Bubble Cycles

Very preliminary, welcome comments

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Abstract

We develop a simple model that explains episodes on bubbles that occurred in the past Japan and the recent United States. The basic idea is that the self-fulfilling change in the saving rate leads to bubble-induced business cycles with co-movement between bubbles and investment. We have a counter-intuitive implication of the stimulus package of fiscal expansion in the bubbly economy.

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1. Introduction

Bubbles create economic booms, but the crash of bubbles leads to prolonged and severe recessions. The story of business fluctuations induced by bubbles explains the current turmoil in the world economy that has been triggered by the sub-prime-loan syndrome in the U.S. economy as well as the “lost decade in Japan” in the 1990s.

The aim of this paper is to construct a simple model that explains business fluctuations that are induced by investments that go hand in hand with bubbles. The starting point of the argument is Tirole (1985) that develops the theory of rational bubbles in the growth model of overlapping generations. His argument has two weak points to explain the recent observations on how bubbles impact on economic activities. First, Tirole (1985) demonstrates that the bubbles arise when capital over-accumulation arise in the bubbleless economy, and move the economy to the Golden Rule. But we find no evidence for capital over-accumulation, but rather for capital under-accumulation (e.g., Abel et al, 1989). Second, in Tirole bubbles are assets that compete with investment in capital in the asset portfolio so that the competition effect of bubbles crowds out investment, but the crowding-out view seems to contradict the recently observed comovement between bubbles and investment.

Central ingredients so as to generate the bubble-induced business cycles are, first the imperfection of capital markets, and second the aggregate saving behavior. The poor working of capital markets brought about by imperfect pledgeability gives rise to the wedge between the rate of return to capital and the real interest rate [e.g., Stiglitz and Weiss (1981), Gale and Hellwig (1985), Williamson (1986), and others], which in turn leads to an interesting situation where the interest rate is smaller than the growth rate so that bubbles can arise, but capital under-accumulation prevails. Furthermore, it induces people to find bubbles as internal wealth valuable. The channel of the collateral effect enables investment to go hand in hand with bubbles (e.g., Venture (2003), Caballero and Krishnamurthy (2006), Tirole (2008)).

The aggregate saving behavior where savings increase rapidly as the interest rate for some region motivates the link between savings and bubbles. Bubbles increase the interest rate, and are followed by the rise in savings so that the self-fulfilling expectation on savings boosts bubbles. The link between bubbles and saving is not unconventional. A large part of the flows of income are consumed, but incomes from capital gains of assets are almost reinvested.

Whether bubbles crowd investment in or out depend on whether the collateral effect is stronger or weaker than the competition effect. Whether savings rise or not is essential to which effect is greater. When savings do not change whether bubbles arise, the collateral effect is always dominated by the competition effect, and then the bubbly steady state features less
investment than the bubbleless steady state. By contrast, when savings change when bubbles arise, the collateral effect can dominate the competition effect, and the bubbly steady state will feature more investment than the bubbleless steady state.

In the latter, we develop a story of business cycles induced by bubbles that are linked to the self-fulfilling behavior of savings. Appreciations in asset prices generate the economic boom, and the crash of asset prices leads to the recession, and importantly bubbles are induced by the self-fulfilling expectations on savings and/or the rate of return to bubbles. When people anticipate the high return of bubbles, they save and bubbles actually boost, whereas people anticipate the low return of bubbles, they dissave and bubbles burst.

This paper has also implications for dynamic efficiency/inefficiency and sustainability of asset bubbles. Like the frictionless Diamond model, the greater rate of return to capital than the growth rate is a sufficient condition under which the bubbleless economy is dynamic efficient. Bubbles can arise even when the bubbleless economy is dynamic efficient. Unlike the Diamond model, however, the smaller rate of return to capital than the growth rate is not a sufficient condition under which the bubbleless economy is dynamic inefficient. When there is capital over-accumulation in the bubbleless economy, asset bubbles can arise, but nonetheless asset bubbles may not improve efficiency. Asset bubbles in the dynamic efficient economy arise from the fact that the initial bubbleless economy is Pareto sub-optimal. Several other papers have demonstrated that, in the presence of wedge between social and private returns to capital, bubbles can arise even when the bubbleless economy is dynamic efficient (e.g., Saint-Paul (1992), Grossman and Yanagawa (1993) and Femminis (2002)).

Closely related are Caballero et al (2006) and Tirole (2008) that develop models so as to explain the comovement between investment and bubbles or asset prices. Caballero et al (2006) develop their argument for the growth-saving feedback so as to explain the high correlation between investment and the stock price, and bubbles, but in theirs the bubbly steady state does not feature more investment than the bubbleless steady state. Tirole (2008) develops a model of imperfect pledgeability where the bubbly steady state features more investment than the bubbleless steady state, but relies on some non-standard assumption for the portfolio choice of entrepreneurs.

