, D71, D61, D82

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1 Introduction

1.1 Purpose

Since the 1990s, governments in numerous countries have conducted auctions to allocate a variety of objects or assets including spectrum rights, vehicle ownership licenses, and land, etc. Although auctions sometimes make a large amount of government revenue, the announced goals of many government auctions are rather to allocate objects “efficiently”, i.e., to agents who benefit most from them.\(^1\) Agents benefiting more are willing to pay higher prices for them, and thus have a better chance to win the auctions. However, as we mentioned below, large-scale auction payments would influence agents’ abilities to utilize objects or benefit from them, thereby complicating efficient allocations. This article analyzes rules that allocate auctioned objects efficiently even when payments are so large that it impairs agents’ abilities to utilize them or realize their benefits. We investigate what types of allocation rules can allocate objects efficiently in such environments.

1.2 Main Result

An allocation rule, or simply a rule, is a function that assigns to each preference profile an allocation, which consists of an assignment of objects and agents’ payments. Each agent receives one object at most. The domain of rules is the class of preference profiles. We assume that preferences satisfy monotonicity, continuity, and “finite compensation”, which means that, given an assignment, any change of assigned object is compensated by a finite amount of money. We call such preferences “classical”. It is well-known that in this model, there is a minimum price Walrasian equilibrium (MPWE),\(^2\) and that the allocation associated with the MPWE coincides with the outcome of a certain type of auctions, called the “simultaneous ascending (SA) auction”.\(^3\) Under SA auctions, bids on all objects start simultaneously, and the sale of any object is not settled as long as new bids are made on some objects. We focus on the rule that assigns an MPWE allocation to each preference profile. We refer to this rule as the “minimum price Walrasian (MPW) rule”.

The MPW rule satisfies four desirable properties. First is (Pareto-)efficiency. An allocation is efficient if no agent can be made better off without either some other agent being made worse off or the government’s revenue being reduced.\(^4\) Note that efficiency is evaluated based on agents’ preferences. Thus, an efficient allocation cannot be chosen without information about preferences. Since preferences are private information, agents may have an incentive to behave strategically to influence the final outcome in their favor. Strategy-proofness is an incentive-compatibility property, which gives a strong incentive for each agent to reveal his true preferences. It says that for each preference profile, in the normal form game induced by the rule, it is a (weakly) dominant strategy for each agent to reveal his true preference. The

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\(^1\)For example, frequency auctions in the United States were introduced to promote “efficient and intensive use of the electromagnetic spectrum”. See McAfee and McMillan (1996, p.160).

\(^2\)See Demange and Gale (1985).

\(^3\)For example, see Demange, Gale, and Sotomayor (1986).

\(^4\)In our auction model, efficiency is defined by taking government revenue into account.
MPW rule satisfies strategy-proofness, and chooses an efficient allocation corresponding to the revealed preferences.

The third property of the MPW rule is individual rationality, which requires that no agent should be made worse off than if he had received no object and paid nothing. This property induces voluntary participation. Fourth is no subsidy for losers. Under the MPW rule, the governments never subsidize losers. This property prevents agents who do not necessarily need objects from flocking to auctions to sponge subsidies.

The primary conclusion of this article is that only the minimum price Walrasian rule satisfies strategy-proofness, efficiency, individual rationality, and no subsidy for losers (Theorem 4.1). Since the outcome of the MPW rule coincides with that of the SA auction (Proposition 5.1), the result supports SA auctions adopted by many governments.

1.3 Related Literature

Holmström (1979) establishes a fundamental result relating to our question that applies when agents’ benefits from auctioned objects are not influenced by their payments, i.e., agents have “quasi-linear” preferences. He assumes that preferences are quasi-linear, and shows that only the Vickrey–Clarke–Groves type (VCG) allocation rules satisfy strategy-proofness and efficiency. His result implies that on the quasi-linear domain, only the Vickrey rule satisfies strategy-proofness, efficiency, individual rationality, and no subsidy for losers. As Marshall (1920) demonstrates, preferences are approximately quasi-linear if payments for goods we analyze are sufficiently low. However, quasi-linearity is not an appropriate assumption for large-scale auctions. Excessive payment for the auctioned objects may damage bidders’ budgets to purchase complements for effective use of the objects, and thus may influence the benefit from the objects. Or bidders may need to obtain loans to bid high amounts, and typically financial costs are nonlinear in borrowings, which makes bidders’ preferences on objects and payments non-quasi-linear. In spectrum license auctions and vehicle ownership license auctions, license prices often equal or exceed bidders’ annual revenues. Thus, bidders’ preferences are non-quasi-linear for such important auctions. As contrasted with Holmström (1979), our result applies to such environments.

Saitoh and Serizawa (2008) investigate a problem similar to ours in the case where the domain includes non-quasi-linear preferences, and there are multiple copies of the same object.
They generalize Vickrey rules by employing compensating valuations from no object and no payment, and characterize the generalized Vickrey rule by strategy-proofness, efficiency, individual rationality, and no subsidy.\footnote{Sakai (2008) also obtains a result similar to theirs.} We stress that when preferences are not quasi-linear, the heterogeneity of objects makes the MPW rule different from the generalized Vickrey rule.\footnote{In Section 6, we give a detailed discussion on this point by contrasting the MPW rule with the generalized Vickrey rule.}

Although the assumption of quasi-linearity neglects the serious effects of large-scale auction payments in actual practice, it is difficult to investigate the above question without this assumption. Quasi-linearity simplifies the description of efficient allocations. More precisely, under quasi-linear preferences, an efficient allocation of objects can be achieved simply by maximizing the sum of realized benefits from objects (agents’ net benefits), and hence is independent of how much agents pay. In this sense, Holmström (1979) characterizes only the payment part of strategy-proof and efficient rules. On the other hand, without quasi-linearity, efficient allocations of objects do depend on payments, and thus cannot be simply identified in the same way as in the quasi-linear case. Furthermore, as mentioned earlier, on non-quasi-linear domains, the MPW rule is different from the generalized Vickrey rule, and the former outperforms the latter in terms of our desirable properties, i.e., strategy-proofness and efficiency are satisfied by the MPW rule, but not by the generalized Vickrey rule. Needless to say, Holmström’s (1979) results cannot be applied to prove our results on a non-quasi-linear domain. It is worthwhile to mention that most standard results of auction theory, such as the Revenue Equivalence Theorem, also depend on assuming quasi-linearity. In this article, we overcome that difficulty.

Since Hurwicz’s (1972) seminal work, many authors have investigated efficient and strategy-proof rules in pure exchange economies.\footnote{For example, see Zhou (1991), Barberà and Jackson (1995), Schummer (1997), Serizawa (2002), and Serizawa and Weymark (2003).} In pure exchange economies, classical\footnote{In pure exchange economies, where consumption spaces are some multidimensional Euclidean space, classical preferences are assumed to satisfy convexity in addition to continuity and monotonicity. Clearly, the class of such preferences contains non-quasi-linear preferences.} preferences are standard, but no rule is strategy-proof, efficient, and individually rational on the classical domain. On the other hand, Demange and Gale (1985) show that, in the model studied in this article, the MPW rule is strategy-proof, efficient, and individually rational on the classical domain.\footnote{More precisely, Demange and Gale (1985) study two-sided matching markets which contain our model as a special case, and show the rules selecting an optimal stable assignment for one side of the market are group strategy-proof for the agents on that side.} Generalizing the MPW rule to the situations where price ranges are bounded, Andersson and Svensson (2012) introduce the “minimum weak price equilibrium rule”, and demonstrate that it satisfies strategy-proofness and a weak variant of efficiency. Miyake (1998) shows that only the MPW rule satisfies strategy-proofness among “Walrasian rules”.\footnote{A “Walrasian rule” is the rule that assigns a Walrasian equilibrium allocation to each preference profile.} Note that the Walrasian rules are a small part of the class of allocation rules satisfying efficiency, individual rationality, and no subsidy for losers. By developing analytical tools different from

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\end{equation}
we extend his characterization in that we establish the uniqueness of the rules satisfying the desirable properties without confinement to Walrasian rules.

Many authors have analyzed SA auctions in quasi-linear settings (Ausubel, 2004, 2006; Ausubel and Milgrom, 2002; de Vries, Schummer, and Vohra, 2007; Gul and Stacchetti, 2000; and Mishra and Parkes, 2007; Andersson, Andersson and Talman, 2012, etc). In non-quasi-linear settings, the MPW rules differ from the generalized Vickrey rules, and it is the MPWE allocation that coincides with the outcome of the SA auction. Alaei, Jain, and Malekian (2013) construct an alternative algorithm computing MPWE in non-quasi-linear setting. Therefore, our result demonstrates that the SA auction and alternative algorithms analyzed by those authors are more important in non-quasi-linear settings.

The problems of allocating objects and money have been studied by many authors. One of the extensively studied problems not referenced above is the one of fair (no-envy) allocation (Svensson, 1983; Maskin, 1987; Alkan, Demange, and Gale, 1991; Tadenuma and Thomson, 1991). In the context of strategy-proofness, fair allocation rules are investigated by Tadenuma and Thomson (1995), Sun and Yang (2003), Ohseto (2006), and Svensson (2004, 2009). Svensson (2004) characterizes the class of strategy-proof and envy-free rules under some auxiliary conditions. In our models, no-envy implies an important feature of Walrasian equilibrium allocation. Given an allocation, take the price vector such that the price of each object is the payment of the agent who receives it. No-envy implies that for this price vector, each agent demands the object he receives in the given allocation. Since we do not impose no-envy on rules, we cannot use this important feature in our proof. On the other hand, Svensson (2004, 2009) does not impose individual rationality and no subsidy for losers. Thus, his results are logically independent from ours.

Other authors have investigated the existence of strategy-proof and nonbossy rules. Miyagawa (2001) characterizes the class of strategy-proof, nonbossy, individually rational, and onto rules. Svensson (2002) characterizes classes of strategy-proof and nonbossy rules that satisfy several additional desirable properties. It is well known that nonbossiness together with strategy-proofness makes the analysis tractable. Since the MPW rules violate nonbossiness, we do not impose this demanding property, and thus cannot apply their proof techniques in our proof.

1.4 Organization

The article is organized as follows. Section 2 sets up the model and introduces basic concepts. Section 3 defines the MPWE and discusses its properties. Section 4 provides our main

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\[\text{\[In Appendix B, we discuss why different analytical tools are necessary.\]}\]

\[\text{\[No-envy (Foley, 1967) is the requirement that no agent should prefer anyone else’s assignment to his own.\]}\]

\[\text{\[Some authors also investigate the problem by other fairness axioms. See, for example, Ashlagi and Serizawa (2012) and Mukherjee (2013) for the axiom of anonymity in welfare, and Sakai (2013) and Adachi (2013) for the axiom of weak envy-freeness for equals.\]}\]

\[\text{\[Svensson (2009) also characterizes the class of group strategy-proof and envy-free rules that satisfy a condition on payment rule, which he call regularity.\]}\]

\[\text{\[Nonbossiness (Satterthwaite and Sonnenschein, 1981) is the requirement that when an agent’s preferences change, if his assignment remains the same, then the chosen allocation should remain the same.\]}\]

\[\text{\[See also Schummer (2000) for the other analysis of strategy-proof and nonbossy rules.\]}\]
Each agent consumes one object at most. We denote the object that agent \( i \) consumes by \( o_i \). Each agent has a complete and transitive preference relation, denoted by \( R_i \). Let \( P_i \) and \( I_i \) be the strict and indifference relation associated with \( R_i \), respectively. Given a preference \( R_i \) and a bundle \( z_i \), let the upper contour set and lower contour set of \( R_i \) at \( z_i \) be \( UC(R_i, z_i) = \{ z' \in \mathbb{R} : z' \geq R_i z_i \} \) and \( LC(R_i, z_i) = \{ z' \in \mathbb{R} : z_i \geq R_i z' \} \), respectively. For each \( i \in N \), agent \( i \)'s consumption set is \( L \times \mathbb{R} \), and a (consumption) bundle for \( i \) is a pair \( z_i \equiv (x_i, t_i) \in L \times \mathbb{R} \). Let \( 0 \equiv (0,0) \).

Each agent \( i \) has a complete and transitive preference relation \( R_i \) on \( L \times \mathbb{R} \). Let \( P_i \) and \( I_i \) be the strict and indifference relation associated with \( R_i \), respectively. Given a preference \( R_i \) and a bundle \( z_i \), let the upper contour set and lower contour set of \( R_i \) at \( z_i \) be \( UC(R_i, z_i) = \{ z' \in \mathbb{R} : z' \geq R_i z_i \} \) and \( LC(R_i, z_i) = \{ z' \in \mathbb{R} : z_i \geq R_i z' \} \), respectively. For each \( i \in N \), agent \( i \)'s preference \( R_i \) satisfies the following properties:

**Money monotonicity:** For each \( x_i \in L \) and each \( t_i, t'_i \in \mathbb{R} \), if \( t'_i < t_i \), then \( (x_i, t'_i) \in R_i (x_i, t_i) \).

**Finiteness:** For each \( t_i \in \mathbb{R} \) and each \( x_i, x'_i \in L \), there exist \( t'_i, t''_i \in \mathbb{R} \) such that \( (x'_i, t'_i) \in R_i (x_i, t_i) \) and \( (x_i, t_i) \in R_i (x'_i, t''_i) \).

**Continuity:** For each \( z_i \in L \times \mathbb{R} \), \( UC(R_i, z_i) \) and \( LC(R_i, z_i) \) both are closed.\( ^{25} \)

Let \( \mathcal{R}^E \) denote the class of continuous, money monotonic, and finite preferences, the extended domain. Given \( R_i \in \mathcal{R}^E \), \( z_i \equiv (x_i, t_i) \in L \times \mathbb{R} \), and \( y_i \in L \), we define the compensating valuation \( cv_i(y_i; z_i) \) of \( y_i \) from \( z_i \) for \( R_i \) by \( (y_i, t_i + cv_i(y_i; z_i)) \in I_i z_i \), and let \( CV_i(y_i; z_i) \equiv t_i + cv_i(y_i; z_i) \). We refer to \( CV_i(y_i; z_i) \) as the compensated valuation of \( y_i \) from \( z_i \) for \( R_i \). Note that by continuity and finiteness, \( CV_i(y_i; z_i) \) exists, and by money monotonicity, \( CV_i(y_i; z_i) \) is unique. The compensated valuation for \( R_i \) is denoted by \( CV_i' \).

We introduce another property of preferences.

**Desirability of objects:** For each \( x_i \in M \) and each \( t_i \in \mathbb{R} \), \( (x_i, t_i) \in R_i (0, t_i) \).\( ^{26} \)

**Definition 2.1.** A preference \( R_i \) is classical if it satisfies continuity, money monotonicity, finiteness, and desirability of objects.

Let \( \mathcal{R}^C \) denote the class of classical preferences, the classical domain. Note that \( \mathcal{R}^C \subset \mathcal{R}^E \).

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\( ^{25} \)Money monotonicity and finiteness imply continuity. All of results hold without assuming continuity.

\( ^{26} \)A preference \( R_i \) satisfies weak desirability of objects if for each \( x_i \in M \), \( (x_i, 0) \in R_i 0 \). All the results in this article still hold if desirability of objects is replaced by weak desirability of objects.
Definition 2.2. A preference \( R_i \) is quasi-linear if there is a “valuation function” \( v_i : L \to \mathbb{R}_+ \) such that (i) \( v_i(0) = 0 \), (ii) for each \( x \in M \), \( v_i(x) > 0 \), and (iii) for each \( z_i = (x_i, t_i) \in L \times \mathbb{R} \), and each \( z_i' \equiv (x_i', t_i') \in L \times \mathbb{R} \), \( z_i \not\succ z_i' \) if and only if \( v_i(x_i) - t_i \geq v_i(x_i') - t_i' \).

Let \( R^Q \) denote the class of quasi-linear preferences, the quasi-linear domain. Note that \( R^Q \subset R^C \).

An object allocation is an \( n \)-tuple \((x_1, \ldots, x_n) \in L^n \) such that for each \( i, j \in N \), if \( x_i \neq 0 \) and \( i \neq j \), then \( x_i \neq x_j \), that is, no two agents receive the same object except when both receive the null object. Let \( X \) be the set of object allocations. A (feasible) allocation is an \( n \)-tuple \( z \equiv (z_1, \ldots, z_n) \equiv ((x_1, t_1), \ldots, (x_n, t_n)) \in [L \times \mathbb{R}]^n \) of bundles such that \((x_1, \ldots, x_n) \in X \). Let \( Z \) be the set of feasible allocations. We denote the object allocation and agents’ payments at \( z' \in Z \) by \( x' \equiv (x_1', \ldots, x_n') \) and \( t' \equiv (t_1', \ldots, t_n') \), respectively.

Let \( R \) be a class of preferences such that \( R \subseteq R^E \). A preference profile is an \( n \)-tuple \( R \equiv (R_1, \ldots, R_n) \in R^n \). Given \( R \equiv (R_1, \ldots, R_n) \in R^n \) and \( N' \subseteq N \), let \( R_{N'} \equiv (R_i)_{i \in N'} \) and \( R_{-N'} \equiv (R_i)_{i \in N \setminus N'} \).

An allocation rule, or simply a rule, on \( R^n \) is a function \( f \) from \( R^n \) to \( Z \). Given a rule \( f \) and a preference profile \( R \in \mathcal{R}^n \), we denote agent \( i \)'s assigned object under \( f \) at \( R \) by \( f_i^x(R) \) and his payment by \( f_i^t(R) \), and we write

\[
  f_i(R) \equiv (f_i^x(R), f_i^t(R)), \quad f(R) \equiv (f_1(R), \ldots, f_n(R)), \quad \text{and} \quad f^x(R) \equiv (f_j^x(R))_{j \in N}.
\]

We introduce basic properties of rules. The efficiency condition defined below takes the auctioneer’s preference into account and assumes that he is only interested in his revenue. An allocation \( z' \in Z \) (Pareto-)dominates \( z \in Z \) for \( R \in \mathcal{R}^n \) if

\[
  (i) \quad \sum_{i \in N} t_i' \geq \sum_{i \in N} t_i, \quad \text{(ii) for each } i \in N, \ z_i' R_i z_i, \quad \text{and (iii) for some } j \in N, \ z_j' P_j z_j.
\]

An allocation \( z \in Z \) is (Pareto-)efficient for \( R \in \mathcal{R}^n \) if there is no feasible allocation that dominates \( z \).

Efficiency: For each \( R \in \mathcal{R}^n \), \( f(R) \) is efficient for \( R \).

Individual rationality says that a rule should never select an allocation at which some agent is worse off than if he had received the null object and paid nothing. No subsidy says that the payments should always be nonnegative. No subsidy for losers says that the payments of agents who obtain the null object should always be nonnegative. No subsidy implies no subsidy for losers.

Individual rationality: For each \( R \in \mathcal{R}^n \) and each \( i \in N \), \( f_i(R) R_i 0 \).

No subsidy: For each \( R \in \mathcal{R}^n \) and each \( i \in N \), \( f_i^t(R) \geq 0 \).

No subsidy for losers: For each \( R \in \mathcal{R}^n \) and each \( i \in N \), if \( f_i^x(R) = 0 \), then \( f_i^t(R) \geq 0 \).

The two properties below have to do with incentives. First, by misrepresenting his preferences, no agent should obtain an assignment that he prefers.

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Strategy-proofness: For each \( R \in \mathcal{R}^n \), each \( i \in N \), and each \( R'_i \in \mathcal{R} \), \( f_i(R) R_i f_i(R'_i, R_{-i}) \).

The second is stronger: by jointly misrepresenting their preferences, no group of agents should obtain assignments that they prefer.

Group strategy-proofness: For each \( R \in \mathcal{R}^n \) and each \( N' \subseteq N \), there is no \( R'_{N'} \in \mathcal{R}^{|N'|} \) such that for each \( i \in N' \), \( f_i(R'_{N'}, R_{-N'}) P_i f_i(R) \).

3 Minimum Price Walrasian Equilibrium

3.1 Definition of Walrasian equilibria

We define “Walrasian equilibrium” and “minimum price Walrasian equilibrium”. Let \( \mathcal{R} \subseteq \mathcal{R}^E \) in this section. All results in this section also hold on the classical domain \( \mathcal{R}^C \).

Let \( p \equiv (p^1, \ldots, p^m) \in \mathbb{R}^m_+ \) be a price vector. The budget set at prices \( p \) is defined as \( B(p) \equiv \{ (x, p^x) : x \in L \} \), where \( p^x = 0 \) if \( x = 0 \). Given \( i \in N \), \( R_i \in \mathcal{R} \) and \( p \in \mathbb{R}^m_+ \), agent \( i \)'s demand set is defined as \( D(R_i, p) \equiv \{ x : x \in L, (x, p^x) R_i (y, p^y) \} \).

Definition 3.1. Let \( R \in \mathcal{R}^n \). A pair \( ((x, t), p) \in Z \times \mathbb{R}^m_+ \) is a Walrasian equilibrium for \( R \) if

(WE-i) for each \( i \in N \), \( x_i \in D(R_i, p) \) and \( t_i = p^{x_i} \), and
(WE-ii) for each \( y \in M \), if for each \( i \in N \), \( x_i \neq y \), then, \( p^y = 0 \).

Condition (WE-i) says that each agent receives an object he demands, and pays its price. Condition (WE-ii) says that an object’s price is zero if it is not assigned.

Fact 3.1. For each \( R \in \mathcal{R}^n \), there is a Walrasian equilibrium for \( R \).

Fact 3.1 is already known.\(^{28}\) Given \( R \in \mathcal{R}^n \), let \( W(R) \) be the set of Walrasian equilibria for \( R \), and \( Z(R) \) and \( P(R) \) be the sets of Walrasian equilibrium allocations and prices for \( R \), respectively, i.e.,

\[
Z(R) \equiv \{ z \in Z : \text{for some } p \in \mathbb{R}^m_+ \text{, } (z, p) \in W(R) \},
\]

\[
P(R) \equiv \{ p \in \mathbb{R}^m_+ : \text{for some } z \in Z \text{, } (z, p) \in W(R) \}.
\]

Next is a First Welfare Theorem for our model.\(^{29}\)

Fact 3.2. Let \( R \in \mathcal{R}^n \) and \( z \in Z(R) \). Then, \( z \) is efficient for \( R \).\(^{30}\)

\(^{27}\)Let \( |A| \) denote the cardinality of set \( A \).

