A Sandwich Theorem for Generic $n \times n$ Two Person Games

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Abstract

We study the structure of Nash equilibria in generic $n \times n$ games. A game is said to have a sandwich structure in Nash equilibria if there is a mixed strategy Nash equilibrium “inside” every collection of pure strategy Nash equilibria. A sufficient condition, which solely relies on the ordinal information of the game, is given for a generic $n \times n$ game to have a sandwich structure in Nash equilibria. We provide a lower bound on the number of Nash equilibria and determine the stability of each equilibrium in games with a sandwich structure in Nash equilibria. Moreover, when the number of pure strategy Nash equilibria is equal to the number of pure strategies available to each player, the exact structure of Nash equilibria can be determined.

JEL classification: C62; C72
Keywords: Nash Equilibrium; Lefschetz-Hopf Theorem; Index; Stability

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1 Introduction

The Nash equilibrium is a central solution concept in game theory in analyzing strategic interaction among rational players. A normal form game is a compact representation of the strategic situation facing the players. For an arbitrarily given normal form game, however, it is difficult to pin down the exact structure of Nash equilibria without getting into tedious and cumbersome calculations. Several papers study the number of Nash equilibria in generic games. Quint and Shubik (1997) prove that generically there are at most $2^3 - 1 = 7$ Nash equilibria in $3 \times 3$ two person games, and conjecture that there are at most $2^n - 1$ Nash equilibria in generic $n \times n$ two person games. The conjecture is shown to be true for $n = 4$ (Keiding (1995) and McLennan and Park (1999)), but a sequence of counterexamples shows that the conjecture is false for $n \geq 6$ (von Stengel (1999)). For generic n-person games, the maximal number of pure strategy Nash equilibria is the total number of pure strategies divided by the maximal number of pure strategies possessed by any player (McLennan (1997)). Gul et al. (1993) use the index theorem to show that if a generic n-person game has $k$ pure strategy Nash equilibria, then the number of mixed strategy Nash equilibria is at least $k - 1$.

In this paper, we study the structure of Nash equilibria in generic $n \times n$ two person games. We start with a simple observation that every generic $2 \times 2$ game with two pure strategy Nash equilibria has a mixed strategy Nash equilibrium. This illustrates the following intuitive strategic incentives: when each player’s best response varies with the other player’s strategy (which is suggested by the multiplicity of pure strategy Nash equilibria), randomly choosing among multiple strategies can be optimal provided that the other player also randomizes. Accordingly, in a generic $2 \times 2$ game, between two pure strategy Nash equilibria, there is a mixed strategy Nash equilibrium. Extending this structural property to $n \times n$ games, we say that a game has a sandwich structure in Nash equilibria if there is a mixed strategy Nash equilibrium “inside” every collection of pure strategy Nash equilibria (Definition 1).

Clearly not every generic $n \times n$ has a sandwich structure in Nash equilibria. We characterize a class of generic $n \times n$ games with a sandwich structure in Nash equilibria and obtain the following results. First, a generic $3 \times 3$ game has a sandwich structure in Nash equilibria if no pure strategy is a player’s second best response against more than one pure strategy of the other player (Proposition 1). Second, a sufficient condition called (restricted) even order distribution, which solely relies on the ordinal information of the game, is given for a generic $n \times n$ game to have a sandwich structure in Nash equilibria (Theorem 1). Roughly speaking, even order distribution requires that the ordinal payoff matrix for each player is balanced. Third, in games with a sandwich structure in Nash equilibria, we provide a sharper lower bound on the number of Nash equilibria. Finally, when the number of pure strategy Nash equilibria is equal to the number of pure strategies available to each player, we show that the exact structure of Nash equilibria can be determined. The Lefschetz index of each mixed strategy Nash equilibrium, which provides information
on equilibrium stability, can be computed as well.

## 2 Nash Equilibrium and Lefschetz Number

Consider two-person games in which each player \( i \in \{1, 2\} \) has a finite set \( S^i \) of pure strategies with \(|S^1| = |S^2| = n \geq 2\). Player \( i \)'s set of mixed strategies is \( \Sigma^i = \{x \in \mathbb{R}_+^n | x_1 + ... + x_n = 1\} \), which is a simplex of dimension \( n - 1 \). The sets of profiles of pure and mixed strategies are \( \Sigma = \Sigma^1 \times \Sigma^2 \) and \( \Sigma = \Sigma \times \Sigma^2 \) respectively.

A game \( G : \Sigma \to \mathbb{R}^2 \) assigns to each profile of pure strategies a payoff to each player. Denote by \( G = (G^1, G^2) \) and \( G^i(s_k^i, s_j^k) = G_{ij}^k \), \( i \in \{1, 2\} \). Each game \( G \) can be viewed as a point in \( \mathbb{R}^{2 \times n \times n} \). Given a strategy profile \( \sigma = (\sigma^1, \sigma^2) \in \Sigma \), player 1's expected payoff at strategy \( s_k^1 \in S^1 \) is \( G_k^1(\sigma) = \sum_{j=1}^n G^1(s_k^1, s_j^2) \sigma_j^2 \), and player 2's expected payoff at \( s_k^2 \in S^2 \) is \( G_k^2(\sigma) = \sum_{j=1}^n G^2(s_j^1, s_k^2) \sigma_j^1 \).