Ventura (2003) provides a setting in which entrepreneurs issue debt and also equity to finance investment, where equity takes the form of bubble creation. Caballero and Krishnamurthy (2006) develop a small-open economy in which asset bubbles are used for
collateral for financing productive investment so that bubbles can promote capital accumulation.

This paper is organized as follows. Section 2 sets up the model and studies the benchmark economy. Section 3 analyzes the economy when there are frictions in financial markets. Section 4 develops the story of bubble cycles.

2. Basic Model

Let us consider an economy of overlapping generations that lasts for infinity. At each period \( t = 0, 1, 2, \ldots, \infty \), the economy is populated by a continuum of agents that live for three periods. They are endowed with \( H_t \) units of the final good in the young age, and supply one unit of labor in the middle age. There is no population growth.

At each period the final good is produced by firms that use labor and capital as inputs according to the constant-returns-to-scale technology described as \( Y_t = F(K_t, A_t, N_t) \), where \( K_t \) and \( N_t \) are aggregate supplies of capital and labor, and \( A_t \) is the time-varying technology level that grows at an exogenous rate \( g \). The endowment \( H_t \) also grows at rate \( g \) so that we denote \( H_t = A_t h \). The labor force is constant over time and normalized unity. Letting \( k_t (\equiv K_t / A_t) \) denote capital per effective worker, the output per effective worker is described as \( y_t = Y_t / A_t = F(K_t / A_t, 1) \equiv f(k_t) \), where \( f(.) \) is thrice continuously differentiable, increasing, concave, satisfying \( f(0) = 0 \), and \( \lim_{k_t \to 0} f'(k_t) = +\infty \). Since the production technology is homogeneous of degree one, output of the final good can be described in terms of the action of a single, aggregate, price-taking firm. From the profit-maximizing behavior of that firm, output is exhausted by the payment to two inputs and each input is paid its marginal product. The rate of return to capital \( R_t \) and the wage rate \( w_t \) are determined to satisfy \( R_t = f'(k_t) \) and \( w_t = f(k_t) - k_t f'(k_t) \equiv W(k_t) \), respectively. The final good is numeraire.

Assume that capital depreciates fully after one period. The price of capital is then equal to \( R_t \).

Within each generation, agents are divided into two types, “entrepreneurs” and “investors”. A fraction \( \alpha \) \((0 < \alpha < 1)\) of agents are entrepreneurs, while the remaining fraction \( 1 - \alpha \) of agents are investors.

Entrepreneurs born at \( t \) differ in their preference. A fraction \( s_e \) \((0 < s_e < 1)\) of them maximize their old age consumption, while the remaining are “impatient” and maximize their linear utility, \( c_{t+2}^m + \beta_e c_{t+3}^o \), where \( c_{t+1}^m(c_{t+2}^o) \) is consumption in the middle (old) age, and \( \beta_e \) is
the discount factor. In the middle age, each of entrepreneurs has access to one linear capital investment technology that transforms one unit of the final good into one unit of capital after one period.

Investors born at $t$ also differ in their preference. A fraction $s_i (0 < s_i < 1)$ of them maximize their old age consumption, while the remaining are “impatient” and maximize their linear utility, $c_{t+1} + \beta_1 c_{t+2}$, where $\beta_1$ is the discount factor. Any of investors cannot have access to the capital investment technology and earns the old age income only by investing their wealth in lending to others or holding assets.

For analytical convenience, we assume $s_i = s_e = s_L$, but allow for $\beta_e = \beta_f$. Let $s_i$ denote the aggregate saving rate. We focus on the case for $s_i = 1$ or $s_i = s_L$, where the former implies that all the “impatient” agents save in the middle age, and the latter that all the “impatient” agents consume in the middle age. The essential feature is the aggregate saving function is that there is a region in which savings increase rapidly as capital rises.

Assume that there is no enforcement mechanism to fulfill financial contracts between debtors and creditors and hence to enforce on borrowers to repay their debt. When debtors breach the contract and refuse to make their repayment, a portion $\lambda (0 < \lambda < 1)$ of their earnings are assumed to be forfeited by the creditor. A low $\lambda$ is interpreted to capture weak bankruptcy procedure, poor bank monitoring, and low contract enforcement, and will be associated with poorly developed financial markets. Much literature on economic growth has argued the importance of institutional quality (e.g., North (1981), Hall and Jones(1999), Acemoglu et al(2001) and others). La Porta et al (1997, 1998), Beck et al(2000), and Levine et al(2000) have linked economic and financial developments by using various indicators of investor rights and protection, and legal enforcement as instrumental variables for financial development. The parameter $\lambda$ is thus interpreted to capture the efficiency of the broadly defined financial system.

