Sunspot Equilibria in a Production Economy:
Do Rational Animal Spirits Cause Overproduction? *

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Abstract
We study a standard two period economy with one nominal bond and one firm. The firm finances the input with the nominal bond in the first period, and its profits are distributed to the shareholders in the second period. We show that in the neighborhood of each efficient equilibrium, a sunspot equilibrium also exists. It is shown that the equilibrium interest rate is lower than the efficient level, and that there is over production in the sunspot equilibrium, under some conditions. However, there is no sunspot equilibrium if the profit share of the firm can be traded as well as the bond. (JEL classification numbers: D52, D53, D61)

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1 Introduction

We consider a simple two period private ownership economy with production. There is a single perishable good in each period, traded competitively in each period. There is a nominal bond market in the first period. There is one firm which uses the first period good as input and produces the second period good. Since the revenue is earned in the second period, the firm needs to issue the bond for purchasing its input in the good market in the first period. The households are risk averse, and the technology is convex, so no economic agent favors randomness per se. The purpose of this paper is to analyze the properties of sunspot equilibria in this economy.

Recall that sunspots are a theoretical device to model random phenomena which do not affect tastes, endowments of goods and resources, or production technology, within the framework of rational expectations. Such phenomena include for instance price uncertainty, fear of inflation, animal spirits of investors, and the psychology of the markets in general. Under the standard convexity assumptions, a sunspot equilibrium is Pareto inefficient. So in other words, we investigate if and how rational animal spirits cause inefficiency in a production economy.

The existence and the real indeterminacy of sunspot equilibria in pure exchange economies have been investigated extensively. But production economies have not been studied systematically in this context to the best of our knowledge. Introducing production in an incomplete market setting is known to be a challenge, because the objective of the firm is not clearly defined in some cases. In this paper, we do not try to resolve how this issue should be addressed: instead, we postulate that the firm maximizes expected profits. Although no formal justification is given, expected profit maximization appears at least very plausible in the simple setup we study.

Given the postulate, many insights from the case of pure exchange are then valid in the production economy as well, and we shall take advantage of them whenever possible in this paper. After providing a formal description of the model and the definition of sunspot equilibria in Section 2, we study the existence problem in Section 3. We shall show that a sunspot equilibrium exists, with only very non-generic exceptions. The idea

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1See the seminal paper Cass and Shell (1983).
2See Cass (1992) for a short overview. See also Gotardi and Kajii (1999), where a real asset instead of a nominal bond is considered.
is similar to that for the exchange economy with a nominal bond. Since market clearing in the bond market is enough to establish general equilibrium in all the markets, one can arbitrarily fix the real values of nominal returns depending on the sunspot states, and let the bond price adjust to clear the market. Except for some knife edge cases, the resulting consumption will be random, and we have a sunspot equilibrium (Proposition 1).

A sunspot equilibrium is Pareto inefficient, and there are two sources for the inefficiency. The first is distributional inefficiency, which is the focus of the pure exchange models in the literature: in effect, households consume according to an extrinsic lottery in the second period, which is welfare worsening by risk aversion. The second is production inefficiency: the firm may be producing too much or too little in a sunspot equilibrium. This point cannot be addressed in the pure exchange models obviously, and hence this paper supplies a set of new insights.

Section 4 is devoted to the question of production inefficiency, which is the main part of this paper. We establish characterization results (Proposition 2 and Proposition 3) which identify exactly when over/under production occurs in sunspot equilibria around efficient equilibria. As is often the case in general equilibrium analyses, the standard set of assumptions which guarantees the existence of an equilibrium is not strong enough to determine if either over production or under production is to take place. We argue nevertheless that in standard settings, a sunspot equilibrium tends to exhibit over production: in a sunspot equilibrium, the price of the bond is too high, i.e., the rate of interest is too low, so that the firm produces too much. That is, rational animal spirits tend to cause overproduction.

In Section 5, we extend the model by introducing a market for the profit share of the firm. Of course, if sunspots do not matter, trading profit share is redundant since the price of the share must be determined in such a way that the bond and the share are equivalent as assets in our set up. This is a classical justification of why we do not consider markets for shares in the textbook model of private ownership economies. But in a sunspot equilibrium, this equivalence might break down and so introduction of the profit share market could generate a different kind of sunspot equilibrium.

Interestingly enough, in this extended set up, we show in Proposition 4 that there is no sunspot equilibrium; the argument is a slight modification of the ingenious idea of Mas-Colell (1992).
consumption is affected by sunspots. The households will trade a “mutual fund” consisting of the bond and the share whose payoff is independent of sunspots. It is also shown that although the returns of the bond and the share might be affected by sunspots, the production level and equilibrium consumption are the same as the textbook case of no sunspot states. That is, if there is a market for the share, in a sharp contrast to the case without it, sunspots do not matter as far as economic welfare is concerned.\textsuperscript{4}

Section 6 contains discussions on several issues. First, we comment on the issue of welfare gains and losses: although a sunspot equilibrium is inefficient, there might be some households who are better off in the sunspot equilibrium than in an efficient equilibrium.\textsuperscript{5} Next, we provide a comparison with models with background income risks, which yield results with a flavor similar to ours. Finally, we give some remarks on extending our model to allow more than one good in each period and multiple firms.

2 The Model

We consider a standard competitive two-period economy with production. There is one perishable consumption good in each period. There is one firm with an increasing and concave production function \( f \) with \( f(0) = 0 \); the firm produces \( f(z) \) units of good in the second period (period 1) from \( z \) units of good in the first period (period 0). There are \( H \geq 2 \) households, labelled by \( h = 1, 2, \ldots, H \). Household \( h \) is endowed with \( e^0_h \) units of the consumption good as well as a profit share \( \theta_h \) of the firm in period 0, and with \( e^1_h \) units of the consumption good in period 1. By assumption \( \sum_h \theta_h = 1 \) holds. We write \( e_h = (e^0_h, e^1_h) \).

In the first period, period 0, a nominal bond which pays off one unit in units of account (say, dollar) in the second period is traded in a competitive market. The bond price in units of the period 0 good is denoted by \( q > 0 \). The firm will finance its input by the bond, which will be held by households in equilibrium. Write \( B \) for the amount of the bond supplied by the firm, and \( z_h \) for the amount demanded by household \( h \). So the firm

\textsuperscript{4}Here neither the bond nor the share is redundant in most equilibria. There are other cases where the existence of some markets makes the economy immune to sunspots, but those markets are redundant in equilibria. In a pure exchange setup, Kajii (1997) shows that if there are an enough number of financial options there is no sunspot equilibrium and the options are all redundant.

raises \( qB \) in units of the period 0 good, which is used as input. Consequently, it produces \( f(qB) \) units of the period 1 good and is liable for the outstanding bond, i.e., \( B \) dollars.

The real returns of the nominal bond will be determined in the markets, which might be random; the households expect that the price of the consumption good in dollars might be random, and then consequently the real value of bond’s nominal payoff is expected to be a random variable. In other words, the households expect inflation, and the rate of inflation may vary according to the state of the economy. This idea is formally described using sunspots as follows.

At the beginning of the second period, the state of economy is revealed. State \( s, s = 1, ..., S \) occurs with probability \( \mu_s > 0 \). We assume that these are sunspot states. That is, by assumption, the state is publicly observable, and the fundamentals of the economy described so far are independent of the realization of the sunspot state. It is often convenient to refer to the first period (period 0) as state \( s = 0 \), and we shall follow this convention throughout the paper.

Write \( p_s > 0 \) for the price of the consumption good in dollars when state \( s \) is realized. Then \( r^s := \frac{1}{p^s} \) is the real value of one dollar in state \( s \). By construction, the real payoff of the bond per unit is also \( r^s \) in state \( s \), so we shall refer to \( r^s \) as the (gross) return of the bond in state \( s \). Since only the relative prices matter and the unit for trading the bond can be adjusted arbitrarily, we will always set \( \sum_{s=1}^S \mu^s r^s = 1 \) without loss of generality, i.e., the expected real payoffs of the bond in units of the period 1 good is normalized to be one from now on. We shall write \( \tilde{r} = (r^1, ..., r^S) \in \mathbb{R}_{++}^S \) for the vector of returns.

Sunspots do not influence production, but nevertheless, since the real returns of the bond may be random, the level of profits depends on sunspots in general. Specifically, the realized profit is \( \Pi^s := f(qB) - r^s B \) in state \( s \) in units of the good, which will be distributed to the households according to the profit share \( \theta_h, h = 1, ..., H \). We assume that the firm maximizes expected profits; since \( \sum_{s=1}^S \mu^s = 1 \) and we normalize \( \sum_{s=1}^S \mu^s r^s = 1 \), this means that the firm takes bond price \( q \) as given and solves the following problem:

\[
\max_{B \geq 0} f(qB) - B,
\]

which is a well defined concave problem under our assumptions.