\(^{28}\)For example, see Alkan and Gale (1990). Our model is a special case of theirs.

\(^{29}\)See also Svensson (1983).

\(^{30}\)To see this, suppose that \( z \equiv (z_1, \ldots, z_n) \) is not efficient for \( R \). Then, there is \( z' \equiv (z'_1, \ldots, z'_n) \) such that

\[
(i) \sum_{i \in N} t'_i \geq \sum_{i \in N} t_i \text{, (ii) for each } i \in N \text{, } z'_i R_i z_i \text{, (iii) for some } j \in N \text{, } z'_j P_j z_j.
\]

Since \( z \in Z(R) \), there is a price vector \( p \in \mathbb{R}^m_+ \) such that \( (z, p) \in W(R) \). Then, by (ii) and (WE-i), for each \( i \in N \), \( t'_i \leq p^{z'_i} \). By (iii) and (WE-i), \( t'_j < p^{z'_j} \). Thus, \( \sum_{i \in N} t'_i < \sum_{i \in N} p^{z'_i} = \sum_{i \in N} t_i \). This contradicts (i).
Figure 1: Illustration of non-quasi-linear preferences and the minimum Walrasian equilibrium

Fact 3.3 says that for each preference profile, there is a unique minimum Walrasian equilibrium price vector. The **minimum price Walrasian equilibrium** (hereafter MPWE) is the Walrasian equilibria associated with the minimum price.

**Fact 3.3** (Demange and Gale, 1985). For each $R \in \mathcal{R}^n$, there is a unique $p' \in P(R)$ such that for each $p \in P(R)$, $p' \leq p$.

Let $p_{\text{min}}(R)$ denote this price vector for $R$.

Given $R \in \mathcal{R}^n$, let $W_{\text{min}}(R)$ be the set of minimum price Walrasian equilibria for $R$, and let

$$Z_{\text{min}}(R) \equiv \{ z \in Z : (z, p_{\text{min}}(R)) \in W_{\text{min}}(R) \}.$$ 

By Facts 3.1 and 3.3, for each $R \in \mathcal{R}^n$, the set $Z_{\text{min}}(R)$ is nonempty. Although the correspondence $Z_{\text{min}}$ is set-valued, it is essentially single-valued, i.e., for each $R \in \mathcal{R}^n$, each pair $z, z' \in Z_{\text{min}}(R)$, and each $i \in N$, $z_i$, $z'_i$.\(^{31}\)

As Demange, Gale, and Sotomayor (1986), etc., show for the quasi-linear domain, and as shown for our domain (Section 5), the SA auctions achieve the MPWE.

### 3.2 Illustration of minimum price Walrasian equilibrium

Fig. 1 illustrates a MPWE for three agents, and two objects, say $A$ and $B$. There are three horizontal lines. The lowest one corresponds to the null object. The middle and highest ones correspond to the real objects $A$ and $B$, respectively. The intersection of the vertical line and each horizontal line denotes the bundle consisting of the corresponding object and no

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\(^{31}\)An allocation $z' \in Z$ is obtained by an indifferent permutation from $z \in Z$ if there is a permutation $\pi$ on $N$ such that for each $i \in N$, $z'_i = z_{\pi(i)}$ and $z'_i I_i z_i$ (Tadenuma and Thomson, 1991). Note that for each pair $z, z' \in Z_{\text{min}}(R)$, $z'$ is obtained by an indifferent permutation from $z$. 

8
payment. For example, the origin \( \mathbf{0} \) denotes the bundle consisting of the null object and no payment. For each point \( z_i \) on one of three horizontal lines, the distance from \( z_i \) to the vertical line denotes payment. For example, \( z_1 \) denotes the bundle consisting of object \( A \) and payment \( p^A \). Indifference between bundles is shown by a curvy line connecting them. Welfare increases with decreasing payments. Thus, in Fig. 1, agent 1 prefers \( z_1 \) to \( \mathbf{0} \).

Assume that preferences are as depicted in Fig. 1. The compensated valuations from the origin are ranked as \( CV_1(A; \mathbf{0}) > CV_3(A; \mathbf{0}) > CV_2(A; \mathbf{0}) \) and \( CV_1(B; \mathbf{0}) > CV_2(B; \mathbf{0}) > CV_3(B; \mathbf{0}) \). In Fig. 1, agent 1’s preference is not quasi-linear, but classical.\(^{32}\) Thus, Fig. 1 also illustrates that \( \mathcal{R}^Q \subset \mathcal{R}^C \).\(^{33}\)

The MPWE for the preference profile \( R = (R_1, R_2, R_3) \) is as follows: Agent 1 receives object \( A \) and pays \( CV_3(A; \mathbf{0}) \), i.e., the price \( p^A \) of object \( A \) is \( CV_3(A; \mathbf{0}) \). His consumption is \( z_1 \). Agent 2 receives object \( B \) and pays \( CV_1(B; z_1) \), i.e., the price \( p^B \) of object \( B \) is \( CV_1(B; z_1) \). His consumption is \( z_2 \). Agent 3’s consumption is \( \mathbf{0} \) and depicted as \( z_3 \).

Let’s see why the allocation \( z = (z_1, z_2, z_3) \) is an MPWE for \( R \). First, note that for each agent \( i = 1, 2, 3 \), \( z_i \) is maximal for \( R_i \) in the budget set \( \{ \mathbf{0}, (A, p^A), (B, p^B) \} \). Thus, \( z \) is a Walrasian equilibrium.

Next, let \( (p^A, p^B) \) be a Walrasian equilibrium price. We show \( p^A \geq p^A \) and \( p^B \geq p^B \). If \( p^A < p^A \) and \( p^B < p^B \), then, all agents prefer \( (A, p^A) \) or \( (B, p^B) \) to \( \mathbf{0} \), that is, all three agents demand \( A \) or \( B \) or both. In that case, one agent cannot receive an object he demands, contradicting the condition for a Walrasian equilibrium. Thus, \( p^A \geq p^A \) or \( p^B \geq p^B \). If \( p^A < p^A \), then \( p^B \geq p^B \), and so, both agents 1 and 3 prefer \( (A, p^A) \) to \( \mathbf{0} \) and \( (B, p^B) \), that is, both demand only \( A \). In that case, agent 1 or 3 cannot both receive the objects they demand, contradicting Walrasian equilibrium. Therefore, \( p^A \geq p^A \). If \( p^B < p^B \), both agents 1 and 2 prefer \( (B, p^B) \) to \( \mathbf{0} \) and \( (A, p^A) \), and so, agent 1 or 2 cannot both receive the objects they demand, contradicting Walrasian equilibrium. Therefore, \( p^B \geq p^B \). Hence, \( (z, p) \) is the MPWE.

3.3 Overdemanded and underdemanded sets

Next, we introduce the concepts of “overdemanded set” and “underdemanded set” (Mishra and Talman, 2010; etc.), and relate these concepts to Walrasian equilibria.

**Definition 3.2.** (i) A set \( M' \subseteq M \) of objects is **(weakly) overdemanded at** \( p \) for \( R \) if

\[ |\{ i \in N : D(R_i, p) \subseteq M' \} | \geq |M'|. \]

(ii) A set \( M' \subseteq M \) of objects is **(weakly) underdemanded at** \( p \) for \( R \) if

\[ \forall x \in M', p^x > 0 \implies |\{ i \in N : D(R_i, p) \cap M' \neq \emptyset \} | \leq |M'|. \]

In Fig. 1, note that \( \{ i \in N : D(R_i, p) \subseteq \{ A \} \} = \emptyset \), \( \{ i \in N : D(R_i, p) \subseteq \{ B \} \} = \{ 2 \} \), \( \{ i \in N : D(R_i, p) \subseteq \{ A, B \} \} = \{ 1, 2 \} \), \( \{ i \in N : D(R_i, p) \cap \{ A \} \neq \emptyset \} = \{ 1, 3 \} \), \( i \in N : \)

\(^{32}\)Suppose that agent 1’s preference is quasi-linear. Then, since \( CV_1(B, \mathbf{0}) > CV_1(A, \mathbf{0}) \), agent 1’s compensated valuation \( CV_1(B, z_1) \) of object \( B \) from the point \( z_1 \) in Fig. 1 must be greater than \( CV_2(B, \mathbf{0}) \). However, in Fig. 1, agent 1 prefers \( z_1 \) to the point \( (B, CV_2(B, \mathbf{0})) \). This is a contradiction.

\(^{33}\)See Saitoh and Serizawa (2008) for other examples of non-quasi-linear preferences.
Fact 3.4 and Theorem 3.1 below are established by Mishra and Talman (2010) for quasi-linear preferences. Fact 3.4 is a characterization of Walrasian equilibria by means of the concepts of overdemanded and underdemanded sets. Their proof also works for Fact 3.4 in the extended domain.

**Fact 3.4 (Mishra and Talman, 2010).** Let \( R \in \mathcal{R}^n \). A price vector \( p \) is a Walrasian equilibrium price for \( R \) if and only if no set is overdemanded and no set is underdemanded at \( p \) for \( R \).

Theorem 3.1 is a characterization of the minimum price Walrasian equilibrium by means of the concepts of overdemanded and weakly underdemanded sets. We emphasize, in contrast to Fact 3.4, that Mishra and Talman’s (2010) proof crucially depends on quasi-linearity. It relies on the simple fact that when preferences are quasi-linear, if a set \( M' \) is weakly underdemanded at a Walrasian equilibrium prices \( p \), then all the prices of \( M' \) can be slightly lowered by the same amount while maintaining the Walrasian equilibrium conditions (WE-i) and (WE-ii).\(^{34}\) However, this is not true when preferences are not quasi-linear. Theorem 3.1 is a novel result.

Theorem 3.1 is the key to obtaining Theorem 4.1 and Proposition 5.1. Since we obtain the existence of Walrasian equilibrium as a byproduct of Proposition 5.1, this theorem is also a key to the existence of a Walrasian equilibrium.

**Theorem 3.1.** Let \( R \in \mathcal{R}^n \). A price vector \( p \) is a minimum Walrasian equilibrium price for \( R \) if and only if no set is overdemanded and no set is weakly underdemanded at \( p \) for \( R \).

Corollary 3.1 says that if the number of objects is greater than or equal to the number of agents, the price of some objects is 0. It is used to prove Fact 4.1. Corollary 3.2 says that each object whose price is positive is “connected” by agents’ demands to the null object or to an object with a price of 0. This corollary is used to prove Theorem 4.1.\(^{35}\) For example, in Fig. 1, object \( B \) has a positive equilibrium price, agent 1’s demand connects objects \( A \) and \( B \), and agent 3’s demand connects object \( A \) and the null object.

**Corollary 3.1 (Existence of Free Object).** Let \( m \geq n \), \( R \in \mathcal{R}^n \), and \( z \in Z_{\min}(R) \). Then, there is \( i \in N \) such that \( p_{\min}^{x_i}(R) = 0 \).

**Corollary 3.2 (Demand Connectedness).**\(^{36}\) Let \( R \in \mathcal{R}^n \) and \( (z, p) \in W_{\min}(R) \). For each \( x \in M \) with \( p^x > 0 \), there is a sequence \( \{i_k\}_{k=1}^K \) of \( K \) distinct agents such that (i) \( x_{i_1} = 0 \) or \( p^{x_{i_1}} = 0 \), (ii) \( x_{i_k} = x \), and (iii) for each \( k \in \{1, \ldots, K-1\} \), \( \{x_{i_k}, x_{i_{k+1}}\} \subseteq D(R_{i_k}, p) \).

Here, we also introduce a concept of “\( d_i \)-truncation” of a preference. This concept is important to prove Theorem 3.1. It says that the welfare position of each bundle \( z_i \in M \times \mathbb{R} \) is lowered as much as \( d_i \) in terms of money, but their relative positions are kept.

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\(^{34}\)For details, refer to the proof of Lemma 3 in Mishra and Talman (2010).

\(^{35}\)See Lemma 10 for details.

\(^{36}\)This structure is discussed by Demange, Gale, and Sotomayor (1986) and Miyake (1998).
Given $R_i \in \mathcal{R}$ and $d_i \in \mathbb{R}$, the $d_i$-truncation of $R_i$ is the preference $R'_i$ such that for each $z_i \in M \times \mathbb{R}$, $CV'_i(0; z_i) = CV_i(0; z_i) + d_i$. Given $R \in \mathcal{R}^n$, the $d$-truncation of $R$ is the preference profile $R'$ such that for each $i \in N$, $R'_i$ is the $d_i$-truncation of $R_i$.\footnote{Given $R_i \in \mathcal{R}$ and $d_i \in \mathbb{R}$, the $d_i$-truncation of $R_i$ may not belong to $\mathcal{R}$. The proof of Theorem 3.1 does not require that the $d_i$-truncation of $R_i$ be in $\mathcal{R}$.}

Next are a remark and a fact pertaining to truncations. These are used to prove Theorem 3.1.

**Remark 3.1.** Let $R_i \in \mathcal{R}$, $d_i \in \mathbb{R}$, and $R'_i$ be the $d_i$-truncation of $R_i$. Then, for each $z_i, \hat{z}_i \in M \times \mathbb{R}$, $z_i R_i \hat{z}_i$ if and only if $z_i R'_i \hat{z}_i$.

**Fact 3.5 (Roth and Sotomayor, 1990).** Let $R \in \mathcal{R}^n$ and $R'$ be a $d$-truncation of $R$ such that for each $i \in N$, $d_i \geq 0$. Then, $p_{min}(R') \leq p_{min}(R)$.

## 4 Main Results

In this section, we provide a characterization of the MPWE by means of properties of rules. Let $\mathcal{R} \subseteq \mathcal{R}^E$.

**Definition 4.1.** A rule $f$ on $\mathcal{R}^n$ is a minimum price Walrasian (MPW) rule if for each $R \in \mathcal{R}^n$, $f(R) \in Z_{min}(R)$.

### 4.1 Properties of the minimum price Walrasian rule

Let $g$ be an MPW rule on $\mathcal{R}^n$. First, by Fact 3.2, for each $R \in \mathcal{R}^n$, $g(R)$ is efficient for $R$. Let $R \in \mathcal{R}^n$. Then, there is a price vector $p \equiv (p^1, \cdots, p^m) \in \mathbb{R}^m_+$ such that for each $i \in N$, (a) $g_i(R) \in B(p)$, and (b) for each $z'_i \in B(p)$, $g_i(R) R_i z'_i$. Let $i \in N$. Note that, for each $x \in M$, $p^x \geq 0$, and $B(p) = \{(0,0), (1,p^1), (2,p^2), \cdots, (m,p^m)\}$. Thus, by (a), $g^x_i(R) \geq 0$, and by (b), $g_i(R) R_i 0$. Therefore, the MPW rule satisfy efficiency, individual rationality, and no subsidy.

**Fact 4.1 (Demange and Gale, 1985).** The minimum price Walrasian rule is group strategy-proof.

Theorem 3.1 allows a direct proof (see Appendix B).

### 4.2 Characterizations

In this subsection, we assume that each agent has a classical preference and the number of agents exceeds the number of objects. Recall that all results established in Section 3 also hold in this case. Theorem 4.1 is our main result of this article, a characterization of the MPW rule.

**Theorem 4.1.** Let $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$. A rule $f$ on $\mathcal{R}^n$ satisfies strategy-proofness, efficiency, individual rationality, and no subsidy for losers if and only if it is a minimum price Walrasian rule: for each $R \in \mathcal{R}^n$, $f(R) \in Z_{min}(R)$.

The proof is in Appendix B. Since the MPW rules are group strategy-proof, Theorem 4.1 implies that only the MPW rules satisfies group strategy-proofness, efficiency, individual rationality, and no subsidy for losers. Since no subsidy implies no subsidy for losers, Theorem
4.1 also implies that only the MPW rules satisfies strategy-proofness, efficiency, individual rationality, and no subsidy.

4.3 Indispensability of the axioms and assumptions

The only if part of Theorem 4.1 fails if we drop any of the four axioms, as shown by the following examples.

**Example 4.1 (Dropping strategy-proofness).** Let \( f \) be the rule that chooses a “maximum” price Walrasian equilibrium allocation for each preference profile. Then, \( f \) satisfies the axioms of Theorem 4.1 except for strategy-proofness.\(^{38}\)

**Example 4.2 (Dropping efficiency).** Let \( f \) be the rule such that for each preference profile, each agent receives the null object and pays nothing. Then, \( f \) satisfies the axioms of Theorem 4.1 except for efficiency.

Next, we introduce variants of Walrasian equilibria, ones with “entry fees”. Given an entry fee \( e_i \in \mathbb{R} \), let \( D(R, p, e_i) \equiv \{ x \in L : \text{for each } y \in L, (x, p^x + e_i) R_i (y, p^y) \} \), where \( p^x = 0 \) if \( x = 0 \). A pair \( (z, p) \in Z \times \mathbb{R}^m \) of a feasible allocation and a price vector is a **Walrasian equilibrium with entry fees** for \( R \in \mathcal{R}^n \) if there is an entry fee vector \( e = (e_1, \cdots, e_n) \in \mathbb{R}^m \) such that

(WE-i*) for each \( i \in N \), \( x_i \in D(R_i, p, e_i) \) and \( t_i = p^{t_i} + e_i \), and

(WE-ii) for each \( y \in M \), if for each \( i \in N, x_i \neq y \), then, \( p^y = 0 \).

Note that, by Proposition 5.1 (Section 5), for each preference profile \( R \in \mathcal{R}^n \) and each \( e = (e_1, \cdots, e_n) \in \mathbb{R}^m \), there is a minimum price Walrasian equilibrium with entry fees \( e \), and it is efficient.

A rule \( f \) is a **minimum price Walrasian rule with entry fees** if there is a list \( \{e_i(\cdot)\}_{i \in N} \) of entry fee functions defined on \( \mathcal{R}^n \), and for each \( R \), \( f(R) \) is a minimum price Walrasian equilibrium with entry fees \( \{e_i(R)\}_{i \in N} \). If for each \( i \in N \), the entry fee function \( e_i(\cdot) \) depends only on the other agents’ preferences, then the associated minimum price Walrasian rule with entry fees satisfies strategy-proofness. Thus, we assume that for each \( i \in N \), his entry fee function \( e_i(\cdot) \) is defined on the class of the other agents’ preference profiles \( \mathcal{R}^{n-1} \).

**Example 4.3 (Dropping individual rationality).** Let a list \( \{e_i(\cdot)\}_{i \in N} \) of entry fee functions be such that for each \( i \in N \) and each \( R \in \mathcal{R}^n \), \( e_i(R_{-i}) > 0 \). Then, the associated minimum price Walrasian rule with entry fees satisfies the axioms of Theorem 4.1 except for individual rationality.

**Example 4.4 (Dropping no subsidy for losers).** Let a list \( \{e_i(\cdot)\}_{i \in N} \) of entry fee functions be such that for each \( i \in N \) and each \( R \in \mathcal{R}^n \), \( e_i(R_{-i}) < 0 \). Then, the associated minimum price Walrasian rule with entry fees satisfies the axioms of Theorem 4.1 except for no subsidy for losers.

\(^{38}\)Demange and Gale (1985) also show that for each preference profile, there is a maximum price Walrasian equilibrium. When there is only one object, the maximum price Walrasian equilibrium corresponds to the first price auction. It is well-know that the first price auction is not strategy-proof.
One might wonder if the minimum price Walrasian rules with entry fees can be characterized by only strategy-proofness and efficiency. Our proof of Theorem 4.1 fails if individual rationality and no subsidy for losers are dropped. However, we have not found an example of a rule that satisfies strategy-proofness and efficiency, but is not a MPW rule with entry fees.

One might also wonder if the assumption that \( n > m \) can be dropped in Theorem 4.1. Our proof of Theorem 4.1 also fails if \( n \leq m \). However, we have not found an example of a rule that satisfies the four axioms of Theorem 4.1, but is not an MPW rule even if \( n > m \) is dropped.

5 Simultaneous Ascending Auction

We define a class of simultaneous ascending auctions, and show that they achieve the MPWE. Let \( \mathcal{R} \subseteq \mathcal{R}^E \), \( R \in \mathcal{R}^n \), and \( p \in \mathbb{R}_+^m \).

**Definition 5.1.** A set \( M' \subseteq M \) is a minimal overdemanded set at \( p \) for \( R \) if \( M' \) is overdemanded at \( p \) for \( R \), and there is no \( M'' \subset M' \) such that \( M'' \) is overdemanded at \( p \).

Under a (continuous time) “simultaneous ascending auction”, there is a constant \( d > 0 \), and at each time, each bidder submits his demand at the current price vector, and the prices of the objects in a minimal overdemanded set are raised at a speed at least \( d \). When there is no overdemanded set, the auction stops.

**Definition 5.2.** A simultaneous ascending (SA) auction is a function \( \hat{p} \) from \( \mathbb{R}_+ \times \mathbb{R}_+^m \times \mathcal{R}^n \) to \( \mathbb{R}_+^m \) such that

(i) for each \( p \in \mathbb{R}_+^m \), each \( R \in \mathcal{R}^n \), and each \( x \in M \), \( \hat{p}^x(0, p, R) = 0 \),
(ii) \( \hat{p} \) is absolutely continuous with respect to \( t \) and \( p \),
(iii) there is \( d > 0 \) such that for each \( t \in \mathbb{R}_+ \), each \( p \in \mathbb{R}_+^m \), each \( R \in \mathcal{R}^n \), and each \( x \in M \),
   (iii-a) if \( \hat{p}^x \) is differentiable at \( (t, p) \), and \( x \) is in a minimal overdemanded, \( d\hat{p}^x(t, p, R)/dt \geq d \),
   and
   (iii-b) \( d\hat{p}^x(t, p, R)/dt = 0 \) otherwise.

**Remark 5.1.** For each \( R \in \mathcal{R}^n \), a SA auction \( \hat{p} \) generates a price path \( p(\cdot) \) such that for each \( x \in M \) and each \( t \in \mathbb{R}_+ \),

\[
p^x(t) = \int_0^t \frac{d\hat{p}^x(s, p(s), R)}{ds} ds.
\]

**Proposition 5.1.** For each \( R \in \mathcal{R}^n \), the price path generated by any simultaneous ascending auction converges to the minimum Walrasian equilibrium price in a finite time.