First of all, we investigate an economy with perfect capital market that is characterized by perfect enforcement. Letting $I_t$ denote the amount of investment, $X_t$ denote the internal wealth of the middle age agents, and $r_{t+1}$ denote the interest rate prevailing between $t$ and $t+1$, any of entrepreneurs earns $f'(k_{t+1})I_t - (1 + r_{t+1})(I_t - X_t)$ by implementing own project, while $(1 + r_{t+1})X_t$ by supplying his wealth to others. Entrepreneurs are willing to start their own projects if

\begin{equation}
(2-1) \quad f'(k_{t+1}) \geq 1 + r_{t+1}.
\end{equation}
We call this inequality the **profitability constraint**. The “AK” structure of the investment project makes the inequality bind with equality. Eventually,

\[(2-2) \quad f'(k_{t+1}) = 1 + r_{t+1} \]

is only sustainable in equilibrium. The aggregate capital is defined as the product of three terms, the population of entrepreneurs \( \alpha \), the aggregate saving rate \( s_t \), and the individual investment size \( I_t \) so that \( K_{t+1} = \alpha s_t I_t \), and hence we have

\[(2-3) \quad (1 + g)k_{t+1} = \alpha s_t I_t / A_t . \]

At the beginning of the middle age, the total income is composed of the wage income \( A_t w_t \) and the interest income \( (1 + r_t)H_{t-1} \). Given the saving rate \( s_t \), the aggregate savings of the middle are \( s_t (A_t w_t + (1 + r_t)H_{t-1}) \). Let \( B_t \) denote units of aggregate bubbles, and \( p_t \) denote the price of bubbles. The aggregate savings are used for financing investment in capital \( \alpha s_t I_t \) and purchasing bubbles \( p_t B_t \), and its relation is described by

\[(2-4) \quad s_t (A_t w_t + (1 + r_t)H_{t-1}) + H_t = \alpha s_t I_t + p_t B_t . \]

Letting \( b_t \left(= p_t B_t / A_t \right) \) denote bubbles per effective worker at time \( t \), and using (2-3), (2-4) is expressed by

\[(2-5) \quad s_t \{W(k_t) + h \frac{1 + r_t}{1 + g}\} + h = (1 + g)k_{t+1} + b_t . \]

Under perfect foresight, bubbles have to earn the same rate of return as that on capital to satisfy

\[p_{t+1}/p_t = 1 + r_{t+1} . \]

Given that the net supply of nominal bubbles is zero, the aggregate bubbles per effective worker thus grow to satisfy

\[(2-6) \quad \frac{b_{t+1}}{b_t} = \frac{1 + r_{t+1}}{1 + g} . \]

Finally we exclude negative bubbles;

\[(2-7) \quad b_t \geq 0 . \]

We define two kinds of equilibria. A **bubbleless economy** is defined as an equilibrium in which \( b_t = 0 \) for any \( t \) or \( b_t \) converges to zero if \( b_t > 0 \) for any \( t \). A **bubbly economy** is defined as an equilibrium in which \( b_t \) does not converge to zero.\(^{1}\)

First of all, we examine the analysis of steady states. The steady state of a bubbleless

\(^{1}\) Note that Tirole (1985) distinguishes between an **asymptotically bubbly equilibrium** and a **bubbly equilibrium** by defining the latter as the one in which \( b_t > 0 \) for any \( t \).
economy is characterized by a pair \( \{ \bar{k}, \bar{r}, \bar{s} \} \), satisfying
\[
(1 + g)\bar{k} = \bar{s}W(\bar{k}) + \frac{1 + \bar{r}}{1 + g}h + h.
\]
1 + \bar{r} = f'(\bar{k}) \text{, and } \bar{s} = s_L \text{ or } 1. \text{ On the other hand, the steady state of a bubbly economy is characterized by a pair } \{ k_{GR}, r_{GR}, s_{GR}, b_{GR} \} \text{, satisfying } 1 + r_{GR} = f'(k_{GR}) \text{,}
\]
\[
(1 + g)k_{GR} + b_{GR} = s_{GR}W(k_{GR}) + \frac{1 + r_{GR}}{1 + g}h + h, \text{ and } r_{GR} = g \text{, with } b_{GR} > 0.
\]

We briefly summarize the properties of the economy under perfect financial markets. If \( g > r \), there exists a unique bubbleless economy and the interest rate converges to \( \bar{r} \), while otherwise, there exists an asymptotically bubbly economy and the interest rate converges to \( g \) [Tirole (1985, Proposition 1)]. Ihori (1978) demonstrates that the government bond, which is intrinsically valueless, carries the long-run capital level to the Golden Rule level of capital when \( \bar{r} < g \).