Notice that the firm’s optimal decision as well as the maximized level of profit is independent of \( \tilde{r} \). Let \( \Pi^*(q) \) be the maximum profit given price \( q \), and \( B^*(q) \) be the
set of profit maximizers. That is, the firm will choose $B \in B^*(q)$ given bond price $q$. By construction, if $B \in B^*(q)$, the expected profit is $\Pi^*(q) = f(qB) - B$ and the level of profit in state $s$ is $\Pi^*(q) + (1 - r^s)B$, $s = 1, ..., S$. Note that although the expected profit must be non-negative since zero production is feasible, ex post profit $\Pi^*$ may be negative for some $s$. Also notice that the choice of $B \in B^*(q)$ is indeterminate as far as the expected profit is concerned, but it does affect the randomness of profits in principle.\footnote{For instance, some shareholders may prefer smaller $B$ for less randomness in profits, but the others may not. Then the shareholders might disagree on the choice among $B$ in $B^*(q)$. Moreover, since some shareholders might even prefer lower expected profits if randomness is reduced, the expected profit maximization might not be the shareholders’ interests. This is a potentially interesting question, but we do not pursue it in this paper.}

When $f$ is strictly concave in the sense of $f'' < 0$ everywhere, $B^*(q)$ is singleton and in such a case we shall abuse notation to denote the single element by $B^*(q)$ as well. Note that $B^*(q)$ is increasing in $q$.

For household $h$, holding $z_h$ units of the bond at the end of period 0, $r^s z_h$ units of the consumption good is delivered at the beginning of period 1 in state $s$. Also, household $h$ receives $\theta_h \Pi^s$ for profit share, thus the consumption of household $h$ is $e^0_h - q z_h$ in period 0 and $e^1_h + r^s z_h + \theta_h \Pi^s$ in state $s$ in period 1. Recall that $\Pi^s < 0$ is possible and so household $h$ may be forced to compensate for firm’s losses; i.e., the liability of equity is unlimited. Also $z_h$ may be negative, i.e., household $h$ may choose to borrow.

By assumption, the households take random profits $\tilde{\Pi}$ as given, as well as the other price parameters. Rational expectations then require that random profits are given by an accounting identity $\tilde{\Pi} = \Pi^*(q) + (1 - \tilde{r})B$. Since we focus on rational expectations, we may as well assume that the households take the bond supply $B$ and the (average) profit $\Pi^*(q)$ as given. Taking this into account, the second period consumption can be written in different ways as follows:

$$e^1_h + \tilde{r} z_h + \theta_h \tilde{\Pi} = e^1_h + \theta_h (\Pi^*(q) + B) + \tilde{r} (z_h - \theta_h B)$$

$$= e^1_h + \theta_h \Pi^*(q) + \tilde{r} z_h + (1 - \tilde{r}) \theta_h B.$$  \hspace{1cm} (2)

The preferences of household $h$ are represented by a von Neumann Morgenstern utility function $u_h : \mathbb{R}^2_{++} \to \mathbb{R}$; that is, given bond price $q$, a vector of rate of returns $\tilde{r} = (r^s)_{s=1}^S \in \mathbb{R}^S_+$ with $\sum_{s=1}^S \mu^s r^s = 1$, and a profile of profits $\tilde{\Pi} := (\Pi^s)_{s=1}^S \in \mathbb{R}^S$, if household
$h$ chooses $z_h$ such that consumption is positive, i.e., $e^0_h - qz_h > 0$ and $e^1_h + r^s z_h + \theta_h \Pi^s > 0$
for every $s = 1, ..., S$, the utility is given by:

$$\sum_{s=1}^{S} \mu^s u_h \left( e^0_h - qz_h, e^1_h + r^s z_h + \theta_h \Pi^s \right).$$  \hfill (4)

Household $h$’s problem is to choose $z_h \in \mathbb{R}$ to maximize the expected utility given in (4). Using the expectation operator $E$ with respect to probability measure $(\mu^s)_{s=1}^{S}$, and denoting with a slight abuse of notation by $\tilde{r}$ and $\tilde{\Pi}$ the random returns and profits, respectively, the utility function (4) can also be written as

$$E \left[ u_h \left( e^0_h - qz_h, e^1_h + \tilde{r} z_h + \theta_h \tilde{\Pi} \right) \right].$$  \hfill (5)

Note that both random returns and random profits contribute to the randomness of income in the second period. But recall the property of the second period consumption (2). We can re-write the utility function (5) further so that the vector $\tilde{r}$ is seen to be the single source of randomness, as follows:

$$E \left[ u_h \left( e^0_h - qz_h, e^1_h + \tilde{r} z_h + \theta_h \tilde{\Pi} \right) \right].$$  \hfill (6)

From this expression we see that, other things being equal, the utility is sensitive to a small change in random returns $\tilde{r}$, unless $z_h = \theta_h B$.

It is assumed that $u_h$ is $C^3$, diﬀerentiably strictly increasing (i.e., for any $x_h \in \mathbb{R}^2_{++}$, the gradient $Du_h(x_h)$ is strictly positive), diﬀerentiably strictly concave (i.e., for any $x_h \in \mathbb{R}^2_{++}$, the Hessian $D^2 u_h(x_h)$ is negative deﬁnite), and for each level set, its closure in $\mathbb{R}^2$ is contained in $\mathbb{R}^2_{++}$. The assumption of thrice diﬀerentiability is needed since the second derivatives of demand functions are important in our analysis in Section 4. But the reader will see that the diﬀerentiability assumption is not essential for the existence problem in Section 3 and for the nonexistence result in Section 5.

Under these assumptions, the function (4) is concave in $z_h$ and the optimal choice is characterized by a solution to the first order condition as follows:

$$\sum_{s=1}^{S} \mu^s \left\{ -q \frac{\partial}{\partial x_0} u_h \left( e^0_h - qz_h, e^1_h + r^s z_h + \theta_h \Pi^s \right) + \frac{\partial}{\partial x_1} u_h \left( e^0_h - qz_h, e^1_h + r^s z_h + \theta_h \Pi^s \right) r^s \right\} = 0,$$

\hfill (7)

where $\frac{\partial}{\partial x_0} u_h$ and $\frac{\partial}{\partial x_1} u_h$ are derivatives with respect to the first period consumption and the second period consumption, respectively. Using the expectation operator, (7) can
also be written as:

\[ E \left[ -q \left( \frac{\partial}{\partial x_0} u_h \right) + \left( \frac{\partial}{\partial x_1} u_h \right) \right] = 0, \tag{8} \]

where the derivatives are evaluated at the profile of the optimal consumption bundle.

The solution to (7) is unique if it exists by the strict concavity of the utility function.

The existence depends on the returns \( \tilde{r} \) among others, but since our relevant analysis will be done locally around a competitive equilibrium where the optimal choice is well defined, we will simply assume that a solution exists in the relevant domain of the analysis. For each household \( h \), denote by \( Z_h (q, \tilde{r}, B) \) the unique solution to (7); that is, \( Z_h (q, \tilde{r}, B) \) is the demand of household \( h \) for the bond given prices \( q \) and \( \tilde{r} \) and the bond supply \( B \) of the firm. Then \( Z(q, \tilde{r}, B) := \sum_{h=1}^{H} Z_h (q, \tilde{r}, B) \) is the total demand for the bond demanded by the households. It may first appear unusual that the demand functions depend on firm’s choice variable \( B \) in addition to prices, but as we have explained above, there is no loss as far as rational expectation equilibria are concerned. Note that since the bond supply function \( B^* (q) \) is a function of \( q \), and so \( Z(q, \tilde{r}, B) \) is effectively just a function of price variables \( (q, \tilde{r}) \) when we search an equilibrium.

The prices endogenously determined in the markets are \( q \) and \( \tilde{r} \). Thus the rational expectation equilibrium of this economy is defined as follows:

**Definition 1** A bond price \( q \) and a vector of returns \( \tilde{r} = (\cdot, \cdot, \cdot, \cdot, \cdot) \in \mathbb{R}^S \) with \( \sum_{s=1}^{S} \mu^s r^s = 1 \) constitute a competitive equilibrium if there exists \( B \in B^* (q) \) such that \( Z(q, \tilde{r}, B) - B = 0 \). A competitive equilibrium is called a sunspot equilibrium if the second period consumption is not constant across the states for some household.

The equilibrium condition above says that the bond market clears. As is usually the case, it can be readily shown that if the bond markets clear, all the good markets clear.

**Remark 1** Consider the case where the technology is strictly convex and so the supply \( B^* (q) \) is a singleton for any \( q \). Denoting the unique element by \( B^* (q) \) by convention, the equilibrium condition above can be written as \( Z(q, \tilde{r}, B^* (q)) - B^* (q) = 0 \).