The proof is in Appendix C. Proposition 5.1 says that for each \( R \in \mathcal{R}^n \), the price path \( p(\cdot) \) generated by an SA auction has a termination time \( T \) such that for each \( t \geq T \), \( p(t) = p(T) = p_{\min}(R) \), and at the final prices \( p(T) \), each agent receives an object from his demand, and pays the final price of the object that he receives. Moreover, this proposition implies the existence of a Walrasian equilibrium.

\[39\] Condition (ii) of Definition 5.2 (absolute continuity) guarantees the integrability of \( d\hat{p}^x(t, p(t), R)/dt \) with respect to \( t \).
6 Generalized Vickrey Rule

In this section, we introduce the generalized Vickrey rules, and contrast them with the MPW rules.

6.1 Generalized Vickrey rule

Each quasi-linear preference $R_i$ can be defined by means of a valuation function $v_i : L \rightarrow \mathbb{R}_+$, and a preference profile $R$ in the quasi-linear domain corresponds to a valuation profile $v(R) = (v_1(R_1), \ldots, v_n(R_n))$. Given a valuation profile $v = (v_1, \ldots, v_n)$, let $(x_1^*(v), \ldots, x_n^*(v)) \in \arg \max_{(x_1, \ldots, x_n) \in X} \sum_i v_i(x_i), \sigma_i(v) \equiv \sum_{j \neq i} v_j(x_j^*(v))$, and $\sigma'_i(v) \equiv \max_{(x_1, \ldots, x_n) \in X} \sum_{j \neq i} v_j(x_j)$. On the quasi-linear domain, the Vickrey rules are defined as follows.

**Definition 6.1.** A rule $f$ on the quasi-linear domain is a **Vickrey rule** if for each valuation profile $v$, $f^*(v) \in \arg \max_{(x_1, \ldots, x_n) \in X} \sum_i v_i(x_i)$, and for each $i \in N$, $f'_i(v) = \sigma'_{-i}(v) - \sigma_i(v)$.

To generalize the above definition to the classical domain, we need to use some valuation function $v_i$ for each classical preference $R_i$. The compensated valuation $CV_i(\cdot; 0)$ from the origin is defined for each classical preference $R_i$ and a generalization of valuation function, and so a natural candidate. Given a classical preference $R_i$, let $v_i(\cdot; R_i)$ be a function defined as: for each $x \in L$, $v_i(x; R_i) \equiv CV_i(x; 0)$. Given a classical preference profile $R$, let $v'(R) \equiv (v_1(\cdot; R_1), \ldots, v_n(\cdot; R_n))$.

**Definition 6.2.** A rule $f$ on the classical domain is a **generalized Vickrey rule** if for each valuation profile $v'(R)$, $f^*(v'(R)) \in \arg \max_{(x_1, \ldots, x_n) \in X} \sum_i v_i(x_i; R_i)$, and for each $i \in N$, $f'_i(v'(R)) = \sigma'_{-i}(v'(R)) - \sigma_i(v'(R))$.

A classical preference $R_i$ is **object-blind** if for each $x, y \in M$ and each $t \in \mathbb{R}$, $(x, t) I_i(y, t)$. We call the class of object-blind preferences the “object-blind domain”. On the object-blind domain, Saitoh and Serizawa (2008) and Sakai (2008) characterize the generalized Vickrey rules.

**Fact 6.1 (Saitoh and Serizawa, 2008; Sakai, 2008).** Let $n > m$. A rule on the object-blind domain satisfies strategy-proofness, efficiency, individual rationality, and no subsidy if and only if it is a generalized Vickrey rule.40

On the quasi-linear domain, the classes of Vickrey rules, generalized Vickrey rules, and MPW rules coincide. Fact 6.1 suggests that the generalized Vickrey rules are natural generalizations of the Vickrey rules on the object-blind domain. On the object-blind domain, the classes of generalized Vickrey rules and MPW rules also coincide. However, Theorem 4.1 and Fact 6.1 are mathematically independent.

6.2 Generalized Vickrey Rule vs. Minimum price Walrasian rule

Notice that in example of Section 3 (Fig. 1), agent 2’s payment in the MPWE allocation $z$ cannot be computed from the compensated valuations $v_i(\cdot; R_i), i = 1, 2, 3$, from the origin $0$.  

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40It is straightforward that on the object-blind domain, strategy-proofness, efficiency, individual rationality, and no subsidy for losers imply no subsidy.
Payments of the MPW rule depend on the compensated valuations from various points. It is worthwhile to mention that for the preference profile in Fig. 1, it is agent 1’s preference $R_1$ that determines whether agent 2 or 3 receives an object in the MPWE allocation. In Fig. 1, agent 1 prefers $(A, CV_3(A; 0))$ to $(B, CV_2(B; 0))$, and agent 2 receives an object. However, if agent 1 prefers $(B, CV_2(B; 0))$ to $(A, CV_3(A; 0))$, agent 3 instead receives an object. Object allocations of the MPW rule also depend on the compensated valuations from various points. Thus, the MPWE allocation $z$ is not the outcome of the generalized Vickrey rule. Accordingly, the MPW rule does not coincide with the generalized Vickrey rule.

One can easily check that the generalized Vickrey rule is neither efficient nor strategy-proof on the classical domain with heterogeneous objects. To check this fact, let $R_1 \in \mathcal{R}^C$, $R_2 \in \mathcal{R}^Q$, and $R_3 \in \mathcal{R}^Q$ be such that $CV_1(A; 0) = 9$, $CV_1(B; 0) = 10$, $(A, 6) P_1 (B, 5)$, $CV_2(A; 0) = 3$, $CV_2(B; 0) = 5$, $CV_3(A; 0) = 6$, and $CV_3(B; 0) = 2$. Then, the outcome of the generalized Vickrey rule for $R$ is $z \equiv ((B, 5), (0, 0), (A, 4))$. Let $z' \equiv ((A, 6), (B, 5), (0, -2))$. Then, $z'$ Pareto-dominates $z$, a violation of efficiency. Let $R_1' \in \mathcal{R}^Q$ be such that $CV_1'(A; 0) = 8$ and $CV_1'(B; 0) = 5$. Then, under the generalized Vickrey rule, the bundle that agent 1 obtains for $(R_1', R_{-1})$ is $(A, 6)$. Since $(A, 6) P_1 (B, 5)$, the generalized Vickrey rule violates strategy-proof.

The generalized Vickrey rule employs only a small part of the information about agents’ preferences (i.e., “compensated valuations from the origin”). On the other hand, the MPW rule employs other information (i.e., “compensated valuations from various points”). As we stated in Section 4, only the MPW rule satisfies strategy-proofness, efficiency, individual rationality, and no subsidy for losers on the domain including non-quasi-linear preferences. Thus, the information about compensated valuations from various points is necessary to design rules satisfying the above four properties on this domain. Proposition 5.1 states that the SA auction achieves the same outcome as the MPW rule.

7 Concluding Remarks

In this article, we mainly focus on the analysis of rules that allocate objects efficiently, and show that only the MPW rules are desirable based on the four properties, strategy-proofness, efficiency, individual rationality, and no subsidy for losers. It would be also important to investigate rules that produce more revenue for the auctioneer. An interesting question relating to this issue is whether there are strategy-proof, efficient, and individually rational rules that produce a greater revenue than the MPW rule for each preference profile. We hope that the results and techniques developed in this article will be useful for the study of this research topic.

---

41When agent 1 prefers $(B, CV_2(B; 0))$ to $(A, CV_3(A; 0))$, i.e., $(B, 5) P_1 (A, 6)$, the MPWE allocation $z^*$ is $z^* \equiv ((B, 5), (0, 0), (A, CV_1(A; z_1)))$. Thus, unless $CV_1(A; z_1) = 4$, the MPWE allocation is different from the outcome of the generalized Vickrey rule.
Appendix: Proofs

In this Appendix, we provide the proofs of all results of the article. In Section A, we prove Theorem 3.1 and Corollaries 3.1 and 3.2. In Section B, we give the proofs of the main results (Fact 4.1 and Theorem 4.1). Section C gives the proof of Proposition 5.1. The proofs of Facts 3.4 and 3.5 appear in the Supplementary Note (Morimoto and Serizawa, 2013).

A Proofs for Section 3 (Theorem 3.1 and Corollaries 3.1 and 3.2)

Let \( \mathcal{R} \subseteq \mathcal{R}^E \) in this section.

Lemma A.1. Let \( R \in \mathcal{R}^n \), \( (z, p) \in W(R) \), and \( R' \) be a d-truncation of \( R \) such that for each \( i \in N \) with \( x_i \neq 0 \), \( d_i \leq -CV_i(0 ; z_i) \), and for each \( i \in N \) with \( x_i = 0 \), \( d_i \geq 0 \). Then, \((z, p) \in W(R')\).

Proof. Since \((z, p) \in W(R)\), \((z, p)\) satisfies (WE-i) and (WE-ii) for \( R \). Since (WE-ii) is independent of preferences, we show only (WE-i) for \( R' \), that is, that for each \( i \in N \) and each \( y \in L \), \( (x_i, p^{x_i}) R'_i(y, p^y) \). Let \( i \in N \) and \( y \in L \).

Case 1. \( x_i \neq 0 \).
If \( y \neq 0 \), then by Remark A.1, \( (x_i, p^{x_i}) R'_i(y, p^y) \). If \( y = 0 \), then by \( d_i \leq -CV_i(0 ; z_i) \), \( (x_i, p^{x_i}) R'_i 0 = (y, p^y) \).

Case 2. \( x_i = 0 \).
If \( y = 0 \), then by \( (y, p^y) = 0 = (x_i, p^{x_i}) \), \( (x_i, p^{x_i}) R'_i(y, p^y) \). If \( y \neq 0 \), then by \( (x_i, p^{x_i}) R'_i(y, p^y) \) and \( d_i \geq 0 \), \( (x_i, p^{x_i}) R'_i 0 = (y, p^y) \).

Lemma A.2. Let \( i \in N \), \( R_i \in \mathcal{R} \), \( d_i \in \mathbb{R} \), and \( R'_i \in \mathcal{R} \) be the d_i-truncation of \( R_i \). Let \( p, q \in \mathbb{R}_+^n \), \( x \in M \), and \( y \in L \) be such that \( x \in D(R_i, p) \) and \( y \in D(R'_i, q) \).

(i) If \( q^x < p^x \) and \( y \in M \), then, \( (y, q^y) P_i(x, p^x) \) and \( q^y < p^y \).

(ii) If \( q^x < p^x \) and \( d_i \leq -CV_i(0 ; (x, p^x)) \), then, \( y \in M \), \( (y, q^y) P_i(x, p^x) \), and \( q^y < p^y \).

Proof. (i) Let \( q^x < p^x \) and \( y \in M \). By \( y \in D(R'_i, q) \), \( (y, q^y) R'_i(x, q^x) \). Since \( R'_i \) is the d_i-truncation of \( R_i \), by Remark 3.1, \( (y, q^y) R_i(x, q^x) \). Then,

\[
(y, q^y) R_i(x, q^x) P_i(x, p^x) R_i \quad (y, p^y),
\]

Thus, \( (y, q^y) P_i(x, p^x) \). Also, \( (y, q^y) P_i(x, p^y) \) implies \( q^y < p^y \).

(ii) Let \( q^x < p^x \) and \( d_i \leq -CV_i(0 ; (x, p^x)) \). Then, \( CV_i(0 ; (x, p^x)) \leq 0 \), and so \( (x, p^x) R'_i 0 \). Thus,

\[
(y, q^y) R'_i(x, q^x) P'_i(x, p^x) R'_i 0,
\]

Then, \( (y, q^y) P'_i 0 \) implies \( y \in M \). Thus, by (i) of Lem. A.2, \( (y, q^y) P_i(x, p^x) \) and \( q^y < p^y \).

Proof of Theorem 3.1. “IF”. Assume that no set is overdemanded, and no set is weakly underdemanded at \( p \) for \( R \). Then, by Fact 3.4, \( p \in P(R) \). Suppose that there is \( q \in P(R) \) such that \( q \leq p \) and \( q \neq p \). Without loss of generality, assume that for each \( x \in M' \), \( q^x < p^x \), and for each \( x \notin M' \), \( q^x = p^x \), where \( M' \equiv \{1, \ldots, m'\} \) and \( 1 \leq m' \leq m \).
Since $M'$ is not weakly underdemanded at $p$ for $R$, there is $N' \subseteq N$ such that $|N'| > |M'|$ and for each $i \in N'$, $D(R_i, p) \cap M' \neq \emptyset$. For each $i \in N'$, let $y_i \in D(R_i, p) \cap M'$. Since for each $x \in M'$, $q^x < p^x$, and for each $x \notin M'$, $q^x = p^x$, it follows that for each $i \in N'$ and each $x \notin M'$, $(y_i, q^x_i) \not\in (y_i, p^x_i)$, $R_i(x, p^x) = (x, q^x)$. Thus, for each $i \in N'$, $D(R_i, q) \subseteq M'$. By $|N'| > |M'|$, $M'$ is overdemanded at $q$. Since $q \in P(R)$, this contradicts Fact 3.4.

"ONLY IF". Let $p \equiv p_{\text{min}}(R)$. Then, by Fact 3.4, no set is overdemanded, and no set is underdemanded, at $p$ for $R$. We show that no set is weakly underdemanded at $p$ for $R$. Suppose that there is a set $M'$ that is weakly underdemanded at $p$ for $R$, i.e., for each $x \in M'$, $p^x > 0$, and $|\{i \in N : D(R_i, p) \cap M' \neq \emptyset\}| \leq |M'|$. Let $N' \equiv \{i \in N : D(R_i, p) \cap M' \neq \emptyset\}$. Without loss of generality, assume that $M'$ is minimal: no proper subset of $M'$ is weakly underdemanded at $p$ for $R$. Since $p \in P(R)$, there is $z \in Z$ such that for each $i \in N$, $x_i \in D(R_i, p)$ and $t_i = p^{x_i}$. Since no set is underdemanded at $p$ for $R$, $|N'| = |M'|$. Without loss of generality, let $M' \equiv \{1, \ldots, m'\}$ and $N' \equiv \{1, \ldots, m'\}$.

Step 1. For each $i \in N'$, $x_i \in M'$.

Proof. Since for each $x \in M'$, $p^x > 0$, (WE-ii) implies that for each $x \in M'$, there is $i(x) \in N'$ such that $x_i(x) = x$. Then, $|N'| = |M'|$, for each $i \in N'$, $x_i \in M'$.

For each $x \in M'$, let $q^x \equiv \max\{CV_j(x; z_j) : j \in N \setminus N'\} \cup \{0\}$. Then, for each $x \in M'$, $q^x < p^x$. Let $R'_{m'+1} \in \mathcal{R}$ be such that for each $x \in M'$, if $q^x > 0$, $CV'_{m'+1}(x; 0) = q^x$, and if $q^x = 0$, $CV'_{m'+1}(x; 0) \in (0, p^x)$. Consider the economy $E'$ with object set $M'$ and agent set $N'' \equiv N' \cup \{m' + 1\}$ with preferences $(R_{N''}, R'_{m'+1})$. Let $z_{m'+1} = 0$ and $z_{N''} \equiv (z_{N''}, z_{m'+1})$.

Step 2. $(z_{N''}, p^{M'})$ is a minimum price Walrasian equilibrium of $E'$.

Proof. Let $(\tilde{z}_{N''}, \tilde{p}^{M'}) \in W^{M'', N''}_{\text{min}}(R_{N''}, R'_{m'+1})$. Since $(z_{N''}, p^{M'}) \in W^{M'', N''}(R_{N''}, R'_{m'+1}), \tilde{p}^{M'} \leq p^{M'}$. Let $M^- \equiv \{x \in M' : \tilde{p}^x < p^x\}$. We show $M^- = \emptyset$. Suppose $M^- \neq \emptyset$. Let $N^- \equiv \{i \in N' : D(R_i, p^{M'}) \cap M^- \neq \emptyset\}$.

Step 2.1. For each $i \in N^-$, $\tilde{x}_i \in M^-$. 

Proof. Let $i \in N^-$. Then, there is $x \in D(R_i, p^{M'}) \cap M^-$. Thus, $x \in M'$ and $\tilde{p}^x < p^x$. Since $(\tilde{z}_{N''}, \tilde{p}^{M'}) \in W^{N''}_{\text{min}}(R_{N''}, R'_{m'+1})$, $\tilde{x}_i \in D(R_i, \tilde{p}^{M'})$. Then, by Lem. A.2-(ii), $\tilde{x}_i \in M'$ and $\tilde{p}^x < \tilde{p}^{x_i}$. Thus, $\tilde{x}_i \in M^-$. □

Step 2.2. $M^- = M'$, $N^- = N'$, and $|M^-| = |N^-|$. 

Proof. Since no two agents in $N^-$ receive the same object, Step 2.1 implies $|M^-| \geq |N^-|$. Suppose $M^- \neq M'$. Then, since $M^- \subseteq M'$ and $M'$ is a minimal weakly underdemanded set

\[\text{To see this, suppose that for some } x \in M', q^x \geq p^x. \text{ Then, there is } j \in N \setminus N' \text{ such that } (x, p^x) R_j z_j. \text{ Since } x_j \in D(R_j, p), \text{ } x \in D(R_j, p). \text{ Thus, } j \in N'. \text{ This contradicts } j \in N \setminus N'.\]

\[\text{Let } W^{M', N''}(\tilde{R}_{N''}) \text{ and } W^{M', N''}_{\text{min}}(\tilde{R}_{N''}) \text{ be the sets of Walrasian and minimum price Walrasian equilibria of the economy with object set } M' \text{ and agent set } N'' \text{ with preference } \tilde{R}_{N''}, \text{ and let } p^{M', N''}(\tilde{R}_{N''}) \text{ and } p^{M', N''}_{\text{min}}(\tilde{R}_{N''}) \text{ be the set of Walrasian price and the minimum Walrasian equilibrium price of the economy, respectively.}\]

17
at \( p \) for \( R, M^- \) is not weakly underdemanded at \( p^{M'} \) for \((R_{N'}, R'_{m'+1})\). Thus, since for each \( x \in M^- \), \( p^x > 0 \), we have \(|N^-| \geq |M^-| + 1 \). This contradicts \(|M^-| \geq |N^-| \). Thus, \( M^- = M' \).

By the def. of \( N^- \), \( M^- = M' \) implies \( N^- = N' \). Since \( M' \) is weakly underdemanded, \(|N'| = |M'| \). By the above results, \(|M^-| = |M'| = |N'| = |N^-| \).

\( \square \)

Step 2.3. For each \( x \in M' \), \( \bar{p}^x \geq q^x \).

**Proof.** Suppose there is \( x \in M' \) such that \( \bar{p}^x < q^x \). Thus, \( q^x > 0 \). Then, by \( \bar{x}_{m'+1} \in D(R'_{m'+1}, \bar{p}^M) \) and \( \bar{p}^x < q^x = CV'_{m'+1}(x; 0) \), \( \bar{x}_{m'+1} \in M' \). By \( M^- = M' \) and \( N^- = N' \) (Step 2.2), Step 2.1 implies for each \( i \in N' \), \( x_i \in M' \). This contradicts \(|M'| = m' \).

\( \square \)

Step 2.4. \((z, \bar{p})\) is a Walrasian equilibrium of the original economy, i.e., \((z, \bar{p}) \in W(R)\).

**Proof.** By Step 2.3, for each \( y \in M' \), \( \bar{p}^y \geq q^y \). Let \( h \in N \setminus N' \). Then, for each \( y \in L \), if \( y \notin M' \), then

\[
(\bar{x}_h, \bar{p}^x_h) = (x_h, p^x_h) \quad R_h \quad (y, p^y) = (y, \bar{p}^y),
\]

and if \( y \in M' \), then

\[
(\bar{x}_h, \bar{p}^x_h) = (x_h, p^x_h) \quad R_h \quad (y, \bar{p}^y) = (y, \bar{p}^y),
\]

Thus, for each \( h \in N \setminus N' \), \( \bar{x}_h \in D(R_h, \bar{p}) \).

Let \( h \in N' \). Then, for each \( y \in L \), if \( y \notin M' \), then

\[
(\bar{x}_h, \bar{p}^x_h) = (\bar{x}_h, \bar{p}^x_h) \quad R_h \quad (x_h, \bar{p}^x_h) \quad R_h \quad (x_h, \bar{p}^x_h) \quad R_h \quad (y, p^y) = (y, \bar{p}^y),
\]

and if \( y \in M' \), then

\[
(\bar{x}_h, \bar{p}^x_h) = (\bar{x}_h, \bar{p}^x_h) \quad R_h \quad (y, \bar{p}^y) = (y, \bar{p}^y),
\]

Thus, for each \( h \in N' \), \( \bar{x}_h \in D(R_h, \bar{p}) \). Since \((z, p)\) and \((\bar{z}_{N'}, \bar{p})\) satisfy (WE-ii), so does \((z, \bar{p})\).

\( \square \)

Recall that \( p = p_{\min}(R) \). However, since \( M^- \neq \emptyset \), \( \bar{p} \leq p \) and \( \bar{p} \neq p \). This is a contradiction. Thus, \( M^- = \emptyset \). This completes the proof of Step 2.