3. The Economy with Financial Market Frictions

We now introduce the financial market friction into the benchmark model. The financial market is competitive in the sense that both entrepreneurs (borrowers) and investors (lenders) take the equilibrium rate \( r_{t+1} \) as given. If any of entrepreneurs borrows the amount \( (I_t - X_t) \) and repays \( (1 + r_{t+1}) (I_t - X_t) \) honestly, his earnings will be \( f'(k_{t+1})I_t - (1 + r_{t+1})(I_t - X_t) \), while if he breaches the promise for repayment, a portion \( \lambda \) of his earning is forfeited, and his earning would be \( (1 - \lambda)f'(k_{t+1})I_t \). The incentive compatibility constraint that induces entrepreneurs to commit the truthful behavior is given by
\[
(3-1) \quad (1 + r_{t+1})(I_t - X_t) \leq \lambda f'(k_{t+1})I_t. \quad 2
\]
Equation (3-1) implies that entrepreneurs can borrow the amount up to some fraction of the project revenue so that it will be called the “borrowing constraint”. \(^3\)

The market clearing in the capital market is given by
\[
\alpha s_t(I_t - X_t) = (1 - \alpha)s_tX_t + H_t - p_tB_t,
\]
where the L.H.S. is the aggregate demand for funds by entrepreneurs, and the R.H.S. is the

\(^2\) Implicit in (3-1) is that entrepreneurs do not use the borrowed fund to buy bubbles. Entrepreneurs will borrow only for capital investment because the rate of return from capital is greater than the one from holding bubbles when (3-1) binds with equality, as argued below.

\(^3\) A number of other incentive considerations allow one to derive the similar borrowing constraint. For example, the literature on credit rationing (e.g. Stiglitz and Weiss (1981), Williamson (1986), Aghion and Bolton (1997), and others) leads eventually to the same specification.
aggregate supply of fund by investors, except for holding bubbles. Incorporating the agent’s wealth \( X_t = A_t w_t + H_{t-1} (1 + r_t) \) into the above equality, we have

\[
(3-2) \quad s_t \{ W(k_t) + h \frac{1+r_t}{1+g} \} + h = (1+g)k_{t+1} + b_t ,
\]

which is the same as (2-5).

The determination of the real interest rate requires careful analysis. Either the profitability constraint or the borrowing constraint should bind with equality. If the borrowing constraint is not binding, the profitability constraint should bind with equality, whereas if the borrowing constraint is binding with equality, the profitability constraint may not be binding. The above argument is summarized by the following;

\[
(3-3) \quad 1 + r_{t+1} = \min \{ f'(k_{t+1}), \lambda f'(k_{t+1}) \frac{I_t}{I_t - X_t} \} .
\]

Without loss of generality, we confine attention on symmetric equilibria in which all entrepreneurs choose the same amount of investment.

When the borrowing constraint binds with equality, (3-3) is replaced by

\[
(3-4) \quad 1 + r_{t+1} = \frac{\lambda f'(k_{t+1})(1+g)k_{t+1}}{(1+g)k_{t+1} - \alpha s_t (w_t + h \frac{1+r_t}{1+g})}.
\]

When the borrowing constraint is binding with equality, (2-6), (3-2), (3-4), and \( s_t = s_L \) or 1 define a bubbly economy with \( b_t > 0 \) for \( t \to \infty \). The steady state is expressed as a pair \( (\tilde{k}_b, \tilde{b}, \tilde{s}, \tilde{r}_b) \), satisfying

\[
(3-5) \quad \tilde{s} \{ W(\tilde{k}_b) + h \} + h = (1+g)\tilde{k}_b + \tilde{b}
\]

\[
(3-6) \quad 1 + \tilde{r}_b = \lambda f'(\tilde{k}_b) \frac{(1+g)\tilde{k}_b}{(1+g)\tilde{k}_b - \alpha \tilde{s} \{ W(\tilde{k}_b) + h \} } ,
\]

\[
(3-7) \quad \tilde{r}_b = g ,
\]

with \( \tilde{b} > 0 \).

Before proceeding to the analysis of the bubbly economy, it is useful to consider the bubbleless economy. Equation (3-3), using (3-2), reduces to

\[
(3-8) \quad 1 + r_{t+1} = \min \{ f'(k_{t+1}), \lambda f'(k_{t+1}) \frac{k_{t+1}}{(1-\alpha)k_{t+1} + \alpha h/(1+g)} \} .
\]
We see that \( \frac{k_{t+1}}{(1-\alpha)k_{t+1} + ah/(1+g)} \) is increasing but below \( \frac{1}{1-\alpha} \) as \( k_{t+1} \) rises. It is obvious to verify that \( \alpha + \lambda < 1 \) is a sufficient condition under which the borrowing constraint is binding, to have

\[
1 + r_{t+1} = \frac{\lambda f'(k_{t+1})}{(1-\alpha)k_{t+1} + ah/(1+g)}.
\]

We focus on the case when the borrowing constraint is binding, and so impose the following:

**Assumption 1** \( \alpha + \lambda < 1 \).

Assumption 1 is intended to describe an economy when financial market imperfections are serious.\(^4\) Incorporating (3-9) into the market clearing (3-2) leads to the following dynamics of \( k_t \):

\[
(1 + g)k_{t+1} = s_t[W(k_t) + h\frac{\lambda f'(k_t)k_t}{(1-\alpha)k_t + ah/(1+g)}] + h.
\]

The steady state is characterized by the triplet \( \tilde{k}, \tilde{r}, \tilde{s} \), satisfying

\[
(1 + g)\tilde{k} = \tilde{s}[W(\tilde{k}) + h\frac{\lambda f'(\tilde{k})\tilde{k}}{(1-\alpha)\tilde{k} + ah/(1+g)}] + h,
\]

\[
1 + \tilde{r} = \frac{\lambda f'(\tilde{k})\tilde{k}}{(1-\alpha)\tilde{k} + ah/(1+g)},
\]

and \( \tilde{s} = s_L \) or 1. Given \( k_0 \), the economy converges monotonically to \( \tilde{k} \).