**Remark 2** Note that an equilibrium with random \( \tilde{r} \) is not necessarily a sunspot equilibrium. From the property of the second period consumption (2), it readily follows that an equilibrium \( (q, \tilde{r}) \) is a sunspot equilibrium if and only if there is some \( h \) such that \( Z_h (q, \tilde{r}, B) \neq 0 \) where \( B \) is the corresponding equilibrium bond supply.
When $S = 1$, our model is a standard two period model of consumption and saving, and so an equilibrium exists and every equilibrium is Pareto efficient. An equilibrium for the case of $S = 1$ is called a \textit{certainty equilibrium}. Under our normalization, the real return of bond is one in any certainty equilibrium.

If $(q, 1) \in \mathbb{R}^2$ is a certainty equilibrium, it can be readily seen that $(\bar{q}, \bar{1})$ is an equilibrium for any $S > 1$, where $\bar{1} = (1, \ldots, 1) \in \mathbb{R}^S_{++}$. This is an equilibrium where the households think that the sunspot states do not affect the real returns of bond; that is, they expect that the return of bond in units of good is one for sure. Such an equilibrium is called a \textit{non-sunspot equilibrium} when $S > 1$. By the fundamental theorem of welfare economics applied to the certainty equilibrium and the (strict) risk aversion of households, a non-sunspot equilibrium is Pareto efficient. From now on, we assume that $S > 1$ to avoid triviality.

Conversely, since there is no uncertainty in production and so the aggregate consumption is independent of sunspots, the risk aversion of the households and the convexity of the technology imply that in any Pareto efficient allocation, the consumption of each household must be independent of sunspots. Hence, a sunspot equilibrium is inefficient in particular. But an efficient equilibrium may not be a non-sunspot equilibrium: it is possible that although the equilibrium returns $\bar{r}$ are random, the households use the bond to completely offset income risks generated by random profits, as will be seen in Example 1 and Proposition 4. But such an equilibrium is equivalent to a certainty equilibrium in allocation. More generally, we have the following.

\textbf{Lemma 1} Suppose a competitive equilibrium $(q, \bar{r})$ is not a sunspot equilibrium. Then $(q, \bar{1})$ is a non-sunspot equilibrium, and the level of production and the consumption in $(q, \bar{r})$ and those in $(q, \bar{1})$ are the same.

\textbf{Proof.} Let $(q, \bar{r})$ be an equilibrium which is not a sunspot equilibrium. There is nothing to show if $\bar{r} = \bar{1}$, so assume that $\bar{r} \neq \bar{1}$. By Remark 2, $Z_h(q, \bar{r}, B) = \theta_h B$ must hold for all $h$, where $B \in B^*(q)$. Since the consumption vector of household $h$ is independent of $s$ in $(q, \bar{r})$, the marginal utilities appearing in the first order condition (8) are constants. Since $E[\bar{r}] = 1$, we have $-q \left( \frac{\partial}{\partial x_0} u_h \right) + \left( \frac{\partial}{\partial x_1} u_h \right) = 0$ for every $h$ from (8). But this means that the first order condition for utility maximization given prices $(q, \bar{1})$ holds at $z_h = Z_h(q, \bar{r}, B)$ for every $h$, thus $Z_h(q, \bar{1}, B) = Z_h(q, \bar{r}, B)$ holds and the corresponding
consumption vectors are identical. The production side only depends on $q$. Therefore, $(q, \bar{1})$ is an equilibrium with the same production level and consumption allocation.

### 3 Existence of sunspot equilibria

We argue that a sunspot equilibrium exists. The intuition for the existence is simple. Basically in this model there are $S$ price variables: bond price $q$ and returns $r^1, \ldots, r^S$, but one degree of freedom is lost by normalization. On the other hand, there is one market, the bond market, which needs to be cleared, since the rest of the markets clear automatically if the bond market clears. So even if the returns $\tilde{r}$ are arbitrarily fixed, the bond price $q$ can be adjusted to clear the market. But if $\tilde{r} \neq \bar{1}$, the income will be random and so will be the consumption, except for some coincidental cases. Formally, we have the following existence result.

**Proposition 1** Let $(q, 1)$ be a certainty equilibrium, and denote by $\bar{z}_h$ the bond holding of household $h$ and by $\bar{B}$ the bond issued in the equilibrium. Then (1) there exists $\varepsilon > 0$ such that for any normalized returns $\tilde{r}$ with $|\tilde{r} - \bar{1}| < \varepsilon$, there is a bond price $q$ such that $(q, \tilde{r})$ is an equilibrium. (2) Moreover, if $\bar{z}_h - \theta_h \bar{B} = 0$ for some $h$, then there exists $\varepsilon > 0$ such that for any normalized returns $\tilde{r}$ with $|\tilde{r} - \bar{1}| < \varepsilon$, there is a bond price $q$ such that $(q, \tilde{r})$ is a sunspot equilibrium.

This result can be shown by a simple continuity argument, so we shall omit a proof. Roughly speaking, if $\tilde{r}$ is close to $\bar{1}$, the aggregate demand function for bond must look very close to the one for the economy with $S = 1$, and hence in particular there must be an equilibrium $(q, \tilde{r})$. Moreover, if $\bar{z}_h - \theta_h \bar{B} = 0$, the continuity implies that $Z_h (q, \tilde{r}, B) \neq \theta_h B$ where $B$ is the corresponding equilibrium bond supply, so it must be a sunspot equilibrium (see Remark 2).

The condition $\bar{z}_h - \theta_h \bar{B} = 0$ in (2) of Proposition 1 is indispensable. That is, it is possible that an equilibrium exists for any fixed $\tilde{r}$ arbitrarily close to $\bar{1}$, but the equilibrium is not a sunspot equilibrium (i.e., the consumption is constant), as the following example shows.

**Example 1** Assume that the households are identical. Then their choices must be identical by strict concavity. This means that in any equilibrium, $\bar{z}_h - \theta_h \bar{B} = 0$ for every $h$. 

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In particular, there is no sunspot equilibrium (see Remark 2).

This example is effectively a model of a representative agent, where there can be no trade for risk sharing purposes. It is nevertheless instructive since it implies that our results in the following sections will be relying on the heterogeneity of households’ characteristics.

But a sunspot equilibrium must exist, generically. As long as there is slight heterogeneity in the economy (e.g., households have the same preferences but endowed differently in goods and profit share), it is intuitively plausible that \( z_h - \theta_h B = 0 \) is unlikely to hold in a certainty equilibrium. In fact, although we do not elaborate on the details, it can be formally defined and established that \( z_h - \theta_h B = 0 \) is a non generic property if \( H > 1 \).\(^7\) So we contend that except for non-generic cases such as the case of completely homogeneous agents, a sunspot equilibrium exists in the neighborhood of a non-sunspot equilibrium.

4 Prudence and Over investment

A sunspot equilibrium is inefficient, and there are two sources for inefficiency. The first is distributional inefficiency: for a given aggregate supply of the good which is independent of sunspots, households’ consumption is nevertheless affected by sunspots. This aspect of inefficiency has been discussed extensively in the literature of exchange economies, so we do not endeavour to clarify further.\(^8\)

The second is production inefficiency, which we focus in this section: the firm may be producing too much or too little. More specifically, starting with a certainty equilibrium where production is done at an efficient level, we study the level of production in nearby sunspot equilibria, whose existence has been established in Proposition 1.

Before proceeding to a formal analysis, let us build up some intuition first. Fix a certainty equilibrium and fix any \( \tilde{r} \) close enough to \( \tilde{r} \) in the sense of Proposition 1. Since the firm’s problem (1) is independent of \( \tilde{r} \), the supply curve of the bond is unchanged, and hence we only need to examine the total demand of the households for the bond at the sunspot equilibrium. Then the key question will be how the demand curve shifts;

\(^7\)See Mas-Colell (1985), section 6.
that is, we need to see if the demand for bond gets larger or smaller under $\tilde{r}$, other things being equal. If the demand gets larger, then the price of bond must go up to clear the bond market, i.e., the (average) interest rate will go down, which then should induce over production. The case of under production can be understood analogously.

The bond is a risky asset in a sunspot equilibrium. It might therefore appear at first sight that the risk aversion should imply that the demand for the bond decreases. It is well known however that the risk aversion alone does not determine the sign in a partial equilibrium setting where the level of income is fixed: in fact, it is the magnitude of the relative prudence which plays an important role.\footnote{See Section 4.5 of Gollier (2001), for instance.}

Notice that there is another general equilibrium effect through profits, since the households’ income depend on the profit level, which is random. Even if the firm’s activity does not change so that the average profit remains the same, ex post profits will be more random which will make the second period income more random. Therefore, in principle this is potentially a complex problem of increasing risks in asset returns as well as background income risks.

On the other hand, risks in returns and profits are perfectly correlated in equilibrium, and hence the problem turns out to be manageable to some extent. Note that in equilibrium the second period income is given by (2): as far as the decision problem in equilibrium is concerned, the household effectively take the average profit $\Pi^* (q)$ as given. Moreover, its share of outstanding bond $\theta_h B$ is also taken as given, and the household solves a simple investment problem, controlling the net investment $z_h - \theta_h B$ whose returns are $\tilde{r}$.