Without loss of generality, let \( x_1 \equiv 1, \ldots, x_{m'} \equiv m' \). Let \( \Pi \) denote the set of permutations of \( M' \) and by \( \{x(k)\}_{k=1}^{m'} \) its generic element. Given \( \{x(k)\}_{k=1}^{m'} \in \Pi \), let \( \{i(k)\}_{k=1}^{m'} \) be such that

\[
x_{i(1)} = x(1), \quad x_{i(2)} = x(2), \ldots, x_{i(m')} = x(m'),
\]

\[\text{\footnotesize{44}}\text{By } M^- \subseteq M', \{i \in N : D(R_i, p) \cap M^- \neq \emptyset\} \subseteq \{i \in N : D(R_i, p) \cap M' \neq \emptyset\} = N'. \text{ Then, } \{i \in N' : D(R_i, p^{M'}) \cap M^- \neq \emptyset\} = \{i \in N : D(R_i, p) \cap M^- \neq \emptyset\}. \text{ Hence, } |N^-| = \{|i \in N' : D(R_i, p^{M'}) \cap M^- \neq \emptyset\}| = |\{i \in N : D(R_i, p) \cap M^- \neq \emptyset\}| > |M^-|.
\]

18
and \( \{t(k)\}_{k=1}^{m'} \) be such that

\[
t(1) \leq CV_{m'+1}(x(1); 0), \quad t(2) = CV_{i(1)}(x(2); z_0(1)), \ldots , t(m') = CV_{i(m'-1)}(x(m'); z_0(m' - 1)),
\]

where for each \( k \in \{1, \ldots , m'\} \), \( z_0(k) \equiv (x(k), t(k)) \). We call such a pair \( \{z_0(k), i(k)\}_{k=1}^{m'} \) an assignment sequence (Fig. 2).

**Step 3.** There is \( b < p^1 \) such that for any assignment sequence \( \{z_0(k), i(k)\}_{k=1}^{m'} \) constructed as above, and for \( k \) with \( x(k) = 1 \), \( t(k) < b \).

**Proof.** For any assignment sequence \( \{z_0(k), i(k)\}_{k=1}^{m'} \), since \( t(1) \leq q^{x(1)} < p^{x(1)} \), the following holds inductively: for each \( k \geq 2 \),

\[
(x(k), t(k)) R_{i(k-1)} z_0(k-1) P_{i(k-1)}^{x(k-1)} (x(k-1), p^{x(k-1)}) R_{i(k-1)}^{x(k-1)} (x(k), p^{x(k)})
\]

and \( t(k) < p^{x(k)} \).

Since the cardinality of \( \Pi \) is finite \( (m')! \), there is \( b < p^1 \) such that for any assignment sequence \( \{z_0(k), i(k)\}_{k=1}^{m'} \), and for \( k \) with \( x(k) = 1 \), \( t(k) < b \). □

Let \( R'_1 \) be a \( d_1 \)-truncation of \( R_1 \) such that \( b < CV_1'(x_1; 0) < p^1 \).\(^{45}\) Consider the economy

\(^{45}\)Note that \( d_1 > 0 \).
with object set $M'$ and agent set $N'' \equiv N' \cup \{m' + 1\}$ with preferences $(R'_1, R'_{m'-1}, R_{N'\setminus\{1\}})$.

Let $(\hat{z}, \hat{p}) \in W^{M',N'}(R'_1, R'_{m'-1}, R_{N'\setminus\{1\}})$.

**Step 4.** $\hat{x}_1 \neq 0$.

**Proof.** Suppose that $\hat{x}_1 = 0$. We use Claim A.1 below. It implies that $m'$ agents (agents $2, \ldots, m' + 1$) receive $m'$ different objects in $M'\setminus\{x_1\}$. By $|M'| = m'$, this is a contradiction. Thus, proving Claim A.1 completes the proof of Step 4.

**Claim A.1.** The following sequences $\{i(k)\}$ and $\{z_0(k) \equiv (x(k), t(k))\}, k = 1, \ldots, m'$, can be constructed:

$$x(1) \equiv \hat{x}_{m'+1}, x_{i(1)} = x(1), \text{ and } t(1) \equiv \hat{p}^{x(1)}, \text{ and}$$

$$\forall k \in \{2, \ldots, m'\}, \quad x(k) \equiv \hat{x}_{i(k-1)}, x_{i(k)} = x(k), \text{ and } t(k) \equiv CV_{i(k-1)}(x(k); z_0(k-1)).$$

Furthermore, for each $k \in \{1, \ldots, m'\}, x(k) \neq 0, x(k) \neq x_1, \hat{p}^{x(k)} \leq t(k) \text{ and } \hat{p}^{x(k)} < p^{x(k)}.$

**Proof.** The proof is by induction.

**Induction base:** First, we show $x(1) \equiv \hat{x}_{m'+1} \neq 0$. Suppose $\hat{x}_{m'+1} = 0$. Then, since two agents (1 and $m' + 1$) in $N''$ receive no object and $|N''| = |M'| + 1$, there is $x \in M$ such that for each $h \in N''$, $\hat{x}_h \neq x$. By (WE-ii), $\hat{p}^x = 0$. Since $CV_{m'+1}(x; \emptyset) > 0$, $(x, \hat{p}^x) \not\in W^{M',N''}(R'_1, R'_{m'+1}, R_{N'\setminus\{1\}})$. This is a contradiction since $\hat{x}_{m'+1} = 0$ and $(\hat{z}, \hat{p}) \in W_{\min}^{M',N''}(R'_1, R'_{m'+1}, R_{N'\setminus\{1\}})$. Thus, $x(1) \neq 0$.

Note that by Step 1, $x(1) \neq 0$ implies that agent $i(1)$ with $x_{i(1)} = x(1)$ uniquely exists. Thus, $x(1), i(1)$, and $t(1)$ are well-defined.

Second, we show that $x(1) \neq x_1$. Suppose that $x(1) = x_1$. Then, by Step 3 and the def. of $R'_1$, $\hat{p}^{x(1)} \equiv t(1) < b \leq CV'_1(x_1; \emptyset)$, that is, $(x(1), \hat{p}^{x(1)}) \not\in W^{M'}{R'_1, \emptyset}$. Thus, by $\hat{x}_1 = 0, \hat{x}_1 \not\in D(R'_1, \hat{p})$.

However, since $(\hat{z}, \hat{p}) \in W_{\min}^{M',N''}(R'_1, R'_{m'+1}, R_{N'\setminus\{1\}}), this is a contradiction. Thus, $x(1) \neq x_1$.

**Induction argument:** Let $k \in \{2, \ldots, m'\}$. Assume that Claim A.1 holds until $k - 1$. Since $x(k - 1) \in D(R_{i(k-1)}, \hat{p}), \hat{x}_{i(k-1)} \in D(R_{i(k-1)}, \hat{p}),$ and $\hat{p}^{x(k-1)} < p^{x(k-1)}$, Lem. A.2-(ii) implies that $x(k) \equiv \hat{x}_{i(k-1)} \neq 0$ and $\hat{p}^{x(k)} < p^{x(k)}$.

Note that by Step 1, $x(k) \neq 0$ implies that agent $i(k)$ with $x_{i(k)} = x(k)$ uniquely exists. Thus, $x(k), i(k)$, and $t(k)$ are well-defined.

If $\hat{p}^{x(k)} > t(k) = CV_{i(k-1)}(x(k); z_0(k-1))$, then

$$(x(k-1), \hat{p}^{x(k-1)}) \not\in W^{M',N''}(R'_1, R'_{m'+1}, R_{N'\setminus\{1\}}).$$

When $x(k) \neq x_1$. Then, by Step 3 and the def. of $R'_1, \hat{p}^{x(k)} \leq t(k) < b \leq CV'_1(x_1; \emptyset)$. Thus, $(x(k), \hat{p}^{x(k)}) \not\in W^{M'}{R'_1, \emptyset}$. Then, by $\hat{x}_1 = 0, \hat{x}_1 \not\in D(R'_1, \hat{p})$. This contradicts $(\hat{z}, \hat{p}) \in W_{\min}^{M',N''}(R'_1, R'_{m'+1}, R_{N'\setminus\{1\}}).$ Thus, $x(k) \neq x_1$.

**Step 5.** Concluding that no set is weakly underdemanded at $p$ for $R$.

Note that by the definition of $R'_1, d_1 > 0$. Since $(\hat{z}, \hat{p}) \in W_{\min}^{M',N''}(R'_1, R'_{m'+1}, R_{N'\setminus\{1\}})$, Step 2 and Fact 3.5 imply that $\hat{p} \leq p^{M'}$. Note that

$$(\hat{x}_1, \hat{p}^{x_1}) \not\in D(R'_1, \hat{p}) \quad \text{the def. of } R'_1 \quad (x_1, CV'_1(x_1; \emptyset)) \quad P'_1$$
By Steps 1 and 4, \( x_1 \neq 0 \) and \( \hat{x}_1 \neq 0 \). By the def. of \( \hat{R}_1 \) and Remark 3.1, \( (\hat{x}_1, \hat{p}^{\hat{x}_1}) P_1 (x_1, p^{\hat{x}_1}) \). Then,

\[
(\hat{x}_1, \hat{p}^{\hat{x}_1}) P_1 (x_1, p^{\hat{x}_1}) R_1 (x_1, p^{\hat{x}_1}).
\]

Thus, \( \hat{p}^{\hat{x}_1} < p^{\hat{x}_1} \). By the def. of \( R'_1 \), \( R_1 \) is the \((-d_1)\)-truncation of \( R'_1 \) and \(-d_1 \leq 0 \leq -CV'_1(0; \hat{z}_1) \). Then, Lem. A.1 implies \( \hat{p} \in P^{M',N'}(R_N',R_{m'+1}) \). However, by Step 2, \( p^{M'} = p^{M',N'}_{\min}(R_N',\hat{R}_{m'+1}) \). By \( \hat{p} \leq p^{M'} \) and \( \hat{p}^{\hat{x}_1} < p^{\hat{x}_1} \), this is a contradiction. This completes the proof of Thm. 3.1.

\[\square\]

**Proof of Corollary 3.1.** Suppose that for each \( i \in N \), \( p^i_{\min}(R) > 0 \). Then, for each \( i \in N \), \( x_i \neq 0 \). Let \( M \equiv \{x_1, \ldots, x_n\} \). Then, \( |M| = |N| \). Since \( |M| = |\{i \in N : D(R_i,p) \cap M \neq \emptyset\}| \), \( M \) is weakly underdemanded at \( p \) for \( R \). This contradicts Thm. 3.1.

\[\square\]

**Proof of Corollary 3.2.** Let \( x \in M \) be such that \( p^x > 0 \). We construct the sequence of agents in two steps.

**Step 1.** By (WE-ii) in Def. 3.1, there is \( j_1 \in N \) such that \( x_{j_1} = x \). By Thm. 3.1, the set \( \{x\} \) is demanded at \( p \) by at least two agents. Thus, there is \( j_2 \in N \setminus \{j_1\} \) such that \( x \in D(R_{j_2},p) \). Let \( N_2 \) be the set of such agents. If \( x_{j_2} = 0 \) or \( p_{j_2}^x = 0 \) for some agent \( j_2' \in N_2 \), then let \( j_2 = j_2' \) and go to Step 2. If \( x_{j_2} \neq 0 \) and \( p_{j_2}^x > 0 \) for each \( j_2' \in N_2 \), pick arbitrarily an agent \( j_2 \in N_2 \).

By Thm. 3.1, the set \( \{x_{j_1}, x_{j_2}\} \) is demanded at \( p \) by at least three agents. Thus, there is \( j_3' \in N \setminus \{j_1, j_2\} \) such that \( D(R_{j_3},p) \cap \{x_{j_1}, x_{j_2}\} \neq \emptyset \). Let \( N_3 \) be the set of such agents. If \( x_{j_3'} = 0 \) or \( p_{j_3}' = 0 \) for some agent \( j_3'' \in N_3 \), then let \( j_3 = j_3'' \) and go to Step 2. If \( x_{j_3} \neq 0 \) and \( p_{j_3}^x > 0 \) for each \( j_3' \in N_3 \), pick arbitrarily an agent \( j_3 \in N_3 \).

Since \( m \) is finite, proceeding inductively, there are \( K' \leq m \) and a sequence \( \{j_k\}_{k=1}^{K'} \) of \( K' \) distinct agents such that (a) \( x_{j_k} = 0 \) or \( p_{j_k}^x = 0 \), (b) \( x_{j_1} = x \), and (c) for each \( k \in \{2, \ldots, K'\} \), \( \{x_{j_1}, \ldots, x_{j_{k-1}}\} \cap D(R_{j_k},p) \neq \emptyset \). Then go to Step 2.

**Step 2.** Let \( i_1 \equiv j_{K'} \). By \( (z,p) \in W(R) \), \( x_{i_1} \in D(R_{i_1},p) \). By (c), there is \( i_2 \in \{j_1, \ldots, j_{K'-1}\} \) such that \( x_{i_2} \in D(R_{i_2},p) \). By \( (z,p) \in W(R) \), \( x_{i_2} \in D(R_{i_2},p) \). By (c), there is \( i_3 \in \{j_1, \ldots, j_{K'-1}\} \) such that \( x_{i_3} \in D(R_{i_2},p) \). Proceeding inductively, we have some \( K \) such that \( i_K = j_1 \). Then, the sequence \( \{i_k\}_{k=1}^{K} \) of \( K \) distinct agents satisfies (i), (ii) and (iii).

\[\square\]

**B  Proofs for Section 4 (Main results: Fact 4.1 and Theorem 4.1)**

**Proof of Fact 4.1.** Let \( R \subseteq R^E \). Let \( g \) be an MPW rule on \( R^n \). By contradiction, suppose that there are \( R \subseteq R^n \), \( N' \subseteq N \), and \( R_{N'}, g_{R_{N'}, R_{N'}} \) such that for each \( i \in N' \), \( g_{i}(R_{N'},R_{N'}) P_i g_r(R) \). Let \( z \equiv g(R) \) and \( z' \equiv g(R_{N'},R_{N'}) \), with associated equilibrium prices \( p \) and \( p' \). Without loss of generality, let \( N' = \{1, \ldots, n'\} \). Let \( M^+ \equiv \{x \in M : 0 < p^x\} \) and \( m^+ \equiv |M^+| \). Note that, if \( n > m \), then \( n > m^+ \), and if \( m \leq n \), then by Cor. 3.1, \( m^+ < n-1 < n \).

In this paragraph, we show that for each \( i \in N' \), \( x_i' \neq 0 \) and \( p^{x_i'} < p^{x_i} \). Let \( i \in N' \). Note that

\[
(x_i', p^{x_i'}) P_i g_{i}(R_{N'},R_{N'}) P_i g_r(R) (x_i, p^{x_i}) R_i (x_i, p^{x_i}) R_i (x_i, p^{x_i}).
\]
Thus, $x'_i \neq 0$. Also,

$$(x'_i, p^{x'_i}) P_i (x_i, p^{x_i}) R_i \forall x_i \in D(R, p) (x'_i, p^{x'_i}).$$

Thus, $(x'_i, p^{x'_i}) P_i (x_i, p^{x_i})$ implies that $p^{x'_i} < p^{x_i}$. 

For each $i \in N'$, since $0 \leq p^{x'_i} < p^{x_i}$, $x'_i \in M^+$. Then, if $m^+ < n'$, more than $m^+$ agents receive the objects in $M^+$, which is a contradiction. Thus, assume that $m^+ \geq n'$. By Thm. 3.1, there is $i' \in N \setminus N'$ such that $D(R_{i'}, p) \cap \{x'_1, \ldots, x'_n\} \neq \emptyset$. Without loss of generality, let $i' \equiv n' + 1$. By Lem. A.2-(ii), $x'_{n'+1} \neq 0$ and $0 \leq p^{x'_{n'+1}} < p^{x'_{n'+1}}$. Thus, $x'_{n'+1} \in M^+$. Then, by Thm. 3.1, there is $i'' \in N \setminus \{1, \ldots, n' + 1\}$ such that $D(R_{i''}, p) \cap \{x'_{1}, \ldots, x'_{n'+1}\} \neq \emptyset$. Without loss of generality, let $i'' \equiv n' + 2$. Thus, by Lem. A.2-(ii), $x'_{n'+2} \neq 0$ and $0 \leq p^{x'_{n'+2}} < p^{x'_{n'+2}}$. Thus, $x'_{n'+2} \in M^+$. Repeat this argument $(m^+ - n' + 1)$ times. Then, more than $m^+$ agents receive objects in $M^+$. This is impossible. \hfill \Box

Next, we prove Theorem 4.1. Let $R \equiv R^C$ and $n > m$. Let $f$ be a rule on $R^n$. Since the if part of Theorem 4.1 follows from the discussion in Subsection 4.1, we only give the proof of the only if part of the theorem.

B.1 Difficulties and overview of the proof of Theorem 4.1

We explain the difficulties of the proof, compared to the previous works, and give an overview of the proofs.

First, we discuss the difficulties of our proof, compared to the literature assuming quasi-linearity such as Holmström (1979), etc. As emphasized in Introduction, without quasi-linearity, efficient allocations of objects depend on agents’ payments. Thus, it is difficult to identify the object allocations of the rules satisfying our desirable properties without knowing how many agents pay. On the other hand, it is also difficult to identify the payments of the rules satisfying our properties without knowing how objects are allocated. Therefore, without quasi-linearity, we need to identify simultaneously the object allocation and payments of the rules. This is similar to solving simultaneous equations, and is much more difficult than identifying only the payments by assuming quasi-linearity.

Second, we discuss the difficulties of our proof, compared to Miyake (1998), who shows that only the MPW rule satisfies strategy-proofness among Walrasian rules. The Walrasian rules are a small part of the class of allocation rules satisfying efficiency, individual rationality, and no subsidy for losers. Thus, his proof does not exclude the possibility that the rules satisfying our properties are not among Walrasian rules. When an agent changes his preference, our properties restrict the possibility of allocation changes. If the rule is assumed to be among Walrasian rules, this possibility of allocation is limited to a small class. This fact makes the identification of the allocation of rules relatively easy. Since we establish the uniqueness of the rules satisfying the four properties without confinement to Walrasian rules, our proof is much more difficult.

Third, we discuss how we overcome the above difficulties. The widely employed methods to solve simultaneous equations are constructing algorithms to reach the solutions step by step. In our proof, we employ a similar method, that is, we analyze how the above four properties restrict the possibility of allocations step by step. Lemma 4 states that payments are bounded
below by the \((m + 1)\)-th highest compensated valuations from the origin. Lemmas 7 and 8 restrict the possibility of object allocations in turn, i.e., Lemma 7 restricts the candidates who obtain a real object; Lemma 8 gives a sufficient conditions that an agent obtains a specific real object. These lemmas enable us to prove that agents pay at most the minimum Walrasian prices (Proposition 1).

As Corollary 3.2 states, the MPWE allocation has a structure, called “demand connectedness.” Lemma 10 states that the allocation chosen by the rules satisfying our four properties has a similar structure for special preference profiles. Lemma 12 states that if an agent obtains an object but pays less the minimum Walrasian price, whenever the object is connected to the origin by the demands of agents who pay the minimum Walrasian price, there is a Pareto-improvement. These lemmas enable us to prove that agents pay at least the minimum Walrasian prices (Proposition 2 and the proof of Theorem 4.1).

Our proof has four parts.

Part 1. The following six lemmas are used in the proof.

First, under individual rationality and no subsidy for losers, whenever an agent receives the null object, he pays nothing.

**Lemma 1 (Zero-payment for losers).** Let \( f \) satisfy individual rationality and no subsidy for losers. Let \( R \in \mathcal{R}^n \) and \( i \in N \) be such that \( f^+_i(R) = 0 \). Then, \( f^+_i(R) = 0 \).

Under efficiency, individual rationality, and no subsidy for losers, each real object should be assigned to someone.

**Lemma 2 (No remaining object).** Let \( f \) satisfy efficiency, individual rationality, and no subsidy for losers. Let \( R \in \mathcal{R}^n \) and \( x \in M \). Then, there is \( i \in N \) such that \( f^+_i(R) = x \).

Given an allocation and a pair \( \{i, j\} \) of agents such that agent \( i \) receives a real object and prefers his assignment at least as desirable as \( j \)'s, but \( j \) prefers \( i \)'s assignment to his own, if the difference between \( j \)'s payment and \( i \)'s compensated valuation (CV) of \( j \)'s assignment of objects from \( i \)'s assignment is less than the difference between \( i \)'s payment and \( j \)'s CV of \( i \)'s assignment of objects from \( j \)'s assignment, then a Pareto-improvement is possible.

**Lemma 3 (Sufficient condition for a Pareto-improvement to be possible).** Let \( R \in \mathcal{R}^n \), \( i, j \in N \), and \( z \in Z \) be such that \( x_i \neq 0 \), \( z_i R_i z_j \), and \( z_i P_j z_j \). Assume that \( (a) \ t_j - CV_i(x_j; z_i) < CV_j(x_i; z_j) - t_i \). Then, there is \( z' \in Z \) that Pareto-dominates \( z \) at \( R \).

We introduce additional notations. Given \( R \in \mathcal{R}^n \), \( x \in M \), and \( z \in (L \times \mathbb{R})^n \), let \( \pi^x(R) \equiv (\pi^x_1(R), \ldots, \pi^x_n(R)) \) be the permutation on \( N \) such that \( CV_{\pi^x_1(R)}(x; z_{\pi^x_1(R)}) \leq \cdots \leq CV_{\pi^x_n(R)}(x; z_{\pi^x_n(R)}) \). For each \( k \in N \), let \( C^k(R, x; z) \equiv CV_{\pi^x_k(R)}(x; z_{\pi^x_k(R)}) \). That is, \( C^k(R, x; z) \) is the \( k \)-th highest compensated valuation (CV) of \( x \) from \( z \). We simply write \( C^k(R, x; (0, \ldots, 0)) \) as \( C^k(R, x) \).

Under the four axioms of Theorem 4.1, if an agent receives \( x \in M \), then he pays at least the \((m + 1)\)-th highest CV of \( x \) from the origin. Thus, the \((m + 1)\)-th highest CV of each object from \( 0 \) is a lower bound for the payment of the agent who obtains the object.

23
Lemma 4 (Payment lower bound). Let $f$ satisfy the four axioms of Theorem 4.1. Let $R \in \mathcal{R}^n$, $i \in N$, and $x \in M$. If $f_i^x(R) = x$, then, $f_i^1(R) \geq C^{m+1}(R, x)$.

No subsidy is implied by our four axioms.

Lemma 5 (No subsidy). The four axioms of Theorem 4.1 imply no subsidy.

Hereafter, we use this implication repeatedly.
Given $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$ with $x_i \neq 0$, let $\mathcal{R}_{NCV}(z_i)$ be the set of preferences $R_i' \in \mathcal{R}$ such that for each $y \in L \setminus \{x_i\}$, $CV_i(y; z_i) < 0$, that is, for each object except for $x_i$, the compensated valuation of $R_i'$ from $z_i$ is negative. We refer to the preferences in $\mathcal{R}_{NCV}(z_i)$ as "$z_i$-favoring".