The equilibrium interest rate differs from the marginal product of capital so that we have an interesting case of \( f'(\tilde{k}) > 1 + g > 1 + \tilde{r} \). Since \( f'(.) \) has a natural interpretation of the

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\(^4\) The fraction of entrepreneurs \( \alpha \) is a measure of separation between creditors and debtors, and matters when the borrowing constraint is crucial. As \( \alpha \to 1 \), outside funds are negligible, and all investment is carried out directly by entrepreneurs, while as \( \alpha \to 0 \), outside funds are more important, and each of entrepreneurs has to borrow the greater amount from investors. Another parameter \( \lambda \) captures the development of the contract enforcement mechanism as argued above. As \( \lambda \to 1 \), the incentive compatibility constraint is always satisfied, and entrepreneurs would be able to borrow as much as possible, taking \( r_{t+1} \) as given. As \( \lambda \to 0 \), entrepreneurs would be able to borrow nothing and hence have to self-finance their investment entirely.

\(^7\) In this economy, the efficiency result is then standard. The bubbleless equilibrium is dynamically efficient if \( \tilde{r} > g \), while otherwise, it is dynamically inefficient and the asymptotically bubbly equilibrium is dynamically efficient [Tirole (1985, Proposition 2)].
“average” rate of return to capital, $f'(\tilde{k}) > 1 + g$ corresponds to the condition for dynamic efficiency proposed by Able et al (1989), saying that dynamic efficiency is satisfied when the capital income is greater than the aggregate investment. Later we argue on dynamic efficiency.

We turn to the analysis of the bubbly economy. Our focus is whether bubbles are complements or substitutes with investment. The central argument is which is stronger between the **collateral effect** and the **competition effect**. The competition effect implies that bubbles crowd out investment via the route of rising interest rate. The collateral effect implies that bubbles promote investment when entrepreneurs put up bubbles as internal wealth used for implementing investment projects.

Tirole (1985) demonstrates that the bubbles arise when capital over-accumulation arise in the bubbleless economy, and move the economy to the Golden Rule so as to restore efficiency. However, when there financial market imperfections, we derive the different property that is useful for the analysis.

**Proposition 1**
If the borrowing constraint is binding with equality, the bubbly steady state features less stock of capital than the Golden Rule.

Proof. We derive the steady-state relation between $k$ and $r$ as

(#) $1 + r = \min\{f'(k), \Lambda(k)\}$,

where $\Lambda(k)$ satisfies

$$\Lambda(k) = \frac{\lambda f'(k)(1 + g)k}{(1 + g)k - \alpha s(W(k) + h \frac{\Lambda(k)}{1 + g})}.$$ 

In the $(k, r)$ plane, the real interest rate should be lower between the two curves. In Figure 1, we illustrates the case when $\Lambda(k)$ is always lower than $f'(k)$. When the borrowing constraint is binding, for any given $r$, capital should be smaller than otherwise. We have $\tilde{k}_B < k_{GR}$ for $r = g$. Q.E.D.

Bubbles move the capital stock to the smaller level than the Golden Rule, and are coexistent with capital under-accumulation. This finding suggests that capital over-accumulation is not necessary in the bubbleless economy for bubbles to emerge.

We investigate dynamic properties. The term $(1 + r)/(1 + g)$ in (3-2) and (3-4) makes the analysis a little complicated. Using (2-6), (3-2), (3-4), rearranging terms, we have
Using (3-13), we rewrite the asset market clearing (3-2) as

\[
(1 + g)k_{i+1} + b_i = s_iw_i + s_ih \frac{\lambda f'(k_i)k_i}{(1-\alpha)(1 + g)k_i + ah} + \frac{\alpha s_ih}{(1-\alpha)(1 + g)k_i + ah} b_i + h.
\]

We find \( b_i \) in two terms. The second term in the LHS captures the competition effect, while the third term in the RHS the collateral effect that emerges when entrepreneurs receive endowment in the first period. To the extent that the coefficient of the latter is large, the collateral effect weakens the competition effect, but we see \( \frac{\alpha s_ih}{(1-\alpha)(1 + g)k_i + ah} < 1 \) so that the competition effect tends to be stronger than the collateral effect, given the saving rate \( s_i \), and the negative correlation will be found between bubbles and investment.