Now we begin a formal analysis. For expositional simplicity, we shall assume that the utility functions are additively time separable in this section. The analysis can be done without the separability: we show the key result Lemma 2 below without separability assumption in Appendix, and the reader will see that the other results can be readily generalized in a similar manner. We write $u_h (x^0, x^1) = u^0_h (x^0) + u^1_h (x^1)$ for each $h$, and the utility maximization in equilibrium (6) can now be written as:

$$\max_{z_h} E \left[ u^0_h (c^0_h - q z_h) + u^1_h (c^1_h + \theta_h (\Pi^* (q) + B) + \tilde{r} (z_h - \theta_h B)) \right].$$

(9)
To describe the corresponding first order condition (8), set:

\[ F_h(q, \tilde{r}, z_h, B) := -u_h^0 (e_h^0 - q z_h) q + \sum_{s=1}^{S} \mu^s u_h^s \left( e_h^1 + \theta_h (\Pi^* (q) + B) + r^s (z_h - \theta_h B) \right) r^s. \]  

(10)

Then the first order condition (8), is now written as \( F_h(q, \tilde{r}, z_h, B) = 0 \).

Fix a certainty equilibrium \((\tilde{q}, 1)\) such that for any \( \tilde{r} \) close enough to \( \tilde{I} \), there is a sunspot equilibrium \((q, \tilde{r})\). Let \( \tilde{z}_h, h = 1, \ldots, H \), and \( \tilde{B} \) be the corresponding bond demand for household \( h \) and the bond supply, respectively, in the certainty equilibrium \((\tilde{q}, 1)\). To avoid the uninteresting case of zero production, assume that \( \tilde{B} > 0 \). Also let \( \tilde{x}_h^0 \) and \( \tilde{x}_h^1 \) be the certainty equilibrium consumption of household \( h \) in period 0 and 1, respectively.

Choose any returns \( \tilde{r} \) close enough to \( \tilde{I} \) so that there is a sunspot equilibrium \((q, \tilde{r})\). We first ask if the demand for the bond increase or decrease as returns change from \( \tilde{I} \) to \( \tilde{r} \), keeping \( \tilde{q} \) and \( \tilde{B} \) fixed. That is, we ask how the demand curve shifts around the certainty equilibrium.

First we shall establish some basic results on how individual household’s excess demand \( Z_h \) changes. We shall calculate changes when returns gets marginally risky. Write \( \tilde{r}^{-S} \) for \((r^1, \ldots, r^{S-1})\), and we shall set \( r^S = \left(1 - \sum_{s=1}^{S-1} \mu^s r^s \right) / \mu^S \) to keep the normalization \( E[\tilde{r}] = 1 \). Using this convention, define \( \tilde{Z}_h \) by the rule:

\[ \tilde{Z}_h (q, \tilde{r}^{-S}, B) := Z_h \left( q, \left( \tilde{r}^{-S}, \frac{1 - \sum_{s=1}^{S-1} \mu^s r^s}{\mu^S} \right), B \right), \]  

(11)

for each \( h \). Then our task is to find the derivatives of \( \tilde{Z}_h \) with respect to \( \tilde{r}^{-S} \), and evaluate them at \( \tilde{r}^{-S} = \tilde{I}^{-S} \).

From now on, the derivatives of utility functions are evaluated at the certainty equilibrium: \((\tilde{q}, 1)\), \( \tilde{B} \), and \((\tilde{x}_h^0, \tilde{x}_h^1)\), \( h = 1, \ldots, H \), unless specified otherwise. Differentiating (10), set

\[ \gamma_h := -\frac{\partial}{\partial z_h} F_h (q, \tilde{I}, \tilde{z}_h, \tilde{B}), \]  

(12)

\[ = -\left( u_h^0 q^2 + u_h^1 \right) \]  

(13)

for each \( h \). Under our assumptions on the utility function, \( \gamma_h > 0 \). Observe that by
symmetry and additive separability, \( \frac{1}{\mu^s} \frac{\partial^2}{\partial (r s)^2} F_h (\bar{q}, \bar{1}, \tilde{z}_h, \bar{B}) \) does not depend on \( s \), so set
\[
\alpha_h := \frac{1}{\mu^s} \frac{\partial^2}{\partial (r s)^2} F_h (\bar{q}, \bar{1}, \tilde{z}_h, \bar{B}) = u_h^{1m} \cdot (\tilde{z}_h - \theta_h \bar{B})^2 + 2 u_h^{1m} \cdot (\tilde{z}_h - \theta_h \bar{B})
\] for each \( h \). We have the following result on the first and the second derivatives of \( \tilde{Z}_h \):

**Lemma 2** At \((\bar{q}, \bar{1}, \bar{B})\), \( \frac{\partial}{\partial r} \tilde{Z}_h = 0 \) for every \( s = 1, \ldots, S - 1 \), and

\[
\left( \frac{\partial^2}{\partial r s \partial r s} \tilde{Z}_h \right)_{s, s'} = \frac{\alpha_h}{\gamma_h} M,
\]

where \( M \) is an \( S - 1 \) dimensional positive definite matrix determined by the probability vector \( \mu \) (thus independent of \( h \)).

A proof is given in Appendix. Since \( \gamma_h > 0 \), Lemma 2 says that as a function of \( \bar{r}^{-S} \), \( \tilde{Z}_h (\bar{q}, \bar{r}^{-S}, \bar{B}) \) is locally minimized at \( \bar{r}^{-S} = \bar{1} \) if \( \alpha_h > 0 \), and it is locally maximized if \( \alpha_h < 0 \). Thus if \( \alpha_h > 0 \), then for \( \bar{r}^{-S} \) close enough to \( \bar{1} \), \( \tilde{Z}_h (\bar{q}, \bar{r}^{-S}, \bar{B}) > \tilde{z}_h \). Similarly, the demand decreases if \( \alpha_h < 0 \).

It is useful to develop some intuition about Lemma 2 here. The first order effect disappears because of the envelope property. The reason why the second derivative plays a role can be understood as follows. Since we are interested in increasing risks in the sense of the second order stochastic dominance, if the function \( r \mapsto u_h^{1m} (e_1^1 + \theta_h (\Pi^* (\bar{q}) + B) + r (\tilde{z}_h - \theta_h \bar{B})) \) is convex, then by the definition (10) we have \( F_h (\bar{q}, \bar{r}, \tilde{z}_h, \bar{B}) > 0 \). In this case, since \( F_h \) is decreasing in \( z_h \), it follows that the demand must increase. It can be readily seen from (15), the parameter \( \alpha_h \) is nothing but the derivative of this function. Dividing (15) by \( -2 u_h^{1m} \cdot (\tilde{z}_h - \theta_h \bar{B})^2 \) we see that \( \alpha_h > 0 \) obtains if and only if
\[
\frac{- u_h^{1m}}{u_h^{1m}} > \frac{2}{\tilde{z}_h - \theta_h \bar{B}}.
\]

If household \( h \) is *absolutely prudent* in the sense of \( u_h^{1m} > 0 \), the inequality (17) holds if \( \tilde{z}_h - \theta_h \bar{B} < 0 \), that is, household \( h \) is a *net* lender. So these households will increase the demand for the bond when \( \bar{r} \) gets random. On the other hand, households with \( \tilde{z}_h - \theta_h \bar{B} > 0 \), a *net borrower*, the effect is ambiguous. So condition (17) above can be stringent in some setup.\(^\text{10}\)

\(^\text{10}\)Thus the logic behind the over production result is different from background risk models. See Section 6.
Next, we shall study the aggregate demand. Set $\tilde{Z} (q, \tilde{r}^S, B) := \sum_{h=1}^{H} \tilde{Z}_h (q, \tilde{r}^S, B)$.

**Lemma 3** If $\sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} > 0$, then for any $\tilde{r}$ with $E [\tilde{r}] = 1$ which is close enough to $\tilde{1}$, we have $Z (q, \tilde{r}, B) > B$. If $\sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} < 0$, for any $\tilde{r}$ with $E [\tilde{r}] = 1$ which is close enough to $\tilde{1}$, we have $Z (q, \tilde{r}, B) < B$.

**Proof.** Applying Lemma 2, at $(q, \tilde{1}, B)$, $\frac{\partial}{\partial \tilde{r}} \tilde{Z} = \sum_{h=1}^{H} \frac{\partial}{\partial \tilde{r}} \tilde{Z}_h = 0$ for every $s = 1, ..., S-1$, and $\left( \frac{\partial^2}{\partial \tilde{r} \partial \tilde{q}^s} \tilde{Z} \right)_{s,s'} = \left( \sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} \right) M$, where $M$ is positive definite. So if $\sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} > 0$, $\tilde{Z} (q, \tilde{r}^S, B)$ is locally minimized at $\tilde{r}^S = \tilde{1}^S$. If $\sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} < 0$, $\tilde{Z} (q, \tilde{r}^S, B)$ is locally maximized at $\tilde{r}^S = \tilde{1}^S$. Hence the result follows.