Under strategy-proofness and no subsidy for losers, given $R \in \mathcal{R}^n$, for each agent who receives a real object, if the agent’s preference is changed to a preference that is $f_i(R)$-favoring, then his assignment remains the same.

Lemma 6 (Invariance property). Let $f$ satisfy strategy-proofness and no subsidy. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $f_i^x(R) \neq 0$. Let $R_i' \in \mathcal{R}_{NCV}(f_i(R))$. Then, $f_i(R_i', R_{-i}) = f_i(R)$.

Part 2. The next proposition says that for each preference profile, the allocation chosen by a rule satisfying the four axioms of Theorem 4.1 should (weakly) dominate the MPWE allocations from the bidders’ perspective. This implies that for a rule satisfying our properties, the agent who receives $x \in M$ pays at most the minimum Walrasian price $p^\pi$. Thus, Proposition 1 implies stringent upper bounds for payments even without knowing how objects are allocated.

Proposition 1. Let $f$ satisfy the four axioms of Theorem 4.1. Let $R \in \mathcal{R}^n$ and $z \in W_{\text{min}}(R)$. Then, for each $i \in N$, $f_i(R) \geq 0$.  

We introduce two lemmas to prove Proposition 1. Hereafter, we maintain the assumption that $f$ satisfies the four axioms of Theorem 4.1. Then, by Lemma 5, $f$ also satisfies no subsidy.

From Lemma 4, we deduce that if an agent receives $x \in M$, then his CV of $x$ from $0$ is no less than the $m$-th highest CV of $x$ from $0$. For each $x \in M$, Lemma 7 restricts who can obtain $x$ without knowing how much agents pay.

Lemma 7 (Necessary condition for receiving $x \in M$). Let $R \in \mathcal{R}^n$, $i \in N$, and $x \in M$. If $f_i^x(R) = x$, then, $CV_i(x; 0) \geq C^{m}(R, x)$.

By Lemma 7, the assumption (a) of Lemma 8 implies that for any real object other than $x \in M$, an agent’s CV from $0$ is less than the $m$-th highest, then he never receives a real object other than $x$. Together with this condition, (b) and (c) of Lemma 8 guarantee that agent $i$ receives $x$.

Given $R \in \mathcal{R}^n$, let $Z^{IR}(R)$ be the set of individually rational allocations, that is, $Z^{IR}(R) \equiv \{z \in Z : \text{ for each } i \in N, z_i \in R_i \}$.  

Lemma 8 (Sufficient condition for receiving $x \in M$). Let $R \in \mathcal{R}^n$, $x \in M$, $i \in N$, and $z \in Z^{IR}(R)$. Assume that (a) for each $y \in M \setminus \{x\}$, $CV_i(y; 0) < C^{m}(R, y)$, (b) for each $j \in N \setminus \{i\}$, $f_j(R) \geq C^{1}(R_{-i}, x; z)$, and (c) $CV_i(x; 0) > C^{1}(R_{-i}, x; z)$. Then, $f_i^x(R) = x$.

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46This result also holds for any Walrasian equilibrium allocation $z$.  

24
Part 3. Given \( z \in Z(R) \), let \( \mathcal{R}^I(z) \) be the set of preferences \( R_i \in \mathcal{R} \) such that for each pair \( i, j \in N, z_i I_i z_j \), that is, all the assignments under \( z \) are indifferent. We refer to the preferences in \( \mathcal{R}^I(z) \) as “\( z \)-indifferent”.

Proposition 2 says that given \((z^*, p) \in W_{\text{min}}(R)\) and a preference profile such that a group \( N' \) of agents have \( z^*\)-indifferent preferences, if for any \( z^*\)-indifferent preferences of \( N' \) and each \( x \in M \), the agent outside \( N' \) who obtains \( x \) pays at least \( p^x \), then for each \( x \in M \), the agent in \( N' \) who obtains \( x \) pays at least \( p^x \). Thus, although in a limited pattern, this proposition implies lower bounds for payments even without knowing how objects are allocated.

**Proposition 2.** Let \( R \in \mathcal{R}^n \), \((z^*, p) \in W_{\text{min}}(R)\), and \( N' \subseteq N \). Assume that (2-i) for each \( \bar{R}_{N'} \in \mathcal{R}^I(z^*)|^{N'} \), each \( i \in N \setminus N' \), and each \( x \in M \), if \( f^x_i(\bar{R}_{N'}, R_{-N'}) = x \), \( f^x_i(\bar{R}_{N'}, R_{-N'}) \geq p^x \). Let \( \bar{R}_{N'} \in \mathcal{R}^I(z^*)|^{N'} \). Then, for each \( i \in N' \) and each \( x \in M \), if \( f^x_i(\bar{R}_{N'}, R_{-N'}) = x \), then \( f^x_i(\bar{R}_{N'}, R_{-N'}) \geq p^x \).

We introduce four lemmas to prove Proposition 2.

Given \((z^*, p) \in W_{\text{min}}(R)\), if a group of agents change their preferences to \( z^*\)-indifferent preferences, then for the new preference profile, (a) \( z^* \in Z_{\text{min}}(R) \), and (b) the allocation chosen by the rule \( f \) (weakly) dominates \( z^* \).

**Lemma 9.** Let \( R \in \mathcal{R}^n \), \((z^*, p) \in W_{\text{min}}(R)\), \( N' \subseteq N \), \( \bar{R}_{N'} \in \mathcal{R}^I(z^*)|^{N'} \), and \( \bar{R} \equiv (\bar{R}_{N'}, R_{-N'}) \). Then, (a) \((z^*, p) \in W_{\text{min}}(R)\), and (b) for each \( i \in N \), \( f_i(\bar{R}) z^*_i \).

Given \( p \in \mathbb{R}^m_{++} \) and \( R \in \mathcal{R}^n \), let \( N(R, p) \) denote the set of demanders of the real objects at prices \( p \), that is, \( N(R, p) \equiv \{ i \in N : D(R_i, p) \cap M \neq \emptyset \} \).

Lemma 10 says that given \((z^*, p) \in W_{\text{min}}(R)\) and \( \bar{R} \in \mathcal{R}^n \) such that a group \( N' \) of agents have \( z^*\)-indifferent preferences, if (10-i) for each object \( x \), the agent outside \( N' \) who obtains \( x \) pays at least \( p^x \), and (10-ii) each agent in \( N' \) receives a real object, then (10-a) each agent demanding only the null object at prices \( p \) receives the null object, (10-b) let \( \bar{z} \) be the allocation defined by setting for each \( i \in N' \), \( z_i \equiv (y_i, p^{y_i}) \), where \( y_i \equiv f^x_i(\bar{R}) \), and for each \( j \) outside \( N' \), \( z_j \equiv f^x_j(\bar{R}) \), then \( \bar{z} \in Z_{\text{min}}(R) \), and (10-c) an object obtained by a \( z^*\)-indifferent agent is connected to the null object by the demands of non \( z^*\)-indifferent agents.

**Lemma 10.** Let \( R \in \mathcal{R}^n \) and \((z^*, p) \in W_{\text{min}}(R)\). Let \( N' \subseteq N \) with \( 1 \leq |N'| \leq m \), \( \bar{R}_{N'} \in \mathcal{R}^I(z^*)|^{N'} \), \( \bar{R} \equiv (\bar{R}_{N'}, R_{-N'}) \) and \( N'' \equiv N(R, p) \setminus N' \). For each \( i \in N' \), let \( \bar{z}_i \equiv (y_i, p^{y_i}) \), where \( y_i \equiv f^x_i(\bar{R}) \). For each \( i \in N \setminus N' \), let \( \bar{z}_i \equiv f^x_i(\bar{R}) \). Assume that (10-i) for each \( i \in N \setminus N' \), and each \( x \in M \), if \( f^x_i(\bar{R}) = x \), then \( f^x_i(\bar{R}) \geq p^x \), and (10-ii) for each \( j \in N' \), \( f^x_j(\bar{R}) \neq 0 \). Then, (10-a) for each \( j \notin N(R, p) \cup N' \), \( f^x_j(\bar{R}) = 0 \), (10-b) \( \bar{z} \in Z_{\text{min}}(\bar{R}) \), and (10-c) there is a sequence \( \{i_k\}_{k=1}^K \) of \( K \) distinct agents such that (i) \( K \in \{2, \ldots, m+1\} \), (ii) \( f^x_i(\bar{R}) = 0 \), (iii) for each \( k \in \{1, \ldots, K-1\} \), \( i_k \in N'' \), and \( i_k \in N' \), and (iv) for each \( k \in \{1, \ldots, K-1\} \), \( \{f^x_{i_k}(\bar{R}), f^x_{i_{k+1}}(\bar{R})\} \subseteq D(R_{i_k}, p) \).

When an agent \( i \) receives \( x \in M \) and his CV of the null object from his assignment is negative, for each agent \( j \neq i \), if \( j \)'s CV of \( x \) from \( 0 \) is greater than the difference between what \( i \) pays and \( i \)'s CV of the null object from his assignment, then agent \( j \) receives a real object.
Lemma 11. Let \( R \in \mathcal{R}^n \), \( i \in N \), and \( x \in M \) be such that \( f_i^x(R) = x \) and \( CV_i(0; f_i(R)) < 0 \). Let \( j \in N \setminus \{i\} \). Assume that (10-i) \(-CV_i(0; f_i(R)) < CV_j(x; 0) - f_i^r(R)\). Then, \( f_j^x(R) \neq x \).

Given a preference profile such that an object \( x \) obtained by an agent \( j \) who pays less than \( p^x \) is connected to the null object by the demands of the agents who pay prices \( p \), if \( p^x \) is greater than the difference between what \( j \) pays and his \( CV \) of the null object from his assignment, then a Pareto-improvement is possible.

Lemma 12. Let \( R \in \mathcal{R}^n \) and \((z^*, p) \in W_{\min}(R)\). For each \( i \in N \), let \( x_i \equiv f_i^x(R) \). Assume that there is a sequence \( \{i_k\}_{k=1}^K \) of \( K \) distinct agents such that (a) \( 2 \leq K \leq m + 1 \), (b) \( f_i^x(R) = 0 \), (c) for each \( k \in \{1, \ldots, K-1\} \), \( \{f_i^x(R), f_{i_{k+1}}^x(R)\} \subseteq D(R_{i_k}, p) \) and \( f_i^x(R) = p^x_i \), and (d) \( f_{i_{k+1}}^x(R) < p^x_i \) and \( -CV_{i_{k+1}}(0; f_{i_{k+1}}(R)) < p^x_i - f_{i_{k+1}}^x(R) \). Then, there is an allocation \( z' \) that Pareto-dominates \( f(R) \) at \( R \).

Part 4. We complete the proof of Theorem 4.1.

Sketch of proof of Theorem 4.1. Let \( R \in \mathcal{R}^n \) and \((z^*, p) \in W_{\min}(R)\).

Let \( \bar{R} \in \mathcal{R}^n \) be a profile of \( z^* \)-indifferent preferences. Then, for each \( x \in M \), the \((m+1)\)-th highest \( CV \) of \( x \) from the origin is equal to \( p^x \). Thus, by Lem. 4, for each \( x \in M \), the agent who obtains \( x \) pays at least \( p^x \). We replace the preferences in \( \bar{R} \) by the original preferences in \( R \) one by one, and inductively show that for each \( x \in M \), the agent who obtains \( x \) pays at least \( p^x \).

Step 1: We replace the preference \( \bar{R}_i \) in \( \bar{R} \) of agent \( i \in N \) by his original preference \( R_i \). Then, if agent \( i \) obtains \( x \) at the new profile \((R_i, \bar{R}_{-i})\), then \( f_i^x(R_i, \bar{R}_{-i}) \geq p^x \). For otherwise since \( \bar{R}_i \) is \( z^* \)-indifferent, \( f_i(R_i, \bar{R}_{-i}) \bar{P}_i f_i(R_i) \), contradicting strategy-proofness. Then, by Prop. 2, for each \( x \in M \), the remaining agent who obtains \( x \) pays also at least \( p^x \).

Step 2: We replace the preference \( \bar{R}_j \) in \((R_i, \bar{R}_{-i})\) of agent \( j \neq i \) by his original preferences \( R_j \). Then, if agent \( i \) obtains \( x \) at the new profile \((R_{i,j}, \bar{R}_{-i,j})\), then \( f_i^x(R_{i,j}, \bar{R}_{-i,j}) \geq p^x \). For otherwise since \( \bar{R}_i \) is \( z^* \)-indifferent, Step 1 implies \( f_i(R_{i,j}, \bar{R}_{-i,j}) \bar{P}_i f_i(R_j, \bar{R}_{-j}) \), contradicting strategy-proofness. Similarly, if agent \( j \) obtains \( x \) at \((R_{i,j}, \bar{R}_{-i,j})\), then \( f_j^x(R_{i,j}, \bar{R}_{-i,j}) \geq p^x \). Then, Prop. 2 implies that for each \( x \in M \), the remaining agent who obtains \( x \) pays also at least \( p^x \).

Proceeding inductively, we conclude that, at \( R \), for each \( x \in M \), the agent who obtains \( x \) pays at least \( p^x \). Then, from Prop. 5.1, we deduce that each agent is assigned an object in his demand set at prices \( p \) and pays its price. Thus, (WE-i) in Def. 3.1 holds. Since \( \mathcal{R} \equiv \mathcal{R}^C \) and \( n > m \), for each \( x \in M \), \( p^x > 0 \). By Lem. 2, each object is assigned to someone. Thus, (WE-ii) in Def. 3.1 also holds. Since \( p = p_{\min}(R) \), \( f(R) \in W_{\min}(R) \).

\[ \square \]

B.2 Formal proof of Theorem 4.1.

Part 1: Proof of Lemma 1. By no subsidy for losers, \( f_i^1(R) \geq 0 \). By individual rationality, \( f_i^1(R) \leq 0 \). Thus, \( f_i^1(R) = 0 \).

Proof of Lemma 2. By contradiction, suppose that for each \( i \in N \), \( f_i^x(R) \neq x \). Then, by \( n > m \), there is \( j \in N \) such that \( f_j^x(R) = 0 \). By Lem. 1, \( f_j^x(R) = 0 \). Let \( z' \in Z \) be such that
\[ z'_i \equiv (x, 0) \text{ and for each } i \in N \setminus \{j\}, \quad z'_i \equiv f_i(R). \] Then, since \((x, 0) P_i (0, 0), z'_j P_j f_j(R)\). Note that for each \(i \in N \setminus \{j\}, z'_i I_i f_i(R), \) and \(\sum_{i \in N} t'_i = \sum_{i \in N} f'_i(R). \) Thus, \(z'\) Pareto-dominates \(f(R)\) at \(R\), which contradicts efficiency. □

**Proof of Lemma 3.** Let \(d \equiv t_j - CV_i(x_j; z_i)\) and \(z' \in Z\) be defined by setting \(z'_i \equiv (x_j, t_j - d), z'_j \equiv (x_i, t_i + d), \) and for each \(k \in N \setminus \{i, j\}, z'_k \equiv z_k. \) By \(z'_i = (x_j, CV_i(x_j; z_i)), z'_i I_i z_i. \) By (a) and \(z'_j = (x_i, t_i + t_j - CV_i(x_j; z_i)), z'_j P_j (x_i, CV_j(x_i; z_j)) I_j z_j. \) For each \(k \in N \setminus \{i, j\}, z'_k I_k z_k\) and \(\sum_{k \in N} t'_k = t_j - d + t_i + d + \sum_{k \neq i,j} t_k = \sum_{k \in N} t_k. \) Thus, \(z'\) Pareto-dominates \(z\) at \(R\). □

**Proof of Lemma 4.** First, for each \(y \in M\) and each \(i \in N, (y, 0) P_i (0, 0). \) Thus, for each \(y \in M, C^{m+1}(R, y) > 0. \) By contradiction, suppose that \(f^*_i(R) = x\) and \(f^*_j(R) < C^{m+1}(R, x). \) Let \(R'_i \in \mathcal{R}^Q\) be such that for each \(y \in M, 0 < CV'_i(y; 0) < C^{m+1}(R, y)\) and \(f^*_j(R'_i, R_{-i}) < CV'_i(y; f_i(R)). \) Since \(CV'_i(0; f_i(R)) < 0, \) no subsidy for losers implies \(y' \neq 0. \)

Since \(|\{j \in N \setminus \{i\} : CV'_i(y'; 0) \geq C^{m+1}(R, y')\}| \geq m, \) there is \(j \in N \setminus \{i\}\) such that \(CV'_j(y'; 0) \geq C^{m+1}(R, y')\) and \(f^*_j(R'_i, R_{-i}) = 0. \) By Lem. 1, \(f^*_j(R'_i, R_{-i}) = 0. \)

Let \(z'_i \equiv (0, CV_i(0; f_i(R'_i, R_{-i}))), z'_j \equiv (y', CV_i(0; 0)), \) and for each \(k \neq i, j, z'_k \equiv f_k(R'_i, R_{-i}). \) Then, \(z'_i I_i f_i(R'_i, R_{-i}), \) and for each \(k \neq i, j, z'_k I_k f_k(R'_i, R_{-i}). \) By \(CV'_i(0; 0) > CV'_i(y'; 0)\), \(z'_j P_j f_j(R'_i, R_{-i}). \) Since \(R'_i \in \mathcal{R}^Q, CV'_i(0; f_i(R'_i, R_{-i})) = f^*_i(R'_i, R_{-i}) - CV'_i(y'; 0). \) Thus, \(t_i + t_j = CV'_i(0; f_i(R'_i, R_{-i})) + CV'_i(y'; 0) = f^*_i(R'_i, R_{-i}). \) Then, \(f^*_j(R'_i, R_{-i}) = 0, \) \(\sum_{k \in N} t_k = \sum_{k \in N} f'_k(R'_i, R_{-i}). \) Thus, \(z'\) Pareto-dominates \(f(R'_i, R_{-i})\) at \((R'_i, R_{-i})\), which contradicts efficiency. □

**Proof of Lemma 5.** Let \(f\) satisfy the four axioms of Theorem 4.1 on \(\mathcal{R}^n. \) Let \(R \in \mathcal{R}^n, i \in N, \) and \(x \equiv f^*_i(R). \) If \(x = 0, \) Lem. 5 follows from no subsidy for losers. Thus, suppose that \(x \neq 0. \) Then, by Lem. 1, \(f^*_i(R) \geq C^{m+1}(R, x). \) Since for each \(y \in M\) and each \(i \in N, (y, 0) P_i (0, 0), \) for each \(y \in M, C^{m+1}(R, y) > 0. \) Thus, \(f^*_i(R) > 0. \) □

**Proof of Lemma 6.** First, we show \(f^*_j(R'_i, R_{-i}) = f^*_j(R). \) Suppose not. Let \(x \equiv f^*_j(R'_i, R_{-i}). \) By strategy-proofness, \(f_i(R'_i, R_{-i}) R'_i f_i(R), \) and so, \(f^*_j(R'_i, R_{-i}) \leq CV_j(x; f_i(R)). \) Since \(R'_i \in \mathcal{R}_{NCV}(f_i(R)), CV'_i(x; f_i(R)) < 0. \) Thus, \(f^*_i(R'_i, R_{-i}) < 0, \) contradicting no subsidy.

Next, we show \(f^*_j(R'_i, R_{-i}) = f^*_j(R). \) Suppose that \(f^*_j(R'_i, R_{-i}) < f^*_j(R). \) (The opposite case can be treated symmetrically.) Then, \(f_i(R'_i, R_{-i}) P_i f_i(R), \) contradicting strategy-proofness. □

**Part 2: Proof of Proposition 1.**

**Proof of Lemma 7.** By contradiction, suppose that \(f^*_i(R) = x\) and \(CV_i(x; 0) < C^{m+1}(R, x)\). Then, by Lem. 4, \(C^{m+1}(R, x) \leq f^*_i(R). \) By individual rationality, \(f^*_i(R) \leq CV_i(x; 0). \) Then, by \(CV_i(x; 0) \leq C^{m+1}(R, x), f^*_i(R) = CV_i(x; 0). \) Since \(|\{j \in N : CV_j(x; 0) \geq C^{m+1}(R, x)\}| = m, \) there is \(j \in N \setminus \{i\}\) such that \(CV_j(x; 0) \geq C^{m+1}(R, x)\) and \(f^*_j(R) = 0. \) By Lem. 1, \(f^*_j(R) = 0. \) Then, by \(CV_i(x; f_i(R)) = 0 \) and \(f^*_j(R) = CV_i(x; 0) < C^{m+1}(R, x) \leq CV_j(x; 0), f^*_j(R) - CV_i(x; f_i(R)) < CV_j(x; 0) - f^*_j(R). \) Note that \(x \neq 0. \) Thus, by Lem. 3, there is \(z \in Z\) that Pareto-dominates \(f(R)\) at \(R\), which contradicts efficiency. □

**Proof of Lemma 8.** (Fig. 3) By contradiction, suppose that \(f^*_i(R) \neq x. \) Then, by Lem. 2, there is \(j \in N \setminus \{i\}\) such that \(f^*_j(R) = x. \) Since \(f_j(R) R_j z_j, f^*_j(R) \leq CV_j(x; z_j) < CV_i(x; 0). \)
Since \( z \in Z^{IR}(R) \), for each \( y \in M \), \( CV_j(y; z_j) \leq CV_j(y; 0) \). Let \( R'_j \in R_{NCV}(f_j(R)) \) be such that (i) \(-CV'_j(0; f_j(R)) < CV_i(x; 0) - f'_j(R)\), and (ii) for each \( y \in M \setminus \{x\} \), \( CV'_j(y; 0) = CV_j(y; 0) \). Then, by Lem. 6, \( f_j(R'_j, R_{-j}) = f_j(R) \). Since \( f'_j(R'_j, R_{-j}) = x \), \( f'_j(R'_j, R_{-j}) \neq x \).