The economy is represented as a two-dimensional dynamic system that is characterized by the two loci, \( k_{i+1} = k_i \) and \( b_{i+1} = b_i \), given \( \bar{s} = s_L \) or 1. Demonstrating the dynamic properties is conventional with a phase diagram. The derivation of the curve \( k_{i+1} = k_i \) that follows from (3-2) and (3-13) is given by

\[
[1 - \frac{\alpha s_ih}{(1-\alpha)(1 + g)k_i + ah}] b_i = s_iw_i - (1 + g)k_i + h s_i \frac{\lambda f'(k_i)k_i}{(1-\alpha)(1 + g)k_i + ah} + h.
\]

The curve \( b_{i+1} = b_i \) follows from (3-8) and (3-11), and \( k_i \) and \( b_i \) satisfy

\[
1 + g = \lambda f'(s_iW(k_i) - b_i + \phi(k_i, b_i) + h) s_iW(k_i) - b_i + \phi(k_i, b_i) + h
\]

Figure 2 illustrates a phase diagram representing the dynamics of the economy with the unchanging saving rate, that is, with \( s_i = s_L \). The curve \( b_{i+1} = b_i \) is typically upward sloping, and crosses the curve \( k_{i+1} = k_i \) from below. The bubbly steady state is attained at W. The bubbly steady state features less investment than the bubbleless steady state.

Dynamic properties are qualitatively the same as those developed by Tirole (1985) and Weil
There exists an upwardly sloping saddle path to the bubbly steady state. Given \( k_0 \), all dynamic paths originating from below \( b_0 \) converge toward the bubbleless steady state. Trajectories starting above \( b_0 \) are infeasible as they all lead to the resource constraint being violated in finite periods.

We are now in a position to describe the dynamics when bubbles emerge. Suppose that the economy is in the bubbleless steady state. As bubbles emerge, the economy jumps upwardly to reach the saddle path of the bubbly steady state. The interest rate rises, investment declines, and bubbles also decline as the economy converges to the bubbly steady state. The emergence of bubbles necessarily crowds out investment.

Let \( \frac{k}{W(k)} f'(k) + \{1 - \frac{k}{W(k)}\}(1 + r) \) denote the rate of return to capital for entrepreneurs. Sufficient conditions under which the aggregate saving rate does not change when bubbles arise are \((1 + g)\beta_i < 1 \) and \( z(g, \tilde{k}_b) \beta < 1 \). Then impatient agents consume in both the bubbleless and bubbly economies. We summarize properties of equilibria in the following.

**Proposition 2**

Suppose that \((1 + g)\beta_i < 1 \) and \( z(g, \tilde{k}_b) \beta < 1 \) both hold so that the saving rate remains unchanged when bubbles arise.

(a) If \( \tilde{r} > g \), the economy is bubbleless and the interest rate converges to \( \tilde{r} \).

(b) If \( f'(\tilde{k}) > 1 + g > 1 + \tilde{r} \), there exists a unique bubbly economy with initial bubble \( b_0 \). The per-effective-worker bubbles converge to \( \tilde{b} \) and the interest rate converges to \( g \). In the bubbly economy, the steady-state per-effective-worker capital, denoted \( \tilde{k}_B \), satisfies \( f'(\tilde{k}_B) > 1 + g = f'(k_{GR}) \), with \( \tilde{k}_B < \tilde{k} < k_{GR} \).

(c) If \( 1 + g > f'(\tilde{k}) > 1 + \tilde{r} \), there exists a unique bubbly economy with initial bubble \( b_0 \). The per-effective-worker bubbles converge to \( \tilde{b} \) and the interest rate converges to \( g \). In the bubbly economy, the steady-state per-effective-worker capital satisfies \( f'(\tilde{k}_B) > 1 + g = f'(k_{GR}) \), with \( \tilde{k}_B < k_{GR} < \tilde{k} \).

A heuristic proof of Proposition 2 is as follows. Agents require that, at the stationary state, the
rate of return on bubbles, $1 + g$, be at least equal to the rate of return on lending, $1 + \tilde{r}_b$, so that it must be the case that $g \geq \tilde{r}_b$ if bubbles should happen. On the other hand, at the steady state, the presence of bubbles decreases the capital stock relative to the bubbleless economy, so that we must have $\tilde{k}_b < \tilde{k}$ and hence $\tilde{r}_b > \tilde{r}$ if $\tilde{b} > 0$. Therefore, the necessary condition for bubbles to be sustainable is $g > \tilde{r}$. Conversely, if $g > \tilde{r}$, bubbles absorb the aggregate savings and reduces the capital stock until the interest rate $1 + \tilde{r}$ is pushed up to $1 + g$.

Finally, if $\tilde{r} > g$, it must be the case that $\tilde{r}_b > \tilde{r}$, but then it follows that the aggregate bubbles per-effective-worker should grow indefinitely, which is infeasible.

Bubbles can arise even if the rate of return to capital is greater than the growth rate (Proposition 2(b)). The sustainability of bubbles depends on the relation between the growth rate and the real interest rate, not the rate of return to capital. The case of Proposition 2(b) is interesting and will be relevant for the current observation of the world economy. The borrowing constraint induced by the weak financial system gives rise to the weak demand for investment and the low real interest rate that calls for asset bubbles, which in turn shrinks further investment. The model of finite-horizon agents with financial frictions explains the low real interest rate, asset bubbles, and capital under-accumulation.