These results leads us to ask whether or not a natural set of assumptions determines the sign of $\sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h}$. Assuming absolute prudence, we would like to assert that it tends to be positive. The reason is as follows: as we have seen in (17), assuming absolute prudence, we have $\alpha_h > 0$ for the net lenders. Of course $\alpha_h < 0$ is not ruled out for the net borrowers, and this number could be large enough in absolute value to upset our assertion. But $\alpha_h < 0$ occurs for households whose prudence parameter is low, and/or whose net trade $\tilde{z}_h - \theta_h B$ is very small. Or to say the least, constructing an example of under production is not simple. Also, we do have $\sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} > 0$ in a special but interesting class of models of “ex post homogeneous” economy.

**Lemma 4** Assume absolute prudence for the second period utility function, $u_h^{1'''} > 0$ for every $h$. If the ratio of the absolute risk aversion, $\frac{u_h^{1''}/u_h^{0''}}{u_h^{0''}/u_h^{0'}}$, is identical for all the households in the certainty equilibrium, then $\sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} > 0$.

**Proof.** From (13) and (15), $\sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} = \sum_{h=1}^{H} \frac{u_h^{1''''} (\tilde{z}_h - \theta_h B)^2 + 2u_h^{1'''''} (\tilde{z}_h - \theta_h B)}{-(u_h^{0''''}/q^2 + u_h^{1''''})} + \sum_{h=1}^{H} \frac{2u_h^{1'''''} (\tilde{z}_h - \theta_h B)}{-(u_h^{0''''}/q^2 + u_h^{1''''})}$. The first term is positive by the absolute prudence. To evaluate the second term, note that by the efficiency of the certainty equilibrium, $u_h^{1''}/u_h^{0''}$ is constant across $h$, so the assumption on the absolute risk aversion implies that $u_h^{0''''}/u_h^{1''''}$ is the same for all $h$. Thus $\frac{2u_h^{1'''''} / u_h^{0''''}/q^2 + u_h^{1''''}}{2u_h^{1''''}} = 2 \left( \frac{u_h^{1''''}}{u_h^{0''''}/q^2 + u_h^{1''''}} \right)^{-1}$ is independent of $h$, and so the bond market clearing $\sum_{h=1}^{H} (\tilde{z}_h - \theta_h B) = 0$ implies that the second term is zero. This establishes the result.

**Remark 3** The condition in Lemma 4 essentially says that $\frac{u_h^{1''''}}{u_h^{0''''}}$ is constant across $h$, as is revealed in the proof. It holds for instance if the households have the same utility function and the equilibrium consumption is identical for the households.
Now we are ready to discuss the issue of over/under production. We shall concentrate on two cases: the case of linear technology and the case of strictly convex technology. The analysis for hybrid cases can be done analogously.

Let us first consider the case of linear technology: we assume that \( f(z) = kz \) for some constant \( k > 0 \). Under our normalization, and since \( \bar{B} > 0 \) by assumption, the profit maximization condition implies that the zero profit condition \( \Pi^* (\bar{q}) = 0 \) must hold and hence \( \bar{q} = k^{-1} \). Now fix \( \bar{r} \) close enough to \( \bar{\Pi} \) so that a sunspot equilibrium exists. As we mentioned above, the firm’s profit maximization condition is unaffected, so the sunspot equilibrium prices must be \((\bar{q}, \bar{r})\).

We have the following result on over/under production in sunspot equilibria.

**Proposition 2** Assume linear technology. If \( \sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} > 0 \), then in any sunspot equilibrium close enough to the certainty equilibrium, the level of production exceeds the efficient level (i.e., over production). Similarly, if \( \sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} < 0 \), then there is under production in the nearby sunspot equilibria.

**Proof.** Suppose \( \sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} > 0 \), and fix \( \bar{r} \) close enough to \( \bar{\Pi} \). Note that the function \( F_h \) in (10) is decreasing in \( B \), since \( \bar{r} > 0 \) and \( \theta_h (\bar{\Pi} - \bar{r}) > 0 \) in (3). Since \( F_h \) is decreasing in \( z_h \), this shows that \( Z_h (\bar{q}, \bar{r}, B) \) is decreasing in \( B \) for each \( h \), so is \( Z (\bar{q}, \bar{r}, B) \). On the other hand, by Lemma 3, we have \( Z (\bar{q}, \bar{r}, B) > \bar{B} \). This means that a sunspot equilibrium obtains, i.e., \( Z (\bar{q}, \bar{r}, B) = B \) only if \( B > \bar{B} \), which means that the production level in the sunspot equilibrium is higher than that in the certainty equilibrium.

The case of \( \sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} < 0 \) can be shown analogously. 

Next we consider the case of strictly convex technology: we assume that \( f \) is a \( C^2 \) function with \( f'' < 0 \). In this case, the bond supply function is well defined, so denote by \( B^*(q) \) the supply of the bond when the bond price is \( q \). It can be readily established that \( B^*(q) \) is increasing in \( q \): a higher the bond price means a lower interest rate, so the firm will produce more. The idea of analysis is essentially the same as before, except that in this case, shifts of demand function is not enough to identify over or under production, since the aggregate excess demand function (i.e., households’ demand minus firm’s supply) may be upward sloping around the certainty equilibrium.
Proposition 3 Assume strictly convex technology, and suppose \( Z(q, \tilde{\theta}, B^*(q)) - B^*(q) \) is decreasing in \( q \) at \( \tilde{q} \). If \( \sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} > 0 \), then in any sunspot equilibrium close enough to the certainty equilibrium, the level of production exceeds the efficient level (i.e., over production). Similarly, if \( \sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} < 0 \), then there is under production in the sunspot equilibria.

Proof. Suppose \( \sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} > 0 \), and fix \( \tilde{r} \) close enough to \( \tilde{\theta} \). By Lemma 3, we have \( Z(q, \tilde{\theta}, B) > B \). Recall that \( B^*(q) \) is increasing and \( \bar{B} = B^*(\tilde{q}) \) by definition. By assumption, \( Z(q, \tilde{\theta}, B^*(q)) - B^*(q) \) is decreasing in \( q \) at \( \tilde{q} \), so is \( Z(q, \tilde{\theta}, B^*(q)) - B^*(q) \) by continuity, if \( \tilde{r} \) is close enough to \( \tilde{\theta} \). Therefore, \( Z(q, \tilde{\theta}, B^*(q)) = B^*(q) \) implies \( q > \tilde{q} \) and \( B^*(q) > B^*(\tilde{q}) \) and so the production level in the sunspot equilibrium is higher than that in the certainty equilibrium. The case of \( \sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} < 0 \) can be shown analogously. □

Remark 4 If \( Z(q, \tilde{\theta}, B^*(q)) - B^*(q) \) is increasing in \( q \) at \( \bar{q} \) instead, i.e., the law of (aggregate) demand is violated at the certainty equilibrium, \( \sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} > 0 \) corresponds to under production and \( \sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} < 0 \) corresponds to over production.

We have argued that \( \sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} > 0 \) is prevalent. With this assertion, Proposition 2 and Proposition 3 roughly indicate that we tend to see over production in a sunspot equilibrium. In general, one has to check if \( \sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} > 0 \) actually holds. One can construct various parametric examples, but here we shall give one example of some generality.

Corollary 1 Assume linear technology \( f(z) = z \), and absolute prudence \( u_h^{1,\mu} > 0 \) for every \( h \). Suppose for each \( h \), \( u_h^0 = u_h^1 \), and \( e_h^0 + e_h^1 = 2\bar{e} \) for every \( h \), where \( \bar{e} > 0 \), and \( \sum_{h=1}^{H} (e_h^0 - e_h^1) > 0 \). Then there is over production in sunspot equilibria.

Proof. By assumption \( \bar{q} = 1 \), and so the income of the households must be the same. Thus, perfect consumption smoothing must take place at any certainty equilibrium, so each household consume \( (\bar{x}_h^0, \bar{x}_h^1) = (\bar{e}, \bar{e}) \). Thus assuming absolute prudence, Lemma 4 implies that \( \sum_{h=1}^{H} \frac{\alpha_h}{\gamma_h} > 0 \), and so there is over production in any nearby sunspot equilibrium by Proposition 2. □

5 The Role of Stock markets

Why do we keep the profit share fixed in the standard textbook general equilibrium model of a private ownership economy? An answer would be that one could introduce a market
for trading shares, but it does not really matter if markets are already complete: the share of a firm is a redundant asset by the definition of complete markets, and so the value the share is determined by the no arbitrage condition. The set of equilibrium allocations of goods does not change if the shares could be traded, and in this sense the share markets are redundant in the analysis of a private ownership economy with complete markets.