Next, we show that \( f'_j(R'_j, R_{-j}) \notin M \setminus \{x\} \). Suppose that there is \( y \in M \setminus \{x\} \) such that \( f'_j(R'_j, R_{-j}) = y \). By (ii), \( C^m(R'_j, R_{-j}, y) = C^m(R, y) \). Since \( CV_i(y; 0) < C^m(R, y) \), \( CV_i(y; 0) < C^m(R'_j, R_{-j}, y) \), which contradicts Lem. 7. Thus, \( f'_j(R'_j, R_{-j}) = 0 \). By Lem. 1, \( f'_j(R'_j, R_{-j}) = 0 \). Then, by (i) and Lem. 3, there is \( z' \in Z \) that Pareto-dominates \( f(R'_j, R_{-j}) \) at \( (R'_j, R_{-j}) \), which contradicts efficiency. \( \square \)

**Proof of Proposition 1.** We only show \( f_1(R) R_1 z_1 \) since the other agents can be treated in the same way. If \( x_1 = 0 \), then \( z_1 = 0 \), and so, by *individual rationality*, \( f_1(R) R_1 z_1 \). Thus, assume that \( x_1 \neq 0 \). Let \( N^+ \equiv \{ j \in N : x_j \neq 0 \} \). Note that \( |N^+| = m \).

By contradiction, suppose that \( z_1 P_1 f_1(R) \). We prove Claim 1 below by induction. (iv) of Claim 1 induces a contradiction by the finiteness of \( N^+ \).

**Claim 1.** For each \( k \geq 0 \), there exist a set \( N(k+1) \) of \( k+1 \) distinct agents, say \( N(k+1) \equiv \{1, \ldots, k+1\} \), and \( R'_{N(k+1)} \in R^{k+1} \) such that

(i) \( z_{k+1} P_{k+1} f_{k+1}(R'_{N(k)}, R_{-N(k)}) \),
(ii) for each \( j \in N(k+1) \) and each \( y \in M \setminus \{x_j\} \), \( CV'_j(y; 0) < C^m(R'_{\{1, \ldots, j-1\}}, R_{-\{1, \ldots, j-1\}}, y) \),
(iii) \( t_{k+1} < CV'_k(x_{k+1}; 0) < CV_{k+1}(x_{k+1}; f_{k+1}(R'_{N(k)}, R_{-N(k)})) \), and
(iv) \( N(k+1) \subset N^+ \), where \( N(k) \equiv \{1, \ldots, k\} \).
Proof. Step 1. Let $k = 0$ and $N(1) \equiv \{1\}$. By $z_1 P_1 f_1(R)$, (i-1) holds, and so, $t_1 < CV_1(x_1; f_1(R))$. Note that for each $y \in M$, $C^n(R, y) > 0$. Thus, there is $R'_1 \in \mathcal{R}$ such that (ii-1): for each $y \in M \setminus \{x_1\}$, $CV_1(y; 0) < C^n(R, y)$, and (iii-1): $t_1 < CV_1(x_1; 0) < CV_1(x_1; f_1(R))$.

Note that $\{1\} \subseteq N^+$. Suppose that $\{1\} = N^+$. Since $|N^+| = m$, $m = 1$. Thus, by $x_1 \neq 0$, for each $j \in N \setminus \{1\}, z_j = 0$. Since $z \in W(R)$, for each $j \in N \setminus \{1\}$, $z_j R_j z_1$, and so, $CV_j(x_1; 0) \leq t_1$. Thus, by (iii-1), $C^1(R_{-1}, x_1; z) \leq t_1 < CV_1(x_1; 0)$. By individual rationality, for each $j \in N \setminus \{1\}$, $f_j(R'_1, R_{-1}) R_j 0 = z_j$. Since $z \in Z^{IR}(R'_1, R_{-1})$, Lem. 8 implies $f'_1(R'_1, R_{-1}) = x_1$. By individual rationality, $f'_1(R'_1, R_{-1}) \leq CV_1(x_1; 0)$. However, by (iii-1), $f'_1(R'_1, R_{-1}) < CV_1(x_1; f_1(R))$. Thus, $f_1(R'_1, R_{-1}) P_1 f_1(R)$, contradicting strategy-proofness. Thus, (iv-1): $\{1\} \not\subseteq N^+$.

Step 2 (Induction argument). Let $k \geq 1$. As induction hypothesis, assume that there exist a set $N(k) \supseteq N(1)$ of $k$ distinct agents, say $N(k) \equiv \{1, \ldots, k\}$, and $R'_{N(k)} \in \mathcal{R}^k$ such that

(i) $z_k P_k f_k(R'_{N(k) \setminus \{k\}}, R_{-N(k) \setminus \{k\}})$,

(ii) for each $j \in N(k)$ and each $y \in M \setminus \{x_j\}$, $CV_j(y; 0) < C^n(R'_{\{1, \ldots, j-1\}, R_{-\{1, \ldots, j-1\}}} y)$,

(iii) $t_k < CV_k(x_k; 0) < CV_k(x_k; f_k(R'_{N(k) \setminus \{k\}}, R_{-N(k) \setminus \{k\}}))$,

(iv) $N(k) \not\subseteq N^+$.

By (iv-1), $N^+ \setminus N(k) \neq \emptyset$. The proof has two steps.

Step 2-1. There is $j \in N^+ \setminus N(k)$ such that $z_j P_j f_j(R'_{N(k)}, R_{-N(k)})$. 

Figure 4: Illustration of (i-(k + 1)), (ii-(k + 1)) and (iii-(k + 1)) in the proof of Proposition 1 for $k = 1$.
Proof. By contradiction, suppose that for each $j \in N^+ \setminus N(k)$, $f_j(R'_{N(k)}, R_{-N(k)}) R_j z_j$. Then, (ii-k) and (iii) follow from (ii-f) and for each $y$ without loss of generality, let $R_k$. We complete the proof of Claim 1. Thus, by (iii-f), $CV_j(x; z'_j) = CV_j(x_k; z'_j) = CV_j(x_k; z_j) \leq t_k$. By (ii-k), for each $j \in N(k) \setminus \{k\}$,

$$CV_j(x_k; z'_j) \leq CV_j(x_k; 0) = CV_j(R_{[1, \ldots, j-1]}, R_{-\{1, \ldots, j-1\}}, x_k) \leq CV_n(R, x_k) \leq t_k.$$ 

Thus, by (iii-k), $CV_j(R'_{N(k)}), R_{-N(k)}, x_k; z') \leq t_k < CV_j(x_k; 0)$.

Since the assumptions of Lem. 8 hold for $(R'_{N(k)}, R_{-N(k)})$ as above, Lem. 8 implies that $f'_k(R'_{N(k)}), R_{-N(k)}) = x_k$. By individual rationality, $f'_k(R'_{N(k)}), R_{-N(k)}) \leq CV_j(x_k; 0)$. However, (iii-k) implies that $f'_k(R'_{N(k)}), R_{-N(k)}) < CV_j(x_k; f_k(R'_{N(k)}), R_{-N(k)}, \{k\}))$. Thus, $f_k(R'_{N(k)}), R_{-N(k)}) P_k f_k(R'_{N(k)}), R_{-N(k)}, \{k\})$, contradicting strategy-proofness. 

Step 2-2. We complete the proof of Claim 1.

Proof. Without loss of generality, let $k + 1 \equiv k'$ and $N(k + 1) \equiv N(k) \cup \{k + 1\}$.

Then, $N(k + 1) \supseteq N(k)$, and (i) follows from $z_k' P_{k'} f_{k'}(R'_{N(k)}), R_{-N(k)})$. By (i), $t_{k+1} < CV_{k+1}(x_{k+1}; f_{k+1}(R'_{N(k)}), R_{-N(k)})$. Also, for each $y \in M$, $C^n(R'_{N(k)}, R_{-N(k)}, y) > 0$. Thus, there is $R'_{k+1} \in \mathcal{R}$ such that

$$t_{k+1} < CV_{k+1}(x_{k+1}; 0) < CV_{k+1}(x_{k+1}; f_{k+1}(R'_{N(k)}), R_{-N(k)}).$$

and for each $y \in M \setminus \{x_{k+1}\}$, $CV_{k+1}(y; 0) < C^n(R'_{N(k)}, R_{-N(k)}, y)$. Let $R'_{k+1} \equiv (R'_{N(k)}, R'_{k+1})$. Then, (ii) and (iii) follow from (ii-k).

By (iv-k) and $\{k + 1\} \subseteq N^+, N(k + 1) \subseteq N^+$. Finally, we show (iv): $N(k + 1) \subseteq N^+$. Suppose that $N(k + 1) = N^+$. Then, $|N(k + 1)| = |N^+| = m$. Thus, for each $j \in N \setminus N(k + 1), z_j = 0$.

By (ii), for each $y \in M \setminus \{x_{k+1}\}$,

$$CV_{k+1}(y; 0) < C^n(R'_{N(k)}, R_{-N(k)}, y) = C^{n-1}(R'_{N(k+1)}, R_{-N(k+1)}, y) \leq C^n(R'_{N(k+1)}, R_{-N(k+1)}, y).$$

Let $z' \in Z$ be such that for each $j \in N, z'_j \equiv 0$. Then, $z' \in Z^{IR}(R'_{N(k+1)}, R_{-N(k+1)})$.

By individual rationality, for each $j \in N \setminus \{k + 1\}, f_j(R'_{N(k+1)}, R_{-N(k+1)}) R_j 0 = z'_j$. Since $z \in W(R)$, for each $j \in N \setminus N(k + 1), CV_j(x_{k+1}; z'_j) = CV_j(x_{k+1}; z_j) \leq t_{k+1}$. By (ii), for each $j \in N(k + 1) \setminus \{k + 1\}$,

$$CV_j(x_{k+1}; z'_j) = CV_j(x_{k+1}; 0) < C^n(R'_{[1, \ldots, j-1]}, R_{-\{1, \ldots, j-1\}}, x_{k+1}) \leq C^n(R, x_{k+1}) \leq t_{k+1}. $$
Thus, by (iii), $CV_{k+1}(x_{k+1}; 0) > t_{k+1} \geq C^1(R_{N(k)}', R_{-N(k)}', x_{k+1}; z')$, and the assumptions of Lem. 8 hold for $(R_{N(k+1)}', R_{-N(k+1)}')$. By Lem. 8, $f_k^*(R_{N(k+1)}', R_{-N(k+1)}') = x_{k+1}$.

By individual rationality, $f_k^*(R_{N(k+1)}', R_{-N(k+1)}') \leq CV_{k+1}(x_{k+1}; 0)$. However, by (iii), $f_k^*(R_{N(k+1)}', R_{-N(k+1)}') < CV_k(x_{k+1}; f_{k+1}(R_{N(k)}', R_{-N(k)}'))$.

Thus, $f_{k+1}(R_{N(k+1)}', R_{-N(k+1)}') P_{k+1} f_{k+1}(R_{N(k)}', R_{-N(k)}')$, contradicting strategy-proofness.

\[\square\]

Part 3: Proof of Proposition 2

**Proof of Lemma 9.** First, we show (a). Let $M' \subseteq M$. Since $z^* \in Z_{\text{min}}(R)$, by Thm. 3.1, (i) $|\{i \in N : D(R_i, p) \subseteq M'\}| \leq |M'|$ and (ii) $|\{i \in N : D(R_i, p) \cap M' \neq \emptyset\}| > |M'|$. Note that for each $i \in N'$, $D(\hat{R}_i, p) = L$ and for each $j \in N \setminus N'$, $D(\hat{R}_j, p) = D(R_j, p)$. Thus, for each $i \in N'$, $D(R_i, p) \not\subseteq M'$ and $D(R_i, p) \cap M' \not= \emptyset$. Then,

$$|\{i \in N : D(\hat{R}_i, p) \subseteq M'\}| \leq |\{i \in N : D(R_i, p) \subseteq M'\}| \leq |M'|,$$

and

$$|\{i \in N : D(\hat{R}_i, p) \cap M' \neq \emptyset\}| > |\{i \in N : D(R_i, p) \cap M' \neq \emptyset\}| > |M'|.$$

That is, no set is overdemanded or weakly underdemanded at $p$ for $\hat{R}$. Thus, (a) follows from Thm. 3.1. Then, (b) follows from Prop. 1.

**Proof of Lemma 10.** (10-a). Suppose that for some $j \notin N(R, p) \cup N'$, $x \equiv f_j^*(\hat{R}) \not= 0$. By $D(\hat{R}_j, p) = 0$, individual rationality implies $f_j^*(\hat{R}) \leq CV_j(x; 0) < p^x$. This contradicts (10-i).

(10-b). We show that $\bar{z}$ satisfies (WE-i) in Def. 3.1 for $\hat{R}$. Note that, by $R_{N'} \in \mathcal{R}^I(z^*)^{|N'|}$, for each $i \in N'$ and each $z_i^* \in B(p)$, $\bar{z}_i \bar{z}_i^*$. Thus, $\bar{x}_i \in D(\hat{R}_i, p)$. By the def. of $\bar{z}$, for each $i \in N'$, $t_i = p^x_i$.

Thus, it suffice to prove that for each $i \in N \setminus N'$, $z_i \in D(R_i, p)$. Let $i \in N \setminus N'$ and $y \equiv f^*_i(R)$. By $(z^*, p) \in W_{\text{min}}(R)$, $R_N' \in \mathcal{R}^I(z^*)^{|N'|}$, and by Lem. 9-(b), $f_i^*(\hat{R}) \leq CV_i(y; z_i^*)$.

By $z_i^* \in D(R_i, p)$, $CV_i(y; z_i^*) \leq p^y$, where $p^y = 0$ if $y = 0$. If $y = 0$, by Lem. 1, $p^y = 0 = f_i^*(\hat{R})$.

If $y \neq 0$, by (10-i), $p^y \leq f_i^*(\hat{R})$. Then, by $f_i^*(\hat{R}) \leq CV_i(y; z_i^*) \leq p^y \leq f_i^*(\hat{R})$, $CV_i(y; z_i^*) = p^y = f_i^*(\hat{R})$.

Thus, $f_i^*(\hat{R}) \in D(R_i, p)$, for each $z_i^* \in B(p)$, $f_i(\hat{R})_I z_i^* R_i z_i^*$. Thus, $f_i^*(\hat{R}) \in D(R_i, p)$.

By $\bar{z}_i \equiv f_i(\hat{R})$, $\bar{x}_i \in D(R_i, p)$ and $t_i = p^x_i$.

We next show that $\bar{z}$ satisfies (WE-ii) in Def. 3.1. Since $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$, for each $x \in M$, $p^x > 0$. Thus, $\bar{z}_i \equiv f_i^*(\hat{R}) = x$.

Thus, $(\bar{z}, p) \in W(\hat{R})$. By $(z^*, p) \in W_{\text{min}}(R)$, $\bar{R}_{N'} \in \mathcal{R}^I(z^*)^{|N'|}$, and Lem. 9-(a), $p = p_{\text{min}}(\hat{R})$.

Thus, $\bar{z} \in Z_{\text{min}}(\hat{R})$.

(10-c). Let $M' \equiv \{x \in M : \text{ for some } j \in N', f_j^*(\hat{R}) = x\}$. By (10-ii) and $1 \leq |N'|$, $M' \not= \emptyset$.

Let $x \in M'$. Since $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$, for each $y \in M$, $p^y > 0$. Then, by Cor. 3.2, there is a sequence $\{i_k\}^K_{k=1}$ of $K$ distinct agents such that (i) $\bar{x}_{i_k} = 0$ or $p^{x_{i_k}} = 0$, (ii) $x_{i_k} = x$, and (iii) for each $k \in \{1, \ldots, K - 1\}$, $\{\bar{x}_{i_k}, \bar{x}_{i_{k+1}}\} \subseteq D(\hat{R}_{i_k}, p)$. Since for each $y \in M$, $p^y > 0$, $\bar{x}_{i_k} = 0$.

By (10-ii), $i_k \not\in N'$. By (10-a), for each $k \in \{1, \ldots, K\}$, $i_k \in N'' \cup N'$. Note that, by $i_K \in N'$, $\{k : i_k \in N'\} \neq \emptyset$. Let $K \equiv \min\{k : i_k \in N'\}$. Then, for each $k \in \{1, \ldots, K - 1\}$, $i_k \in N''$, and $i_K \in N'$. Thus, the sequence $\{i_k\}^K_{k=1}$ satisfies (i), (ii), (iii), and (iv) of (10-c).

**Proof of Lemma 11.** Suppose that $f_i^*(R) = 0$. By Lem. 1, $f_i^*(R) = 0$. By assumption (10-i), $-CV_i(0; f_i^*(R)) < CV_i(x; 0) - f_i^*(R)$. Then, by Lem. 3, there is $z' \in Z$ that Pareto-dominates $f(R)$ at $R$, which contradicts efficiency.

\[\square\]
Proof of Lemma 12. Let \( z' \in Z \) be such that \( z_{i_K}' = (0, CV_{i_K}(0; f_{i_K}(R))), z_{i_{K-1}}' = (x_{i_K}, f_{i_K}(R) - CV_{i_K}(0; f_{i_K}(R))) \), for each \( k \in \{1, \ldots, K-1\} \); \( z_{i_k}' = f_{i_k+1}(R) \), and for each \( i \in N \setminus \{i_k\}_{k=1}^K \), \( z_i' \equiv f_i(R) \). (See Fig. 5)

We show that \( z' \) Pareto-dominates \( f(R) \) at \( R \). By the def. of \( CV_{i_K}(0; f_{i_K}(R)), z_{i_K}' I_{i_K} f_{i_K}(R) \).

Also,

\[
\sum_{i \in N} t_i = CV_{i_K}(0; f_{i_K}(R)) + f_{i_K}(R) - CV_{i_K}(0; f_{i_K}(R)) + \sum_{k=1}^{K-2} f_{i_{k+1}}(R) + \sum_{i \in N \setminus \{i_k\}_{k=1}^K} f_i(R)
\]

\[
= f_{i_K}(R) + \sum_{k=2}^{K-1} f_{i_k}(R) + \sum_{i \in N \setminus \{i_k\}_{k=1}^K} f_i(R)
\]

\[
= \sum_{i \in N} f_i(R). \quad \text{by (b)}
\]

Thus, \( z' \) Pareto-dominates \( f(R) \) at \( R \).

\( \square \)
Proof of Proposition 2. Let $R' \equiv (R'_{N'}, R_{-N'})$. Without loss of generality, let $N' \equiv \{1, 2, \ldots, n'\}$. We only show that if $f^i_1(R') = x \in M$, $f^i_1(R') \geq p^x$ since we can treat similarly the other agents in $N'$. Let $f^i_1(R') \equiv x \in M$. By contradiction, suppose that $f^i_1(R') < p^x$. Let $N'' \equiv N(R, p) \setminus N'$.  

Case 1. $|N''| \geq m + 1$.  

Since $f^i_1(R') < p^x$, there is $R_i \in RNCV(f_1(R'))$ such that (ii) $-\overline{CV}_1(0; f_1(R')) < p^y - f_1(R')$. Then, by Lem. 6, $f_1(R_i, R'_{-i}) = f_1(R')$. Note that for each $j \in N'' \setminus \{1\}$, 

$$-\overline{CV}_1(0; f_1(R')) < p^y - f_1(R') = CV_j(x; 0) - f^i_1(R'),$$

where the inequality follows from (ii) and the equality from $R'_j \in R^I(z^*)$. Thus, by Lem. 11, for each $j \in N'' \setminus \{1\}$, $f^x_1(R_i, R'_{-i}) \neq 0$. Since $|N''| \geq m + 1$, this is a contradiction.

Case 2. $|N''| \leq m$.  

We show a contradiction in two steps.

Step 1. There is $\bar{R}_{N''} \in R^I(z^*)^{\lvert N'' \rvert}$ such that 

(a) for each $i \in N'$ and each $z_i \equiv (y, t) \in M \times \mathbb{R}$ with $t < p^y$, $-\overline{CV}_i(0; z_i) < p^y - t$, and
(b) for some $j \in N''$, $f^y_j(\bar{R}_{N''}, R'_{-N''}) \equiv y \neq 0$ and $f^y_j(\bar{R}_{N''}, R'_{-N''}) < p^y$.

Proof. We replace the preference $R'_i$ of each agent $i$ in $N'$ with $\bar{R}_i$, inductively. Since $f^y_1(R') \equiv x \neq 0$ and $f^y_1(R') < p^x$, there is $\bar{R}_i \in R^I(z^*) \cap RNCV(f_1(R'))$ such that

for each $z_i \equiv (y, t) \in M \times \mathbb{R}$ with $t < p^y$, $-\overline{CV}_1(0; z_i) < p^y - t$.

Then, by Lem. 6, $f_1(\bar{R}_i, R'_{-i}) = f_1(R')$. Since $f^y_1(R') \equiv x \neq 0$ and $f^y_1(R') < p^x$, $f^y_1(\bar{R}_i, R'_{-i}) = x \neq 0$ and $f_1(\bar{R}_i, R'_{-i}) < p^x$.

Induction argument: Let $s \leq \lvert N'' \rvert - 1$. As induction hypothesis, assume that there are $S \subseteq N'$ and $\bar{R}_S \in R^I(z^*)^{\lvert S \rvert}$ such that $\lvert S \rvert = s$,

(2-a) for each $i \in S$ and each $z_i \equiv (y, t) \in M \times \mathbb{R}$ with $t < p^y$, $-\overline{CV}_i(0; z_i) < p^y - t$, and
(2-b) for some $j_s \in S$, $f^y_{j_s}(\bar{R}_S, R'_{-S}) \equiv y_s \neq 0$ and $f^y_{j_s}(\bar{R}_S, R'_{-S}) < p^y$.

Then, by (2-a), (2-b), and Lem. 11, for each $i \in N'$, $f^x_i(\bar{R}_S, R'_{-S}) \neq 0$. Thus, by (2-i) of Prop. 2, (10-i) and (10-ii) of Lem. 10 hold for $(\bar{R}_S, R'_{-S})$. In addition, $\lvert N'' \rvert \leq m$. Thus, by (10-c) of Lem. 10, there is a sequence $\{i_k\}_{k=1}^K$ of $K$ distinct agents such that (i) $2 \leq K \leq m + 1$, (ii) $f^x_{i_k}(\bar{R}_S, R'_{-S}) = 0$, (iii) for each $k \in \{1, \ldots, K - 1\}$, $i_k \in N''$, and $i_K \in N'$, and (iv) for each $k \in \{1, \ldots, K - 1\}$, $f^y_{i_k}(\bar{R}_S, R'_{-S}), f^y_{i_{k+1}}(\bar{R}_S, R'_{-S}) \subseteq D(R'_{i_k}, p)$.