4. Increase in Savings and Bubble Cycles

Having studied so far the case when bubbles emerge but the saving rate does not change, we turn to the case when the saving rate changes.

The rise in the interest rate in the bubbly economy may increase savings, which, in turn, will generate the positive feedback between bubbles and investment. When $(1 + g)\beta_s > 1$ and $(1 + g)\beta_s > 1$ are met, all agents save all the middle-aged wealth at least at the steady state of the bubbly economy. If $(1 + \tilde{r})\beta_s < 1 < (1 + g)\beta_s$ and $z(\tilde{r}, \tilde{k})\beta_s < 1 < z(g, \tilde{k})\beta_s$ jointly hold, then impatient agents consume in the bubbleless economy, but save in the bubbly economy. When bubbles emerge, the boost of savings weakens the competition effect, and reinforces the collateral/investment feedback so as to generate the comovement between bubbles and investment. The rise in savings enables the collateral effect to be stronger than the competition effect, and bubbles move with investment in the same direction.

Figure 3 illustrates a phase diagram representing the dynamics of the economy when the
saving rate changes when bubbles arise. The curve \( k_t = k_{t+1} \) shifts outwardly when bubbles arise so that the bubbly steady state features more investment than the bubbleless steady state.

Suppose that the economy is in the bubbleless steady state. As bubbles emerge, the economy jumps upwardly to reach the saddle path of the bubbly steady state, \( V \). In response to the rise in the interest rate, savings increase, investment booms, and bubbles also appreciate as the economy converges to the bubbly steady state. Bubbles crowd investment in. We summarize properties of equilibria in the following.

**Proposition 3**

Suppose that \((1+\tilde{r})\beta < (1+g)\beta\) and \(z(\tilde{r}, \tilde{k})\beta < z(g, \tilde{k})\beta\) so that the saving rate rises when bubbles arise.

(a) If \(\tilde{r} > g\), the economy is bubbleless and the interest rate converges to \(\tilde{r}\).

(b) If \(f'(\tilde{k}) > 1 + g > 1 + \tilde{r}\), there exists a unique bubbly economy with initial bubble \(b_0\). The per-effective-worker bubbles converge to \(\tilde{b}\) and the interest rate converges to \(g\). In the bubbly economy, the steady-state per-effective-worker capital, denoted \(\tilde{k}_B\), satisfies

\[
f'(\tilde{k}_B) > 1 + g = f'(k_{GR}), \text{ with } \tilde{k} < \tilde{k}_B < k_{GR}.
\]

(c) If \(1 + g > f'(\tilde{k}) > 1 + \tilde{r}\), there exists a unique bubbly economy with initial bubble \(b_0\). The per-effective-worker bubbles converge to \(\tilde{b}\) and the interest rate converges to \(g\). In the bubbly economy, the steady-state per-effective-worker capital satisfies

\[
f'(\tilde{k}_B) > 1 + g = f'(k_{GR}), \text{ with } \tilde{k}_B < k_{GR} < \tilde{k}.
\]

The emergence of bubbles is closely linked to the saving behavior of agents. When people anticipate bubbles, all people save, and bubbles are self-fulfilling. We provide a story of business cycles that are driven by the boost and burst of bubbles.

Suppose that the sunspot variable is a Markov chain with two states, \(\{b, c\}\). When the state is \(b\), all agents coordinate their expectations on the bubbly path. When the state is \(c\), they do on the path toward the low capital steady state \(D\). Let the transition probability in which the state \(b\) (or \(c\)) occurs next period given that the current state is \(b\) (or \(c\)) is \(\delta\), with \(\delta \rightarrow 1\). Figure 4 illustrates a cycle of bubbles of \(D \rightarrow V \rightarrow W \rightarrow Z \rightarrow D\), and resembles business fluctuations of the Kardor-Kalecki type.
We interpret the change in savings in a number of ways. Basically, the link between bubbles and saving is not unconventional. A large part of the flows of income are consumed, but incomes from capital gains of assets are almost reinvested. When the boom is accompanied by assets appreciations, the saving rate should have risen, whereas when the bust is accompanied by assets depreciations, the saving rate should have fallen.

Our theory should predict the Granger causality both from growth to saving and from saving to growth. However, Carroll and Weil (1994) find that growth Granger-causes saving with a positive sign, but that saving does not Granger-cause growth. The discrepancy may arise from the fact that the data does not typically account for capital gains as income so that the calculated saving rate may be undervalued (overvalued) in the period of boom (bust).

Although we have constructed the closed economy, the increased savings may be interpreted to include capital inflows from foreign countries. Foreign saving can flow into the home country, and generate bubbles accompanied by the rise in the domestic interest rate, which in turn gives rise to the investment boom. The major source of funding for the investment boom in the United States was the current account.