In our setup, if the share can be exchanged competitively in addition to the bond, it is still the case that the share is redundant in any non-sunspot equilibrium. But it does make a difference: there will be no sunspot equilibrium where production takes place; that is, although returns might be random, the consumption is not random in any equilibrium. We shall also see that the certainty equilibria represent all the equilibria, essentially, as far as the level of production and the consumption allocation are concerned.

Let $q_S$ be the market price of the share. Denote by $\hat{h}$ the share after trade. Thus the induced utility function of household $h$ is now:

$$
\sum_{s=1}^{S} \mu^s u_h \left( e^0_h - qz_h - q_S \left( \hat{\theta}_h - \theta_h \right), e^1_h + r^s z_h + \hat{\theta}_h \Pi^s \right).
$$

Assume that the firm takes prices as given and maximizes the expected profits as before. Thus in particular, the firm takes $q_S$, which is the market value of the firm, as given. A competitive equilibrium can now be defined analogously: $(q, q_S, \hat{r})$ constitutes an equilibrium if both the bond market and the share market clear. An equilibrium is a sunspot equilibrium if the consumption is random for some households in the second period.\(^\text{11}\)

Note that similarly to (2), the second period consumption in (18) may also be written as:

$$
e^1_h + r^s z_h + \hat{\theta}_h \Pi^s = e^1_h + \hat{\theta}_h \left( \Pi^s(q) + B \right) + \hat{r} \left( z_h - \hat{\theta}_h B \right),
$$

when the firm chooses $B$.

Clearly, a certainty equilibrium is an equilibrium in this setup: let $(\bar{q}, 1)$ be a certainty equilibrium where the firm supplies $\bar{B}$. One unit of share yields $f(\bar{q}\bar{B}) - \bar{B}$, so simply set $q_S = \bar{q} \left( f(\bar{q}\bar{B}) - \bar{B} \right)$. Then the bond and the share are equivalent as assets, so set $\hat{\theta}_h = \theta_h$ for all $h$. Then $(\bar{q}, q_S, \bar{1})$ is an equilibrium. Conversely, if $(q, q_S, \bar{1})$ is an equilibrium where the firm supplies $\bar{B}$, $q_S = q \left( f(qB) - B \right)$ must hold by no arbitrage, and so $(q, 1)$ is a certainty equilibrium. This however does not mean that the demand for bond is the same

\(^{11}\)Note that by assumption the firm’s behavior is independent of the composition of shareholders.
in \((q, q_S, \bar{1})\) and \((q, 1)\); since the bond and the share are equivalent assets, households may be trading shares instead of bond, for instance.

The reader may wonder why the indeterminacy of real returns \(\bar{r}\), which is still present in this setup, is not enough to generate a sunspot equilibrium as is shown in Proposition 1. In fact, it is true that an equilibrium exists for any \(\bar{r}\) around \(\bar{1}\), and it is still true in this setup that the consumption must be random if the income is solely generated by the net holding \(z_h - \theta_h B\) of bond, just as in Proposition 1. But since the share can be traded, each household is not obliged to keep its share holding at \(\theta_h\). In particular, household \(h\) might want to adjust \(\theta_h\) to reduce effects from sunspots on consumption if it is inexpensive enough to do so. To say the least, the intuition behind Proposition 1 does not provide a complete picture.

Now we are ready to establish a non-existence result. The non-existence result is established, by an argument which is roughly the same as that of the standard first fundamental theorem of welfare economics as follows:\textsuperscript{12} when the share can be traded, it is always possible to construct a portfolio of the share and the bond whose payoffs are independent of sunspots. So if a sunspot equilibrium existed, using this portfolio, every household’s utility could be improved by avoiding random income. Then such portfolio must be too expensive for every household, but in aggregate this is inconsistent with the market clearing conditions.

**Proposition 4** If the profit share can be traded in period 0, there is no sunspot equilibrium such that the firm produces a positive amount of good.

**Proof.** Suppose there is a sunspot equilibrium, and let \(q\) and \(q_S\) be the bond price and the equity price in equilibrium, respectively. Let \(B\) be the bond issued by the firm in this equilibrium. By assumption, \(f(qB) \geq 0\).

Let \(x^s_h\) be the consumption of household \(h\) in state \(s = 0, 1, \ldots, S\) in the sunspot equilibrium. The feasibility implies that in the second period \(\sum_{h=1}^{H} x^s_h = \sum_{h} e^1_h + f(qB)\) for every \(s\). Let \(\bar{x}^1_h = \sum_{s=1}^{S} \mu^s x^s_h\), i.e., the expected consumption, for every \(h\). Note that

\textsuperscript{12}The argument is a modification of the ingenious idea in Mas-Colell (1992). The main difference is that the returns of assets are fixed exogenously in Mas-Colell (1992), whereas they are endogenously determined in this model. Also, an efficient equilibrium must be a non-sunspot equilibrium in Mas-Colell (1992), which is not the case in our model; see Remark 5.
from the feasibility of the equilibrium consumption \( x \), we have

\[
\sum_{h=1}^{H} \hat{x}_h = \sum_{s=1}^{S} \sum_{h=1}^{H} \mu^s x^s_h,
\]

\[
= \sum_{s=1}^{S} \mu^s \sum_{h=1}^{H} x^s_h,
\]

\[
= \sum_{s=1}^{S} \mu^s \left( \sum_{h=1}^{H} e_h^1 + f(qB) \right),
\]

\[
= \sum_{h=1}^{H} e_h^1 + f(qB),
\]

(20)

thus \((\hat{x}_h)_{h=1}^{H}\) can be attained by reallocating the good available for consumption, \(\sum_h e_h^1 + f(qB)\).

By risk aversion \( u_h(x_0, \hat{x}_h) \geq \sum_{s=1}^{S} \mu^s u_h(x^0_h, x^s_h) \) and the inequality is strict for at least one \( h \) whose consumption is random. Now consider the following portfolio: buy 1 unit of share and \( B \) units of bond. The cost of this portfolio is \( q^* := qB + q_s \). Recall that in state \( s \), the share yields the total profit of the firm \( \Pi^*(q) + (1 - r^s)B \), and the bond pays off \( r^sB \). Since \( \Pi^*(q) = f(qB) - B \) from (1), the payoff of this portfolio is \( \Pi^*(q) + B = f(qB) > 0 \), which is independent of states.

If household \( h \) sells the whole \( \theta_h \) units of the initially owned share and buys \( \frac{1}{f(qB)} (\hat{x}_h - e_h^1) \) units of the portfolio above, then household \( h \)'s consumption is exactly \( \hat{x}_h \) in every state. Period 0 consumption is \( e_h^0 + q_s \theta_h - q^* \left( \frac{1}{f(qB)} (\hat{x}_h - e_h^1) \right) \). But since \((x_0^h, \hat{x}_h^1)\) is more desirable, such plan cannot be budget feasible; if household \( h \) follows this activity in the bond market and the stock market, household \( h \)'s consumption in period 0 must not increase, and must decrease if \( h \) strictly prefers \((x_0^h, \hat{x}_h^1)\) to \( x_h \). In conclusion, we have \( x_0^h \geq e_h^0 + q_s \theta_h - q^* \left( \frac{1}{f(qB)} (\hat{x}_h - e_h^1) \right) \), and the inequality is strict for some \( h \).

Summing up these inequalities and using the resource feasibility conditions \( qB + \)
\[
\sum_{h=1}^{H} x_h^0 = \sum_{h=1}^{H} e_h^0 \text{ in period 0 and (20) in period 1, we have}
\]
\[
-qB = \sum_{h=1}^{H} (x_h^0 - e_h^0)
\]
\[
> \sum_{h=1}^{H} \left( qS \theta_h - q^* \left( \frac{1}{f(qB)} \left( \hat{x}_h - e_h^1 \right) \right) \right)
\]
\[
= qS \left( \sum_{h=1}^{H} \theta_h \right) - (qB + qS) \frac{\sum_{h=1}^{H} (\hat{x}_h - e_h^1)}{f(qB)}
\]
\[
= qS - (qB + qS)
\]
\[
= -qB,
\]
which is a contradiction. ■

Proposition 4 implies that any equilibrium with positive production must be equivalent to a certainty equilibrium in production and allocation of consumption goods, by the reason analogous to Lemma 1. Formally, we have the following result.

**Corollary 2** Let \((q, qS, \bar{r})\) be an equilibrium with positive production. Then \((q, qS, \bar{1})\) is an equilibrium which is efficient, and the production level and the consumption allocation of \((q, qS, \bar{r})\) coincide with those in the certainty equilibrium \((q, 1)\).

**Proof.** Let \((q, qS, \bar{r})\) be an equilibrium with positive production. If \(\bar{r} = \bar{1}\), we have already seen that it is equivalent to \((q, 1)\). So assume \(\bar{r} \neq \bar{1}\).