We show $f^x_{i_K}(\bar{R}_S, R'_{-S}) < p^{\pi_{i_K}}$, where $\pi_{i_K} \equiv f^x_{i_K}(\bar{R}_S, R'_{-S})$. If $i_K = j_s$, $f^y_{i_K}(\bar{R}_S, R'_{-S}) < p^{\pi_{i_K}}$ follows from (2-b). Thus, let $i_K \neq j_s$. By Lem. 9-(b), $f^y_{i_K}(\bar{R}_S, R'_{-S}) \leq p^{\pi_{i_K}}$. By contradiction, suppose that $f^y_{i_K}(\bar{R}_S, R'_{-S}) = p^{\pi_{i_K}}$. Then, we construct a sequence of agents satisfying the assumption of Lem. 12 by adding agent $j_s$ to the above sequence $\{i_k\}_{k=1}^K$ as the $(K + 1)$-th agent. Thus, there is an allocation $z'$ that Pareto-dominates $f(\bar{R}_S, R'_{-S})$ at $(\bar{R}_S, R'_{-S})$. This contradicts efficiency. Thus, $f^y_{i_K}(\bar{R}_S, R'_{-S}) < p^{\pi_{i_K}}$.

Next, we show $i_K \notin S$. By contradiction, suppose that $i_K \in S$. Then, by $f^y_{i_K}(\bar{R}_S, R'_{-S}) < p^{\pi_{i_K}}$, the above sequence $\{i_k\}_{k=1}^K$ satisfies the assumptions of Lem. 12. Thus, there is an
allocation \( z' \) that Pareto-dominates \( f(\bar{R}) \) at \( \bar{R} \), which contradicts efficiency. Thus, \( i_K \notin S \), and so \( i_K \in N' \setminus S \).

Let \( j_{s+1} = i_K \) and \( S' \equiv S \cup \{ j_{s+1} \} \). Then, by \( i_K \notin S \), \( |S'| = s + 1 \), and \( f^2_{j_{s+1}}(R_S, R'_{-S}) \equiv y_{s+1} \neq 0 \) and \( f^1_{j_{s+1}}(\bar{R}_S, R'_{-S}) < p^{y_{s+1}} \). Note that \( f^1_{j_{s+1}}(\bar{R}_S, R'_{-S}) < p^{y_{s+1}} \) implies that there is \( \bar{R}_{j_{s+1}} \in R^I(z^*) \cap R_{NCV}(f_{j_{s+1}}(\bar{R}_S, R'_{-S})) \) such that

for each \( z_{j_{s+1}} \equiv (y, t) \in M \times \mathbb{R} \) with \( t < p^y \), \( -\nabla V_{j_{s+1}}(0; z_{j_{s+1}}) < p^y - t \).

Thus, by (2-a),

for each \( i \in S' \) and each \( z_i \equiv (y, t) \in M \times \mathbb{R} \) with \( t < p^y \), \( -\nabla V_i(0; z_i) < p^y - t \).

By Lem. 6, \( f_{j_{s+1}}(R_S, R'_{-S}) = f_{j_{s+1}}(\bar{R}_S, R'_{-S}) \). Since \( f^2_{j_{s+1}}(R_S, R'_{-S}) \equiv y_{s+1} \neq 0 \) and \( f^1_{j_{s+1}}(R_S, R'_{-S}) < p^{y_{s+1}} \), we have \( f^2_{j_{s+1}}(\bar{R}_S, R'_{-S}) \equiv y_{s+1} \neq 0 \) and \( f^1_{j_{s+1}}(\bar{R}_S, R'_{-S}) < p^{y_{s+1}} \).

Hence, Step 1 holds.

\( \square \)

Step 2. Concluding that \( f^1_i(R') \geq p^x \).

By (a) and (b) of Step 1, and Lem. 11, for each \( i \in N' \), \( f^1_i(\bar{R}_{N'}, R'_{-N'}) \neq 0 \). Then, it follows from (2-i) of Prop. 2 that (10-i) and (10-ii) of Lem. 10 hold for the profile \( (\bar{R}_{N'}, R'_{-N'}) \).

By (10-c) of Lem. 10, there is a sequence \( \{i_k\}_{k=1}^K \) of \( K \) distinct agents such that (i) \( 2 \leq K \leq m+1 \), (ii) \( f^1_i(\bar{R}_{N'}, R'_{-N'}) = 0 \), (iii) for each \( k \in \{1, \ldots, K-1\} \), \( i_k \in N'' \), and \( i_K \in N' \), and (iv) for each \( k \in \{1, \ldots, K-1\} \), \( f^1_i(\bar{R}_{N'}, R'_{-N'}) \equiv f^1_{i_k}(\bar{R}_{N'}, R'_{-N'}) \subseteq D(R'_{i_k}, p) \).

Therefore, similarly to Step 1, we can show \( i_K \notin N' \), which contradicts \( i_K \in N' \).

\( \square \)

Part 4: Proof of Theorem 4.1.

Proof of Theorem 4.1. Let \( R \in R^n \) and \( (z^*, p) \in W_{\min}(R) \). By Lem. 4, for each \( \bar{R} \in R^I(z^*), \) each \( i \in N \), and each \( x \in M \), if \( f^x_i(\bar{R}) = x \), then, \( f^x_i(\bar{R}) \geq p^x \).

Claim 2. Let \( k \in \{1, \ldots, n\} \) and \( N_k \subseteq N \) be such that \( |N_k| = k \). Then, for each \( \bar{R}_{-N_k} \in R^I(z^*|N \setminus N_k|), \) each \( i \in N_k, \) and each \( x \in M \), if \( f^x_i(\bar{R}_{N_k}, \bar{R}_{-N_k}) = x \), then, \( f^x_i(\bar{R}_{N_k}, \bar{R}_{-N_k}) \geq p^x \).

Proof. We proceed Claim 2 by induction on \( k \). Let \( k = 1 \). Let \( N_1 \subseteq N \) with \( |N_1| = 1 \). Let \( \bar{R}_{-N_1} \in R^I(z^*|N \setminus N_1|), \) each \( i \in N_1, \) and each \( x \in M \) be such that \( f^x_i(R_{N_1}, \bar{R}_{-N_1}) = x \). Suppose that \( f^x_i(R_{N_1}, \bar{R}_{-N_1}) < p^x \). Let \( \bar{R}_i \in R^I(z^*) \) and \( \bar{x} \equiv f^x_i(\bar{R}) \). Then, \( f^x_i(\bar{R}) \geq p^x \). Let \( f^x_i(\bar{R}_{N_1}, \bar{R}_{-N_1}) = x \), then, \( f^x_i(\bar{R}_{N_1}, \bar{R}_{-N_1}) \geq p^x \). Then, by Prop. 2, for each \( \bar{R}_{-N_1} \in R^I(z^*|N \setminus N_1|), \) each \( i \in N \setminus N_1, \) and each \( x \in M \), if \( f^x_i(R_{N_1}, \bar{R}_{-N_1}) = x \), then, \( f^x_i(R_{N_1}, \bar{R}_{-N_1}) \geq p^x \).

Let \( k \in \{2, \ldots, n\} \). As induction hypothesis, assume that

2.1: for each \( N_{k-1} \subseteq N \) with \( |N_{k-1}| = k - 1 \), each \( \bar{R}_{-N_{k-1}} \in R^I(z^*|N \setminus N_{k-1}|), \) each \( i \in N_k, \) and each \( x \in M \), if \( f^x_i(R_{N_{k-1}}, \bar{R}_{-N_{k-1}}) = x \), then, \( f^x_i(R_{N_{k-1}}, \bar{R}_{-N_{k-1}}) \geq p^x \).

Let \( N_k \subseteq N \) be such that \( |N_k| = k \). Let \( \bar{R}_{-N_k} \in R^I(z^*|N \setminus N_k|), i \in N_k, \) and each \( x \in M \) be such that \( f^x_i(R_{N_k}, \bar{R}_{-N_k}) = x \). Suppose that \( f^x_i(R_{N_k}, \bar{R}_{-N_k}) < p^x \). Let \( N_{k-1} \equiv N_k \setminus \{i\} \).

34
Let \( \hat{x} \equiv f_i^x(R_{N_{k-1}}, \bar{R}_{N_{k-1}}) \). Then, by the induction hypothesis (2.1), \( f_i^x(R_{N_{k-1}}, \bar{R}_{N_{k-1}}) \geq p^x \). Thus, \( f_i(R_{N_{k}}, R_{N_{k}}) \leq f_i^x(R_{N_{k-1}}, \bar{R}_{N_{k-1}}) \), which contradicts strategy-proofness. Thus, for each \( \bar{R}_{N_{k}} \in \mathcal{R}^I(z^*)\lceil N \setminus N_{k} \rceil \), each \( i \in N_{k} \), and each \( x \in M \), if \( f_i^x(R_{N_{k}}, \bar{R}_{N_{k}}) = x \), then, \( f_i^x(R_{N_{k}}, \bar{R}_{N_{k}}) \geq p^x \). Then, by Prop. 2, for each \( \bar{R}_{N_{k}} \in \mathcal{R}^I(z^*)\lceil N \setminus N_{k} \rceil \), each \( i \in N \setminus N_{k} \), and each \( x \in M \), if \( f_i^x(R_{N_{k}}, \bar{R}_{N_{k}}) = x \), then, \( f_i^x(R_{N_{k}}, \bar{R}_{N_{k}}) \geq p^x \). □

We show that \( f(R) \) satisfies (WE-i) in Def. 3.1. Let \( i \in N \) and \( y \equiv f_i^x(R) \). By Prop. 1, \( f_i^x(R) \leq CV_i(y; z_i^*) \). By \( z_i^* \in D(R_i, p) \), \( CV_i(y; z_i^*) \leq p^y \), where \( p^y = 0 \) if \( y = 0 \). If \( y = 0 \), \( p^y = 0 = f_i^x(R) \). If \( y \neq 0 \), by Claim 2, \( p^y \leq f_i^x(R) \). Then, by \( f_i^x(R) \leq CV_i(y; z_i^*) \leq p^y \leq f_i^x(R) \). Thus, \( f_i(R) I_i z_i^* \). Since \( z_i^* \in D(R_i, p) \), for each \( z_i^* \in B(p) \), \( f_i(R) I_i z_i^* R_i z_i^* \). Thus, for each \( i \in N \), \( f_i^x(R) \equiv y \in D(R_i, p) \) and \( f_i^x(R) = p^y \).

Next, we show that \( f(R) \) satisfies (WE-ii) in Def. 3.1. Since \( R \equiv \mathcal{R}^C \) and \( n > m \), for each \( x \in M \), \( p^x > 0 \). By Lem. 2, for each \( x \in M \), there is \( i \in N \) such that \( f_i^x(R) = x \).

Thus, \( (f(R), p) \in W(R) \). Since \( p = p_{\min}(R) \), we conclude that \( f(R) \in Z_{\min}(R) \). □

C Proofs for Section 5 (Proposition 5.1)

Proof of Proposition 5.1. Let \( \mathcal{R} \subseteq \mathcal{R}^E \) and \( R \in \mathcal{R}^n \). Consider the simultaneous ascending (SA) auction defined in Section 5. By the def. of the auction, the price path \( p(t) \) is non-decreasing with respect to time \( t \). Next, for each \( x \in M \), let \( \bar{p}^x > C^I(R, x) \). Then, each agent demands only the null object at \( \bar{p} \), that is, no overdemanded set exists at \( \bar{p} \). Thus, the price path \( p(\cdot) \) is bounded above, that is, for each \( t \in \mathbb{R}_+ \), \( p(t) \leq \bar{p} \). Note that prices are raised at a speed at least \( d > 0 \). Thus, there is a price vector \( p^x \) such that the price path \( p(\cdot) \) converges to \( p^* \) in a finite time.

Let \( T \) be the termination time of the SA auction. We show that the final price \( p(T) = p_{\min}(R) \). By the def. of SA auctions, no overdemanded set exists at the price \( p(T) \). If no weakly underdemanded set exists at \( p(T) \), then the desired conclusion follows from Thm. 3.1. Thus, we show that no weakly underdemanded set exists at \( p(T) \). The proof is in two steps.

**Step 1.** Let \( t' \in (0, T] \). Assume that there is a set \( M' \) that is weakly underdemanded at \( p(t') \). Let \( N' \equiv \{ i \in N : D(R_i, p(t')) \cap M' \neq \emptyset \} \). Then, \( (5-a) \mid N' \mid \geq 2 \), and \( (5-b) \) there exist \( t'' \in (0, t') \) and \( M'' \subseteq M' \) such that \( \text{and} N'' \equiv \{ i \in N : D(R_i, p(t'')) \cap M'' \neq \emptyset \} \subseteq N' \) and \( M'' \) is underdemanded at \( p(t'') \).

**Proof.** Since \( M' \) is weakly underdemanded at \( p(t') \), for each \( x \in M' \), \( p^x(t') > 0 \) and \( |N'| \leq |M'| \). For each \( i \in N \), let \( z'_i \equiv (x'_i, t'_i) \in D(R_i, p(t')) \). Note that for each \( i \in N \setminus N' \) and each \( x \in M' \), \( CV_i(x; z'_i) < p^x(t') \). For each \( x \in M' \), let \( q^x \equiv \max\{ CV_j(x; z'_j) : j \in N \setminus N' \} \cup \{ 0 \} \). Let \( e > 0 \) be such that for each \( x \in M' \), \( q^x < p^x(t') - e \equiv p^x \). Let \( t'' \equiv \max\{ t \in \mathbb{R}_+ : \text{for some} \ x \in M', p^x(t) \leq p^x \} \). Then, there is \( x' \in M' \) such that \( dp^x(t'')/dt > 0 \) and \( p^x(t'') = p^x(t') \). Since \( dp^x(t'')/dt > 0 \), there is a minimal overdemanded set \( M \) at \( p(t'') \) that includes \( x' \).

Let \( M' \equiv M \cap M' \). Since \( x' \in M' \), \( M' \neq \emptyset \). Let \( N' \equiv \{ i \in N' : D(R_i, p(t'')) \cap M' \neq \emptyset \} . D(R_i, p(t'')) \in M \} \).
We show that $|\hat{N}'| > |\hat{M}'|$. If $\hat{M} \subset M'$, then $\hat{M}' = \hat{M}$ and for each $i \in \hat{N}'$, $D(R_i, p(t'')) \subseteq \hat{M}'$. Since $\hat{M}$ is overdemanded at $p(t'')$, the desired conclusion holds. Thus, assume that $\hat{M} \not\subseteq M'$. Let $\hat{M}'' \equiv \hat{M} \setminus M'$ and $\hat{N}'' \equiv \{i \in N : D(R_i, p(t'')) \subseteq \hat{M}''\}$. Then,

$$\{i \in N : D(R_i, p(t'')) \subseteq \hat{M}\} = \{i \in N : D(R_i, p(t'')) \subseteq \hat{M}''\} \cup \{i \in N : D(R_i, p(t'')) \cap \hat{M}' \neq \emptyset \text{ and } D(R_i, p(t'')) \subseteq \hat{M}\} = \hat{N}'' \cup \hat{N}'$$

where the first equality follows from $\hat{M}'' \cup \hat{M}' = \hat{M}$ and $\hat{M}'' \cap \hat{M}' = \emptyset$, and the second from the fact that for each $i \in N \setminus N'$, $D(R_i, p(t'')) \cap \hat{M}' = \emptyset$. Note that for each $x \in M'$, $q^x < p^x \leq p^x(t'')$. Thus, for each $i \in N \setminus N'$ and each $x \in M'$,

$$(x_i', p^{x_i'}(t')) R_i (x_i', p^{x_i'}(t')) R_i (x, q^x) P_i (x, p^x(t'')).$$ 

Since $\hat{M}' \subseteq M'$, for each $i \in N \setminus N'$, $D(R_i, p(t'')) \cap \hat{M}' = \emptyset$. Thus, $\hat{N}'' \cap \hat{N}' = \emptyset$. Then,

$$|\hat{N}''| + |\hat{N}'| = |\{i \in N : D(R_i, p(t'')) \subseteq \hat{M}\}| > |\hat{M}| \quad (\hat{M} \text{ is overdemanded at } p(t''))$$

$$= |\hat{M}''| + |\hat{M}'|.$$ 

Note that $\hat{M}'' \subseteq \hat{M}$. Since $\hat{M}$ is a minimal overdemanded set at $p(t'')$, $\hat{M}''$ is not overdemanded at $p(t'')$, and so, $|\hat{N}''| \leq |\hat{M}''|$. Thus, $|\hat{N}'| > |\hat{M}'|$. 

We show (5-a). Since $\hat{M}' \neq \emptyset$, $1 \leq |\hat{M}'|$. By $|\hat{N}'| > |\hat{M}'|$ and $\hat{N}' \subseteq N'$, we have $1 \leq |\hat{M}'| < |\hat{N}'| \leq |N'|$, and thus, $|N'| \geq 2$.

Next, we show (5-b). Let $M'' \equiv M' \setminus \hat{M}'$. Since $\hat{M}' \subseteq M'$, $M'' \neq \emptyset$. By $\hat{M}' \neq \emptyset$, $M'' \not\subseteq M'$. First, we show that $N'' \subseteq N' \setminus \hat{N}'$, that is, for each $i \in N''$, $i \in N'$ and $i \notin \hat{N}'$. Let $i \in N''$. Then, $D(R_i, p(t'')) \cap M'' \neq \emptyset$. Since for each $x \in M'$, $q^x < p^x(t'')$ and $M'' \subseteq M'$, for each $j \in N \setminus N'$, $D(R_j, p(t'')) \cap M' = \emptyset$. This implies $i \notin N'$. Since $\hat{M}' = M' \cap M$ implies $M'' = M' \setminus \hat{M}$, $D(R_i, p(t'')) \cap M'' \neq \emptyset$ implies $D(R_i, p(t'')) \setminus \hat{M} \neq \emptyset$. Since $\hat{N}' \subseteq \{j \in N : D(R_i, p(t'')) \subseteq \hat{M}\}$, this implies $i \notin \hat{N}'$. Thus, $N'' \subseteq N' \setminus \hat{N}'$.

Since $|\hat{N}'| > |\hat{M}'| \geq 1$, $|\hat{N}'| \geq 2$, and so, $N'' \not\subseteq N'$. Finally, it follows from the inequalities below that $M''$ is underdemanded at $p(t'')$.

$$|N''| \leq |N'| - |\hat{N}'| \quad \text{by } \hat{N}' \subseteq N'$$

$$< |N'| - |\hat{M}'| \quad \text{by } |\hat{N}'| > |\hat{M}'|$$

$$\leq |M'| - |\hat{M}'| \quad \text{by } |N'| \leq |M'|$$

$$= |M''|.$$

\[\square\]

\[\text{47To see this, suppose that } \hat{M}' = M'. \text{ Since } M' \text{ is weakly underdemanded at } p(t''), |N'| \leq |M'|. \text{ By } \hat{M}' = M' \text{ and } |\hat{N}'| > |\hat{M}'|, |N'| \leq |M'| = |\hat{M}'| < |\hat{N}'| \leq |N'|, \text{ which is a contradiction.}\]
Step 2. There is no weakly underdemanded set at $p(T)$.

Proof. By contradiction, suppose that there is a set $M_1$ that is weakly underdemanded at $p(T)$.

Let $N_1 \equiv \{ i \in N : D(R_i, p(T)) \cap M_1 \neq \emptyset \}$. Then, by Step 1, $|N_1| \geq 2$, and there exist $t_1 < T$ and $M_2 \subsetneq M_1$ such that $N_2 \equiv \{ i \in N : D(R_i, p(t_1)) \cap M_2 \neq \emptyset \} \subsetneq N_1$ and $M_2$ is underdemanded at $p(t_1)$. Since $M_2$ is underdemanded at $p(t_1)$, Step 1 also implies that $|N_2| \geq 2$, and there exist $t_2 < t_1$ and $M_3 \subsetneq M_2$ such that $N_3 \equiv \{ i \in N : D(R_i, p(t_2)) \cap M_3 \neq \emptyset \} \subsetneq N_2$ and $M_3$ is underdemanded at $p(t_2)$. Proceeding this argument inductively, we obtains a sequence $\{ N_k \} \subsetneq N_1$ such that for each $k \geq 2$, $|N_k| < |N_{k-1}|$ and $|N_k| \geq 2$. However, since $N_1$ is finite and for each $k \geq 2$, $N_k \subsetneq N_1$, this is a contradiction. □

References


Supplementary Note for “Strategy-proofness and Efficiency with Non-quasi-linear Preferences: A Characterization of Minimum Price Walrasian Rule”

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In this supplement, we provide the proofs that we have omitted in “Strategy-proofness and Efficiency with Non-quasi-linear Preferences: A Characterization of Minimum Price Walrasian Rule”. In Section D, we provide the proof of Fact 3.4 in Section 3. The proof is the same as Mishra and Talman’s (2010), but we provide it for completeness. Fact 3.5 is already shown by Demange and Gale (1985) and Roth and Sotomayor (1990). We also give the proof of Fact 3.5 in Section E for completeness.

D Proof of Fact 3.4.

The following theorem is used to prove Fact 3.4.

Hall’s Theorem (Hall, 1935). Let \( N = \{1, \ldots, n\} \) and \( M = \{1, \ldots, m\} \). For each \( i \in N \), let \( D_i \subseteq M \). Then, (i) there is a one to one mapping \( x' \) from \( N \) to \( M \) such that for each \( i \in N \), \( x'(i) \in D_i \) if and only if (ii) for each \( N' \subseteq N \), \( |\bigcup_{i \in N'} D_i| \geq |N'| \).

Fact 3.4 (Mishra and Talman, 2010). Let \( \mathcal{R} \subseteq \mathcal{R}^E \) and \( R \in \mathcal{R}^n \). A price vector \( p \) is a Walrasian equilibrium price for \( R \) if and only if no set is overdemanded and no set is underdemanded at \( p \) for \( R \).

Proof. “ONLY IF”. Let \( p \in P(R) \). Then, there is an allocation \( z = (x_i, t_i)_{i \in N} \) satisfying conditions (WE-i) and (WE-ii) in Def. 3.1. Let \( M' \subseteq M \).