5. Dynamic Efficiency

We now turn to the question of dynamic efficiency. We use the dynamic efficiency criterion by Cass (1972) who defines that the economy is dynamically efficient if there does not exists another feasible sequence of capital which provides at least as much aggregate consumption at all dates and strictly higher aggregate consumption in at least one date. We should note that our economy is Pareto sub-optimal in the sense that some intra-generational transfer of income can make all agents better off. If an intra-generational transfer of income would be permitted between investors and entrepreneurs through government intervention at the first period of their lives, an appropriate tax-subsidy scheme will move the economy substantially to the Diamond-Tirole model. It turns out that we investigate dynamic efficiency of the economy that is not Pareto optimal.

First, if \( \tilde{r} > g \), the efficiency result is straightforward because the sustainable bubbles are ruled out under perfect foresight. Secondly, we consider the case for \( f'(\tilde{k}) > 1 + g > 1 + \tilde{r} \), where the per-effective-worker aggregate consumption is less than the Golden Rule at the initial equilibrium without bubbles. Bubbles move the capital stock down to an even smaller level than
the Golden Rule, and decrease the consumption. Finally, we consider the case of capital over-accumulation in the bubbleless equilibrium, with $1 + g > f'(\tilde{k}) > 1 + \tilde{r}$. As shown in Figure 5, there exists a $k$, under which the per-effective-worker aggregate consumption is the same as the one at $\tilde{k}$, satisfying $f(k) - (1 + g)\tilde{k} = f(\tilde{k}) - (1 + g)\tilde{k}$ and less than $k_{GR}$. The welfare implications differ according to whether the steady state of the asymptotically bubbly equilibrium $\tilde{k}_B$ lies greater than $k$ or not. We summarize the following.

**Proposition 4**

(a) Suppose that the saving rate remains unchanged when bubbles arise. If $\tilde{r} > g$, the bubbleless economy is dynamically constrained efficient.

(b) If $f'(\tilde{k}) > 1 + g > 1 + \tilde{r}$, the asymptotically bubbly economy does not improve efficiency. The bubbleless economy is dynamically constrained efficient.

(c) If $1 + g > f'(\tilde{k}) > 1 + \tilde{r}$, the asymptotically bubbly economy may or may not improve efficiency. If $\tilde{k}_B < k$, the asymptotically bubbly economy does not improve efficiency, and the bubbleless economy is dynamically constrained efficient, while if $\tilde{k}_B > k$, it improves efficiency so that the bubbleless economy is dynamically constrained inefficient.\(^8\)

Proposition 3(a) and 3(b) say that $f'(\tilde{k}) > 1 + g$ is a sufficient condition under which the bubbleless economy is dynamic efficient. Combined with Proposition 2(b), it turns out that bubbles can arise even when the bubbleless economy is dynamic efficient. Proposition 3(c) says that $1 + g > f'(\tilde{k})$ is not a sufficient condition under which the bubbleless economy is dynamic inefficient.

The mechanism under which asset bubbles arise even when the bubbleless economy is dynamically efficient is closely related to the fact that the bubbleless economy is Pareto sub-optimal. Several other papers have demonstrated that, in the presence of wedge between social and private returns to capital, bubbles can arise even when the bubbleless economy is dynamic efficient (e.g., Saint-Paul (1992), Grossman and Yanagawa (1993) and Femminis (2002)).\(^9\)

\(^8\) The proof of © is left to Appendix.

\(^9\) Saint-Paul (1992), Grossman and Yanagawa (1993) provide endogenous growth models in which bubbles arise even when the private return to capital falls short of the growth rate, and
Abel et al (1989), Zilcha (1992), and Bohn (1995) investigate dynamic efficiency in stochastic models with infinitely-lived agents. Their implication is that dynamic efficiency depends on the relation between the growth rate and the rate of return on “risky” capital, not the “safe” interest rate, and that the Non-Ponzi condition is satisfied and so sustainable bubbles are excluded.

In our economy, dynamic efficiency is satisfied even when the “average” rate of return to capital is smaller than the growth rate so that when the Abel’s criterion is violated. Furthermore, asset bubbles arise even when the economy is dynamically efficient so that the Abel’s criterion is satisfied.11

11 Kraay and Ventura (2005) provide a counterexample in which the Abel’s condition is satisfied but the economy is dynamically inefficient.
References
Ventura, J., 2003, Economic growth with bubbles,
Figure 1    Capital under-Accumulation in the Bubbly Economy

\[ f'(k) \]

\[ \Lambda(k) \]

\[ 1 + r \]

\[ 1 + g \]

\[ \tilde{k}_B \]

\[ k_{GR} \]
Figure 2  Equilibrium Dynamics with Unchanging Saving Rate
Figure 3  Equilibrium Dynamics with Increased Saving Rate
Figure 4  Bubble Cycle

bubbleless steady state  bubbly steady state
Figure 5   Bubble Dilution