The demand for bond and that for share must be chosen optimally given \((q, qS, \bar{r})\). By differentiating utility function (18) with respect to \(z_h\) and \(\hat{\theta}_h\), we have the first order necessary conditions
\[
E \left[ -q \left( \frac{\partial}{\partial x_0} u_h \right) + \left( \frac{\partial}{\partial x_t} u_h \right) \bar{r} \right] = 0 \quad \text{and} \quad E \left[ -qS \left( \frac{\partial}{\partial x_0} u_h \right) + \left( \frac{\partial}{\partial x_t} u_h \right) \bar{\Pi} \right] = 0
\]
for every \(h\), where the derivatives are evaluated at the equilibrium consumption of \(h\) in \((q, qS, \bar{r})\).

Now by Proposition 4, consumption in \((q, qS, \bar{r})\) cannot be random. It implies that
\[
z_h - \hat{\theta}_h B = 0 \text{ for every } h \text{ in view of (19), and the first order conditions above imply that}
\]
\[
-q \left( \frac{\partial}{\partial x_0} u_h \right) + \left( \frac{\partial}{\partial x_t} u_h \right) = 0 \quad \text{and} \quad -qS \left( \frac{\partial}{\partial x_0} u_h \right) + \left( \frac{\partial}{\partial x_t} u_h \right) \Pi^* (q) = 0 \text{ hold for every } h \text{ at the same } z_h \text{ and } \hat{\theta}_h.
\]
Then just as in Lemma 1, these imply that \((q, qS, \bar{1})\) constitutes an equilibrium, and the underlying consumption allocation and production level are the same as in certainty equilibrium \((q, 1)\). ■
Remark 5 Proposition 4 and Corollary 2 do not imply that any equilibrium is a non-sunspot equilibrium \((q, q_S, \tilde{1})\). In fact, it can be readily established that there is an equilibrium \((q, q_S, \tilde{r})\) for any \(\tilde{r}\) close enough to \(\tilde{1}\), similarly to Proposition 1. If \(\tilde{r} \neq \tilde{1}\), both the bond and the share are risky assets, but the households effectively trade a common “mutual fund” which enable them to achieve non-random consumption in \((q, 1)\).\(^{13}\)

An equilibrium with zero production is of course not interesting for the purpose of this paper. Moreover, it can be readily confirmed that the equilibrium production level must be positive in a typical environment where the total endowment in the second period is low enough compared to the first period total endowment and/or the marginal product is large enough at zero production. So we contend that Proposition 1 is strong enough for our purpose.

But it is still instructive to show that one can construct an economy where a sunspot equilibrium exists even the share can be traded. By Proposition 4, the firm must be inactive and so the share must be worthless in such an equilibrium. For instance, start with an exchange economy with one nominal bond, and say that we have an efficient equilibrium with bond price \(\bar{q}\). Introduce a firm with linear (and unproductive) technology \(f(z) = kz\) where \(k > 0\) is small. Then at bond price \(\bar{q}\), no production is a unique profit maximizing plan, and so the initial exchange economy equilibrium is now an equilibrium in the production economy with this firm. The initial endowments can be arranged so that part (2) of Proposition 1 is applicable, thus a sunspot equilibrium \((q, \tilde{r})\) exists. But if \(\tilde{r}\) is close enough to \(\tilde{1}\), no production is still a unique profit maximizing production plan. Now consider the market of share but set \(q_S = 0\). Then it can be readily checked that the equilibrium market clearing conditions are satisfied under \((q, q_S, \tilde{r})\).

6 Remarks

6.1 Welfare Gains and Losses

Consider a certainty equilibrium and a sunspot equilibrium close to it. Although the sunspot equilibrium must necessarily be inefficient, some households may nevertheless be better off in the sunspot equilibrium than in the certainty equilibrium. This point is

\(^{13}\)To be more precise, in \((q, q_S, \tilde{1})\), household \(h\) chooses \(z_h = (x_h - c_h^1) \frac{B}{\gamma(qB)}\) and \(\tilde{\theta}_h = (x_h - c_h^1) \frac{1}{\gamma(qB)}\), which constitutes the risk free portfolio used in the proof of Proposition 4.
first raised by Goenka - Préchac (2006) in a simple symmetric pure exchange setting, and then it is elaborated in a general exchange economy setup by Kajii (2007). These papers however do not take production into account. Here we shall discuss how the question of welfare gains and losses can be addressed in the model with production.

There are three effects which determines the economic welfare in a sunspot equilibrium, relative to the certainty equilibrium. First, sunspots make the returns of asset more random, which is bad for all households since they loose a perfect saving device.

Secondly, the equilibrium bond price is different from the efficient one. As we have argued, the equilibrium bond price tends to be higher in the non-linear technology case, making the expected real interest rate lower in sunspot equilibria. This is bad news for those who save. Consider a typical setup where households are endowed with good in period 0 only, so all the households save in equilibrium. Then this second effect is also bad for all the households.

The third effect is more delicate. A lower real interest rate is good news for the firm, and the firm tends to be more profitable in the sunspot equilibrium. The additional profits are distributed to the shareholders, so this is welfare improving. Especially for those households with relatively large share, the positive welfare effect from this channel can be large enough to offset the first two negative effects.

To sum up the discussion, we conclude that: (1) a household whose share $\theta_h$ is zero must be worse off in the sunspot equilibrium. (2) if the technology exhibits constant returns to scale, then all the households must be worse off in the sunspot equilibrium because expected profit is always zero. A formal analysis including other cases appears to be a very interesting research agenda.

### 6.2 Comparison with background income risk models

The sunspot model we developed in this paper has some flavor of the so called background income risk model. More specifically, imagine that the second period endowments gets slightly riskier, thus states are no longer sunspots, and the real return of the bond is fixed at one. Then by the precautionary saving argument, the saving of each household will increase assuming that $u_h'' > 0$ for every household. Therefore, the price of bond must go up and the level of production also goes up, and so this background risk model also explains a higher level of production.
However, a higher level of production in this model does not mean that there is over production. Notice that since the background risks cannot be insured, one cannot hope for full efficiency to begin with. And more importantly, one cannot necessarily say that the higher level of production under background risk is excessive, since there is no benchmark efficient level of production described within the model. In our sunspot model, the certainty equilibrium is a benchmark for comparison, and the meaning of over/under production is very clear.

In the background risk model, a relevant exercise close to ours is to check the constrained efficiency of the equilibrium.\textsuperscript{14} For instance, suppose the government can control the level of input and output by some criterion different from profit maximization, letting all the other variables be endogenously determined in the markets. Should the government find reducing the level of output beneficial to the economy, one can then argue that there is over production.\textsuperscript{15}

6.3 Extensions

To conclude, let us provide a few remarks concerning the single good assumption in our analysis. If there are multiple consumption goods, the set of sunspot equilibria is still parametrized by $\tilde{r}$, and we believe that the existence of sunspot equilibria can be established analogously. A potential complication arises due to changes in equilibrium relative prices of goods within each spot markets. This will make the analysis potentially involved, but it appears to us that the nature of the analysis will not change as far as the existence is concerned.\textsuperscript{16} The issue of under/over production will become less clear cut, obviously. Nevertheless, we believe that analogous exercise can be done to see if the real interest rate goes down or not due to sunspots.

In the case of multiple goods, it is natural to think of many firms as well. In the standard complete markets setup, one could regard these firms as one firm which does a joint production because of the equivalence of individual firms’ profit maximization.

\textsuperscript{14}Various sorts of constrained efficiency exercises are possible in the incomplete market models. See Citanna, Kajii, and Villanacci (1998) for a general treatment and an overview of the literature.

\textsuperscript{15}Davila \textit{et al} (2007) considered a model of background individual uninsurable risks with no aggregate risk and asked if a competitive equilibrium is constrained efficient. They found that over production tends to occur if the households are relatively homogeneous.

\textsuperscript{16}See Gottardi and Kajii (1999) for the case of pure exchange.
and profit maximization of the aggregated firm. Then even in the sunspot set up, as far as we assume expected profit maximization, the same argument would work. However, for the non-existence result (Proposition 4), such aggregation is not neutral. If each consumption good is produced by one firm, and if all the firms’ shares are traded in their respective markets, then the non-existence result will still hold. Then, it means that the aggregation of production side does not work as in the complete markets. There seems to be many interesting directions for further research.
Appendix: Proof of Lemma 2

We shall give a proof without time additive separability assumption. The reader then will see that the other results reported in the main text can readily be shown without the separability assumption.