We show that \( M' \) is not overdemanded at \( p \) for \( R \). Let \( N' \equiv \{i \in N : D(R_i, p) \subseteq M'\} \). Since for each \( i \in N' \), \( x_i \in D(R_i, p) \subseteq M' \), and each indivisible object is consumed at most one agent, \( |N'| = |\{x_i : i \in N'\}| \). Since \( \{x_i : i \in N'\} \subseteq M' \), \( |\{x_i : i \in N'\}| \leq |M'| \). Thus, \( |N'| \leq |M'| \).

We show that \( M' \) is not underdemanded at \( p \) for \( R \). Let \( N' \equiv \{i \in N : D(R_i, p) \cap M' \neq \emptyset\} \). Suppose that for each \( x \in M' \), \( p^x > 0 \) and \( |N'| < |M'| \). Note that \( |N'| < |M'| \) implies that there is \( x \in M' \) such that for all \( i \in N \), \( x_i \neq x \). Then, condition (WE-ii) implies that \( p^x = 0 \). This is a contradiction. Thus, \( |N'| \geq |M'| \).

“IF”. Assume that no set is overdemanded and no set is underdemanded at \( p \) for \( R \).

Let \( Z^* \equiv \{z = (x_i, t_i)_{i \in N} \in Z : \text{for each } i \in N, x_i \in D(R_i, p) \text{ and } t_i = p^{x_i}\} \). First, we show \( Z^* \neq \emptyset \). Suppose that there is \( N' \subseteq N \) such that for each \( i \in N' \), \( 0 \notin D(R_i, p) \) and \( |\bigcup_{i \in N'} D(R_i, p)| < |N'| \). Then \( \bigcup_{i \in N'} D(R_i, p) \) is overdemanded at \( p \) for \( R \). Thus, for each \( N' \subseteq N \), if for each \( i \in N' \), \( 0 \notin D(R_i, p) \), then \( |\bigcup_{i \in N'} D(R_i, p)| \geq |N'| \). Then, by Hall’s Theorem, there is \( z' \in Z \) such that for each \( i \in N \), if \( 0 \notin D(R_i, p) \), then \( x'_i \in D(R_i, p) \) and \( t'_i = p^{x'_i} \). Thus, \( Z^* \neq \emptyset \).
By the def. of $Z^*$, for each $z \in Z^*$, $(z,p)$ satisfies (WE-i). We show that there is $z \in Z^*$ such that $(z,p)$ satisfies (WE-ii). Let $M^+(p) \equiv \{x \in M : p^x > 0\}$. Let
\[ z = \arg \max_{z' \in Z^*} |\{y \in M^+(p) : \text{for some } i \in N, x'_i = y\}|, \tag{1} \]
that is, $z$ maximizes over $Z^*$ the number of objects in $M^+(p)$ that are assigned to some agents. Then, by the def. of $Z^*$, $(z,p)$ satisfies (WE-i).

Let $M^0 \equiv \{y \in M^+(p) : \text{for each } i \in N, x_i \neq y\}$. Note that, if $M^0 = \emptyset$, $(z,p)$ also satisfies (WE-ii). Thus, we show that $M^0 \neq \emptyset$. By contradiction, suppose that $M^0 = \emptyset$.

Let $N^0 \equiv \{i \in N : D(R_i(p) \cap M) \neq \emptyset\}$. For each $k = 1, 2, \ldots$, let $M^k \equiv \{y \in M : \text{for some } i \in N^k, x_i = y\}$ and $N^k \equiv \{i \in N : D(R_i(p) \cap M^k) \neq \emptyset\} \setminus \{ \bigcup_{k' = 0}^{k - 1} N^{k'} \}$. We claim by induction that for each $k \geq 0$, $M^k \subseteq M^+(p)$ and $N^k \neq \emptyset$.

**Induction argument:**

**Step 1.** By the def. of $M^0$, $M^0 \subseteq M^+(p)$. Since $M^0$ is not underdemanded at $p$ for $R$, $|N^0| \geq |M^0|$. Thus, $M^0 \neq \emptyset$ implies that $N^0 \neq \emptyset$.

**Step 2.** Let $K \geq 1$. As induction hypothesis, assume that for each $k \leq K - 1$, $M^k \subseteq M^+(p)$ and $N^k \neq \emptyset$.

First, we show that $M^K \subseteq M^+(p)$. Suppose that there is $x \in M^K \setminus M^+(p)$. Then, $p^x = 0$. By the induction hypothesis, there is a sequence $\{x(s), i(s)\}_{s=1}^K$ such that
\[
\begin{align*}
x(1) &= x, & x_{i(1)} &= x(1), \\
x(2) &\in D(R_{i(2)}(p) \cap M^{K-1}), & x_{i(2)} &= x(2), \\
x(3) &\in D(R_{i(3)}(p) \cap M^{K-2}), & x_{i(3)} &= x(3), \\
& \vdots & & \vdots \\
x(K) &\in D(R_{i(K)}(p) \cap M^1), & x_{i(K)} &= x(K).
\end{align*}
\]

Let $x(K + 1) \in D(R_{i(K)}(p) \cap M)$. For each $s \in \{1, 2, \ldots, K\}$, let $z'_i(s) \equiv (x_{i(s+1)}, p^{x_{i(s+1)}})$, and for each $j \in N \setminus \{i(s)\}_{s=1}^K$, let $z'_j \equiv z_j$. Then, $z' \in Z^*$ and
\[
|\{y \in M^+(p) : \text{for some } i \in N, x'_i = y\}| = |\{y \in M^+(p) : \text{for some } i \in N, x_i = y\}| + 1.
\]
This is a contradiction to (1). Thus, $M^K \subseteq M^+(p)$.

Next, we show that $N^K \neq \emptyset$. By $M^K \subseteq M^+(p)$ and the induction hypothesis, $\bigcup_{k=1}^K M^k \subseteq M^+(p)$. Thus, since $\bigcup_{k=0}^K M^k$ is not underdemanded at $p$ for $R$,
\[
|\bigcup_{k=0}^K N^k| \geq |\bigcup_{k=0}^K M^k|. \tag{2}
\]
By the def. of $M^k$ and $N^k$, for each $k, k' \in \{0, 1, \ldots, K\}$ with $k \neq k'$, $N^k \cap N^{k'} = \emptyset$, which also implies that $M^k \cap M^{k'} = \emptyset$. Thus,
\[
|\bigcup_{k=0}^K N^k| = \sum_{k=0}^K |N^k| \text{ and } |\bigcup_{k=0}^K M^k| = \sum_{k=0}^K |M^k|.
\]
Then, by (2),
\[
\sum_{k=0}^{K-1} |N^k| + |N^K| = \sum_{k=0}^{K} |N^k| \geq \sum_{k=0}^{K} |M^k| = \sum_{k=1}^{K} |M^k| + |M^0|.
\]
(3)

For each \( k \geq 1 \), by \( M^k \subseteq M^+(p) \), \( |M^k| = |N^{k-1}| \). Thus, \( \sum_{k=0}^{K-1} |N^k| = \sum_{k=1}^{K} |M^k| \). Then, by (3),
\[
|N^K| \geq |M^0|.
\]
Thus, by \( M^0 \neq \emptyset \), \( |N^K| \geq 1 \), and so \( N^K \neq \emptyset \).

Since \( M^+(p) \) is finite, by the above induction argument, for large \( K \), \( \bigcup_{k=0}^{K} M^k \supseteq M^+(p) \). Since \( \bigcup_{k=0}^{K} M^k \subseteq M^+(p) \), this is impossible. \( \square \)

### E  Proof of Fact 3.5.

Let \( \mathcal{R} \subseteq \mathcal{R}^E \).

**Lemma E.1.** Let \( i \in N \) and \( R_i \in \mathcal{R} \). Let \( p, q \in \mathbb{R}_+^m \) and \( x, y \in L \) be such that \( x \in D(R_i, p) \) and \( (y, q^y) P_i (x, p^x) \). Then, \( y \in M \) and \( q^y < p^y \).

**Proof.** Since \( (y, q^y) P_i (x, p^x) \) and \( x \in D(R_i, p) \), we have \( (y, q^y) P_i (x, p^x) R_i 0 \). Thus, \( y \in M \).

Also, by \( x \in D(R_i, p) \), \( (y, q^y) P_i (x, p^x) R_i (y, p^y) \). Thus, \( (y, q^y) P_i (y, p^y) \) implies \( q^y < p^y \). \( \square \)

Given \( R, R' \in \mathcal{R}^n \), \( (z, p) \in W(R) \), and \( (z', p') \in W(R') \), let
\[
N^1 \equiv \{ i \in N : z' P_i z_i \},
\]
\[
M^1 \equiv \{ x \in M : p^x > p'^x \},
\]
\[
X^1 \equiv \{ x \in L : \text{ for some } i \in N^1, x_i = x \}, \text{ and } X'^1 \equiv \{ x \in L : \text{ for some } i \in N^1, x_i' = x \}.
\]

**Lemma E.2: Decomposition** (Demange and Gale, 1985). Let \( R \in \mathcal{R}^n \) and \( (z, p) \in W(R) \). Let \( R' \in \mathcal{R}^n \) be a d-truncation of \( R \) such that for each \( i \in N \), \( d_i \leq -CV_i(0; z_i) \), and let \( (z', p') \in W(R') \). Then, \( X^1 = X'^1 = M^2 \).

**Proof.** First, we show \( X'^1 \subseteq M^2 \). Let \( x \in X'^1 \). Then, there is \( i \in N^1 \) such that \( x_i' = x \). By \( i \in N^1 \), \( (x_i', p^{x_i'}) P_i (x_i, p^{x_i}) \). Thus, by \( x_i \in D(R_i, p) \) and Lem. E.1, \( x_i' \in M \) and \( p^{x_i'} < p^{x_i} \), and so, \( x = x_i' \in M^2 \). Thus, \( X'^1 \subseteq M^2 \).

Next, we show \( M^2 \subseteq X^1 \). Let \( x \in M^2 \). Then, \( x \in M \) and \( 0 \leq p^x < p^x \). Thus, by (WE-ii), there is \( i \in N \) such that \( x_i = x \). By \( d_i \leq -CV_i(0; z_i) \) and Lem. A.2-(ii), \( (x_i', p^{x_i'}) P_i (x_i, p^{x_i}) \). Thus, \( i \in N^1 \), and so, \( x = x_i \in X^1 \). Thus, \( M^2 \subseteq X^1 \).

Note that by the def. of \( X^1 \) and \( X'^1 \), \( |X^1| \leq |N^1| \) and \( |X'^1| \leq |N^1| \). Since \( X^1 \subseteq M^2 \subseteq M \), each agent in \( N^1 \) receives a different object, and so, \( |X^1| = |N^1| \geq |X^1| \). Since \( X^1 \subseteq M^2 \subseteq X^1 \), \( |X^1| \leq |M^2| \leq |X^1| \). Thus, \( |X^1| = |M^2| = |X^1| \). By \( |X^1| = |M^2| \) and \( X^1 \subseteq M^2 \), \( X^1 = M^2 \).

By \( |M^2| = |X^1| \) and \( M^2 \subseteq X^1 \), \( M^2 = X^1 \). \( \square \)

**Lemma E.3: Lattice Structure** (Demange and Gale, 1985). Let \( R \in \mathcal{R}^n \) and \( (z, p) \in W(R) \). Let \( R' \) be a d-truncation of \( R \) such that for each \( i \in N \), \( d_i \leq -CV_i(0; z_i) \), and let \( (z', p') \in W(R') \). Then, \( (i) \hat{p} \equiv p \land p' \in P(R) \), and \( (ii) \check{p} \equiv p \lor p' \in P(R') \).

\( \text{Denote } p \land p' \equiv (\min\{p^x, p'^x\})_{x \in M} \text{ and } p \lor p' \equiv (\max\{p^x, p'^x\})_{x \in M}. \)
Proof. Let $N^1 \equiv \{ i \in N : z'_i P_i z_i \}$ and $M^2 \equiv \{ x \in M : p^x > p^x \}$.

(i). Let $\hat{z}$ be defined by setting for each $i \in N^1$, $\hat{z}_i \equiv z'_i$, and for each $i \in N \setminus N^1$, $\hat{z}_i \equiv z_i$. We show that $(\hat{z}, \hat{p}) \in W(R)$.

Step 1. $(\hat{z}, \hat{p})$ satisfies (WE-i).

Let $i \in N$ and $x \in L$. In the following two cases, we show $(\hat{x}_i, \hat{p}^x) R_i (x, \hat{p}^x)$, which implies $\hat{x}_i \in D(R_i, \hat{p})$.

Case 1. $i \in N^1$.

By $\hat{x}_i = x'_i$ and Lem. E.2, $\hat{x}_i \in M^2$, and so, $\hat{x}_i \in M$ and $p'^x \prec p^x_i$. Thus, $\hat{p}^x_i = p'^x_i$.

First, assume that $x \in M^2$. Then, by $\hat{p}^x = p^x$,

$$(\hat{x}_i, \hat{p}^x_i) = z'_i \quad R'_i (x, p'^x) = (x, \hat{p}^x).$$

Since $R'_i$ is a $d_i$-truncation of $R_i$, $\hat{x}_i \not\in 0$, and $x \not\in 0$, Remark 3.1 implies $(\hat{x}_i, \hat{p}^x) R_i (x, \hat{p}^x)$.

Next, assume that $x \not\in M^2$. Then, by $\hat{p}^x = p^x$,

$$(\hat{x}_i, \hat{p}^x_i) = z'_i \quad P_i z_i \quad R_i (x, p^x) = (x, \hat{p}^x).$$

Case 2. $i \not\in N^1$.

By $\hat{x}_i = x_i$ and Lem. E.2, $\hat{x}_i \not\in M^2$. Thus, $\hat{p}^x_i \not\leq p^x_i$ or $\hat{x}_i = 0$. First, we assume that $x \in M^2$. Then, $\hat{p}^x = p^x$. Note that $i \not\in N^1$ implies $(\hat{x}_i, \hat{p}^x) = z_i R_i z'_i$.

Case 2-1. $x'_i \not\in 0$.

By $x'_i \in D(R'_i, p')$, $z'_i R'_i (x, p'^x) = (x, \hat{p}^x)$. Since $R'_i$ is a $d_i$-truncation of $R_i$, $x'_i \not\in 0$, and $x \not\in 0$, Remark 3.1 implies $z'_i R_i (x, p'^x)$. Thus,

$$(\hat{x}_i, \hat{p}^x_i) = z_i R_i z'_i R_i (x, p'^x) = (x, \hat{p}^x).$$

Case 2-2. $x'_i = 0$.

Then, $z'_i = 0$. Since $x'_i \in D(R'_i, p')$, $CV'_i(x; 0) \leq p'^x_i$. Thus, if $CV_i(x; 0) \leq CV'_i(x; 0)$, then, $z'_i R_i (x, p'^x)$, which implies that

$$(\hat{x}_i, \hat{p}^x_i) = z_i R_i z'_i R_i (x, p'^x) = (x, \hat{p}^x).$$

Next, assume that $CV_i(x; 0) > CV'_i(x; 0)$. Then, since $R'_i$ is a $d_i$-truncation of $R_i$, $d_i > 0$, which implies that $x_i \not\equiv 0$.\footnote{To see this, suppose that $x_i = 0$. Then, $d_i \leq -CV_i(0; z_i) = 0$, which contradicts $d_i > 0$.} Then, by $d_i \leq -CV_i(0; z_i)$, $CV_i(x; z_i) \leq CV'_i(x; 0) \leq p'^x$, which implies that $z_i R_i (x, p'^x)$. Thus,

$$(\hat{x}_i, \hat{p}^x_i) = z_i R_i (x, p'^x) = (x, \hat{p}^x).$$

Next, assume that $x \not\in M^2$. Then, $\hat{p}^x = p^x$. Since $\hat{x}_i = x_i \in D(R_i, p)$,

$$(\hat{x}_i, \hat{p}^x_i) = z_i R_i (x, p^x) = (x, \hat{p}^x).$$
Step 2. \((\hat{z}, \hat{p})\) satisfies (WE-\textit{ii}).

Let \(x \in M\) be such that \(\hat{p}^x > 0\). We show that there is \(i \in N\) such that \(\hat{x}_i = x\). Since \(\hat{p} = p \land p', \hat{p}^x > 0\) implies \(p^x > 0\) and \(p'^x > 0\).

**Case 1.** \(x \in M^2\).

By Lem. E.2, there is \(i \in N^1\) such that \(x'_i = x\). By \(i \in N^1\) and the def. of \(\hat{z}\), \(\hat{x}_i = x'_i\). Thus, \(\hat{x}_i = x\).

**Case 2.** \(x \notin M^2\).

By \(p^x > 0\), there is \(i \in N\) such that \(x_i = x\). By Lem. E.2, \(i \notin N^1\). Thus, \(\hat{x}_i = x_i\), and so, \(\hat{x}_i = x\).

(ii). Let \(\varepsilon\) be defined by setting for each \(i \in N^1\), \(z_i \equiv z_i\), and for each \(i \in N \setminus N^1\), \(\varepsilon_i \equiv \varepsilon'_i\). We show \((\varepsilon, \hat{p})\) \(\in W(R')\).

**Step 1.** \((\varepsilon, \hat{p})\) satisfies (WE-\textit{i}).

Let \(i \in N\) and \(x \in L\). In the following two cases, we show \((\varepsilon_i, \hat{p}^{\varepsilon_i}) R_i'(x, \hat{p}^x)\), which implies \(\varepsilon_i \in D(R_i', \hat{p})\).

**Case 1.** \(i \in N^1\).

By \(\varepsilon_i = x_i\) and Lem. E.2, \(\varepsilon_i \in M^2\), and so, \(\varepsilon_i \in M\) and \(p^\varepsilon_i < p^\hat{z}_i\). Thus, \(\hat{p}^\varepsilon_i = p^\varepsilon_i\). First, assume that \(x \in M^2\). Since \(\varepsilon_i = x_i \in D(R_i, p)\) and \(\hat{p}^x = p^x\),

\[
(\varepsilon_i, \hat{p}^\varepsilon_i) = z_i R_i (x, p^x) = (x, \hat{p}^x).
\]

Since \(R_i'\) is a \(d_i\)-truncation of \(R_i\), \(\varepsilon_i \neq 0\), and \(x \neq 0\), Remark 3.1 implies \((\varepsilon_i, \hat{p}^{\varepsilon_i}) R_i'(x, \hat{p}^x)\). Next, assume that \(x \notin M^2\). Then, \(p^x \leq p'^x\) or \(x = 0\).

**Case 1-1.** \(x \neq 0\).

Since \(\varepsilon_i = x_i \in D(R_i, p)\) and \(\hat{p}^x = p^x\geq p^x\),

\[
(\varepsilon_i, \hat{p}^\varepsilon_i) = z_i R_i (x, p^x) R_i (x, \hat{p}^x).
\]

Since \(R_i'\) is a \(d_i\)-truncation of \(R_i\) and \(\varepsilon_i \neq 0\), \((\varepsilon_i, \hat{p}^{\varepsilon_i}) R_i'(x, \hat{p}^x)\).

**Case 1-2.** \(x = 0\).

Since \(R_i'\) is a \(d_i\)-truncation of \(R_i\) and \(d_i \leq -CV_i(0; z_i)\),

\[
(\varepsilon_i, \hat{p}^\varepsilon_i) = z_i R_i' 0 = (x, \hat{p}^x).
\]

**Case 2.** \(i \notin N^1\).

By \(\varepsilon_i = x'_i\) and Lem. E.2, \(\varepsilon_i \notin M^2\). Thus, \(p^\varepsilon_i \leq p'^\varepsilon_i\) or \(\varepsilon_i = 0\). If \(\varepsilon_i = 0\),

\[
(\varepsilon_i, \hat{p}^\varepsilon_i) = 0 = z'_i R_i' (x, p^xy) R_i' (x, \hat{p}^x).
\]

Thus, assume that \(\varepsilon_i \neq 0\). Then,

\[
(\varepsilon_i, \hat{p}^\varepsilon_i) = z'_i R_i' (x, p^xy) R_i' (x, \hat{p}^x).
\]
Step 2. \((\bar{x}, \bar{p})\) satisfies \((\text{WE-ii})\).

Let \(x \in M\) be such that \(\bar{p}^x > 0\). We show that there is \(i \in N\) such that \(\bar{x}_i = x_i\). Since \(\bar{p} = p \land p', \bar{p}^x > 0\) implies \(p^x > 0\) or \(p'^x > 0\).

Case 1. \(x \in M^2\).

By Lem. E.2, there is \(i \in N^1\) such that \(x_i = x_i\). Since \(i \in N^1\), \(\bar{x}_i = x_i\). Thus, \(\bar{x}_i = x_i\).

Case 2. \(x \notin M^2\).

If \(p'^x = 0\), then \(p'^x > 0 < p^x\). Thus, \(x \in M^2\), which is a contradiction. Thus, \(p'^x > 0\). Then, there is \(i \in N\) such that \(x'_i = x\). By Lem. E.2, \(i \notin N^1\), which implies that \(\bar{x}_i = x'_i\). Thus, \(\bar{x}_i = x\).

The following is a corollary of Lem. E.3.

Corollary E.1. Let \(R \in \mathcal{R}^n\) and \(p, p' \in P(R)\). Then, (i) \(p \land p' \in P(R)\) and (ii) \(p \lor \hat{p} \in P(R)\).

Fact 3.5 (Roth and Sotomayor, 1990). Let \(R \in \mathcal{R}^n\) and \(R'\) be a \(d\)-truncation of \(R\) such that for each \(i \in N\), \(d_i \geq 0\). Then, \(p_{\min}(R') \leq p_{\min}(R)\).

Proof. Let \((z', p') \in W(R')\). Then, for each \(i \in N\), since \(CV'_i(0; z'_i) \leq 0\) and \(d_i \geq 0\), \(-d_i \leq 0 \leq -CV'_i(0; z'_i)\). Since \(R\) is the \((-d)\)-truncation of \(R'\), Lem. E.3 implies \(\hat{p} \equiv p' \land p_{\min}(R) \in P(R')\). Thus, since \(p_{\min}(R') \leq \hat{p}\), \(p_{\min}(R') \leq p_{\min}(R)\).

References