Fix a certainty equilibrium \((\tilde{q}, 1)\) and denote by \(\tilde{B}\) and \(\tilde{z}_h, h = 1, ..., H\), the bond supply and the demand in the equilibrium, respectively. For each \(h\), let

\[
F_h (q, r, z_h, B) := \sum_{s=1}^{S} \mu^s \{ -\tilde{q} \frac{\partial}{\partial x_0} u_h + \frac{\partial}{\partial x_1} u_h \cdot r^s \},
\]

where derivatives are evaluated at \((e^0_h - \tilde{q} z_h, (e^1_h + \theta_h (\Pi^* (q) + B) + r^s (z_h - \theta_h B))^S_{s=1})\).

That is, \(F_h (q, r, z_h, B) = 0\) is the first order condition for utility maximization. Thus by construction, \(F_h (\tilde{q}, \tilde{r}, \tilde{z}_h, \tilde{B}) = 0\) for every \(h\). Note also that by the additive separability and the symmetry across the states, we have for any pair of states \(s\) and \(s'\):

\[
\frac{1}{\mu^s} \frac{\partial}{\partial r^s} F_h = \frac{1}{\mu^{s'}} \frac{\partial}{\partial r^{s'}} F_h; \tag{21}
\]

\[
\frac{\partial^2}{\partial r^s \partial r^{s'}} F_h = 0, \text{ if } s \neq s'; \tag{22}
\]

\[
\frac{1}{\mu^s} \frac{\partial^2}{\partial (r^s)^2} F_h = \frac{1}{\mu^{s'}} \frac{\partial^2}{\partial (r^{s'})^2} F_h, \tag{23}
\]

where the derivatives are evaluated at \((\tilde{q}, \tilde{r}, \tilde{z}_h, \tilde{B})\).

To keep the normalization \(E [\tilde{r}] = 1\), as in the main text write \(\tilde{r}^{-S}\) for \((r^1, ..., r^{S-1})\), and define \(\Phi_h (\tilde{r}^{-S}, z_h)\) for each \(h\) by the rule:

\[
\Phi_h (\tilde{r}^{-S}, z_h) := F_h \left( \tilde{q}, \left( \tilde{r}^{-S}, 1 - \sum_{s=1}^{S-1} \mu^s r^s \right), z_h, \tilde{B} \right).
\]

Under our maintained assumptions on the utility function, the change in the modified demand \(\hat{Z}_h (\tilde{q}, , \tilde{B})\) (see (11)) is given by the implicit function theorem applied to the identity \(\Phi_h (\tilde{r}^{-S}, z_h) = 0\).

First, we shall show that \(\frac{\partial}{\partial \tilde{r}^s} \hat{Z}_h = 0\) at \(\tilde{r}^{-S} = \tilde{1}^{-S}\). Indeed, evaluated at \(\tilde{r}^{-S} = \tilde{1}^{-S}\) and \(z_h = \tilde{z}_h\), we have \(\frac{\partial}{\partial \tilde{r}^s} \Phi_h = \frac{\partial}{\partial \tilde{r}^s} F_h + \frac{\partial}{\partial \tilde{r}^s} F_h \cdot \left( -\frac{\mu^s}{\mu^s} \right)\) for each \(s = 1, ..., S - 1\), and \(\frac{\partial}{\partial \tilde{r}^s} \Phi_h = \frac{\partial}{\partial \tilde{r}^s} F_h\). Therefore, by differentiating the identity \(\Phi_h (\tilde{r}^{-S}, z_h) = 0\), we have

\[
\frac{\partial}{\partial \tilde{r}^s} \Phi_h + \frac{\partial}{\partial \tilde{r}^s} \Phi_h \frac{\partial}{\partial \tilde{r}^s} \hat{Z}_h = \frac{\partial}{\partial \tilde{r}^s} F_h - \left( \frac{\mu^s}{\mu^s} \right) \frac{\partial}{\partial \tilde{r}^s} F_h + \frac{\partial}{\partial \tilde{r}^s} F_h \frac{\partial}{\partial \tilde{r}^s} \hat{Z}_h = 0, \text{ for each } s = 1, ..., S - 1.
\]

The symmetry relation (21) then implies that this equation is reduced to \(\frac{\partial}{\partial \tilde{r}^s} F_h \frac{\partial}{\partial \tilde{r}^s} \hat{Z}_h = 0\), so \(\frac{\partial}{\partial \tilde{r}^s} \hat{Z}_h = 0\) must hold for each \(s = 1, ..., S - 1\), since \(\frac{\partial}{\partial \tilde{r}^s} F_h\) is not zero.
Next, we calculate the second order effect, \( \frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z}_h \). Set \( \gamma_h := -\frac{\partial}{\partial z_h} F_h \) (\( = -\frac{\partial}{\partial z_h} \Phi_h \)), and set \( \alpha_h \) to be the common constant in (23), so \( \frac{\partial^2}{\partial(r^s)^2} F_h = \mu^s \alpha_h \), \( s = 1, \ldots, S - 1 \). We shall show that \( \left( \frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z}_h \right)_{s,s'} = \frac{\alpha_h}{\gamma_h} M \), where \( M \) is an \( S - 1 \) dimensional positive definite matrix determined by probability \( \mu \) (thus in particular independent of \( h \)).

Differentiate the identity \( \frac{\partial}{\partial r^s} \Phi_h + \frac{\partial}{\partial z_h} \Phi_h \frac{\partial}{\partial r^s} \hat{Z}_h = 0 \) with respect to \( r^s' \), and evaluate the result at \( \tilde{r}^{-S} = \tilde{1}^{-S} \) and \( z_h = \bar{z}_h \). Both \( \frac{\partial}{\partial r^s} \Phi_h \) and \( \frac{\partial}{\partial z_h} \Phi_h \) are functions of \( z_h \), but since \( \frac{\partial}{\partial z_h} \hat{Z}_h = 0 \) when \( \tilde{r}^{-S} = \tilde{1}^{-S} \) and \( z_h = \bar{z}_h \), these indirect effects vanish. Also the cross effect \( \frac{\partial^2}{\partial z_h \partial r^{s'}} \Phi_h \) is multiplied by \( \frac{\partial}{\partial r^{s'}} \hat{Z}_h \) and so this also vanishes. Thus the resulting equation is simplified as follows:

\[
\frac{\partial^2}{\partial r^s \partial r^{s'}} \Phi_h + \frac{\partial}{\partial z_h} \Phi_h \frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z}_h = 0. \tag{24}
\]

So solving (24) we have

\[
\frac{\partial^2}{\partial r^s \partial r^{s'}} \Phi_h = - \left( \frac{\partial}{\partial z_h} \Phi_h \right)^{-1} \frac{\partial^2}{\partial r^s \partial r^{s'}} \Phi_h = \frac{1}{\gamma_h} \frac{\partial^2}{\partial r^s \partial r^{s'}} \Phi_h. \]

Now by construction, 

\[
\frac{\partial^2}{\partial r^s \partial r^{s'}} \Phi_h = \frac{\partial}{\partial z_h} \left( \frac{\partial}{\partial r^s} F_h + \frac{\partial}{\partial \mu^s} F_h \cdot \left(-\frac{\mu^s}{\mu^s}\right) \right) = \left( \frac{\partial^2}{\partial r^s \partial r^{s'}} F_h + \frac{\partial}{\partial \mu^s} \frac{\partial}{\partial r^s} F_h \cdot \left(-\frac{\mu^s}{\mu^s}\right) \right) + \left( -\frac{\mu^s}{\mu^s} \right) \left( \frac{\partial^2}{\partial r^s \partial r^{s'}} F_h + \frac{\partial}{\partial \mu^s} \frac{\partial}{\partial r^s} F_h \cdot \left(-\frac{\mu^s}{\mu^s}\right) \right)
\]

holds by (22) and \( \frac{\partial^2}{\partial (r^s)^2} F_h = \mu^s \alpha_h \). Thus if \( s \neq s' \), 

\[
\frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z}_h = \frac{1}{\gamma_h} \left( \mu^s \alpha_h + \alpha_h \left( \frac{\mu^s}{\mu^s} \right)^2 \right) = \frac{\alpha_h}{\gamma_h} \left( \mu^s + \left( \frac{\mu^s}{\mu^s} \right)^2 \right).
\]

Writing these in a matrix form, we obtain the following:

\[
\left( \frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z}_h \right)_{s,s'} = \frac{\alpha_h}{\gamma_h} \begin{bmatrix}
\mu^1 & 0 & \cdots & 0 \\
0 & \mu^2 & \vdots & \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \mu^{S-1}
\end{bmatrix} + \frac{1}{\mu^S} \begin{bmatrix}
\mu^1 \\
\vdots \\
\mu^{S-1}
\end{bmatrix} \begin{bmatrix}
\mu^1 \\
\vdots \\
\mu^{S-1}
\end{bmatrix}.
\]

The two matrices consisting of \( (\mu^1, \ldots, \mu^S) \) in (25) are both positive definite, and they are determined by probabilities only. Thus we have established the desired property of

\[
\left( \frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z}_h \right)_{s,s'}.
\]
References