We follow the differential topology approach adopted in Gul et al. (1993) to compute the Lefschetz fixed point index of Nash equilibrium. Define \( f : \Sigma \to \mathbb{R}^{2 \times n} \) by \( f(\sigma) = G(\sigma) + \sigma = (G_1^1(\sigma) + \sigma_1, ..., G_n^1(\sigma) + \sigma_1, G_1^2(\sigma) + \sigma_2^1, ..., G_n^2(\sigma) + \sigma_2^1) \). Let \( r : \mathbb{R}^{2 \times n} \to \Sigma \) be the retraction that maps each point in \( \mathbb{R}^{2 \times n} \) to the nearest point in \( \Sigma \); that is, \( r(x) \) is the unique point in \( \Sigma \) that satisfies \( (x - r(x))(y - r(x)) \leq 0 \) for all \( y \in \Sigma \). Then \( \sigma \in \Sigma \) is a Nash equilibrium of \( G \) if and only if it is a fixed point of \( h = (h_{\sigma_1}, ..., h_{\sigma_n}) = r \circ f : \Sigma \to \Sigma \) (Corollary 2, Gul et al. (1993)). \( h \) is called a Nash map.

A game \( G \in \mathbb{R}^{2 \times n \times n} \) is said to be generic if every Nash equilibrium \( \sigma \in \Sigma \) of \( G \) satisfies the following two conditions: (i) for each player, every best response is used with positive probability,\(^1\) and (ii) \( \det(I - h'(\sigma)) \neq 0 \), where \( h' = \left\{ \frac{\partial h_k^i}{\partial \sigma_j} \right\} \) is the Jacobian matrix of \( h \). By Sard’s theorem, the set of generic games is open and its complement has Lebesgue measure 0 in \( \mathbb{R}^{2 \times n \times n} \) (Theorem 3, Gul et al. (1993)).

Given a generic game \( G \) and the associated Nash map \( h : \Sigma \to \Sigma \), the (fixed-point) index of \( h \) at a Nash equilibrium \( \sigma \) can be determined as\(^2\)

\[
\begin{align*}
i(h, \sigma) &= sign \det \begin{bmatrix} I - h'(\sigma) & \Lambda^T \\ -\Lambda & 0 \end{bmatrix}, \text{ where} \\
\Lambda &= \begin{bmatrix} e_{1 \times n} & 0 \\ 0 & e_{1 \times n} \end{bmatrix} \text{ with } e_{1 \times n} = (1, ..., 1).
\end{align*}
\]

If \( \sigma \) is a pure strategy Nash equilibrium, \( i(h, \sigma) = +1 \), for a small perturbation in payoffs does not change each player’s best strategy and hence \( h'(\sigma) = 0 \). A Nash equilibrium \( \sigma' \) with \( i(h, \sigma) = -1 \) is unstable under any Nash dynamics (Demichelis and Germano (2002)).

\(^1\)In particular, this condition implies that every pure strategy Nash equilibrium is a strict Nash equilibrium.

\(^2\)The index is well-defined over a large class of Nash maps, not just the one used in Gul et al. (1993).
Recall that for a map $f : X \to X$, the Lefschetz number of $f$ is defined as

$$L(f) = \sum_n (-1)^n tr(f_* : H_n(X) \to H_n(X)),$$

where $H_n(X)$ is the $n$th homology group of the chain complex of $X$, and $tr(f_* : H_n(X) \to H_n(X))$ is the trace of the homomorphism induced by $f$ on $H_n(X)$. If $f : X \to X$ has finitely many fixed points, we have the following result:

**Lefschetz-Hopf theorem** $\sum_{x = f(x)} i(f, x) = L(f)$.

The Lefschetz-Hopf theorem links the global topological structure of a function to the local topological structure of its fixed points. Applying the Lefschetz-Hopf theorem to the Nash map $h : \Sigma \to \Sigma$ associated with a generic game $G$, we have

$$\sum_{\sigma \text{ is a Nash equilibrium of } G} i(h, \sigma) = L(h).$$

Since the Lefschetz number is a homotopy invariant, $L(h) = L(\tilde{h})$ for any $\tilde{h} : \Sigma \to \Sigma$ homotopic to $h : \Sigma \to \Sigma$. As $\Sigma$ is contractible, all (continuous) maps are homotopic and hence have the same Lefschetz number. The constant map $\bar{h} : \Sigma \to \Sigma$ has one fixed point and $\bar{h}' = 0$. Thus,

$$L(\bar{h}) = \text{sign} \det \begin{bmatrix} I & \Lambda^T \\ \Lambda & 0 \end{bmatrix} = +1.$$

Accordingly, $L(h) = L(\bar{h}) = +1$. As a pure strategy Nash equilibrium has index $+1$, $L(h) = +1$ immediately implies that a generic game with $k$ pure strategy Nash equilibria has at least $k - 1$ mixed strategy Nash equilibria (Gul et al. (1993)).

### 3 A Sandwich Theorem

To illustrate the intuition behind the main result, consider the following Stag Hunt game:

**Example 1 Stag Hunt**

<table>
<thead>
<tr>
<th></th>
<th>Stag</th>
<th>Hare</th>
</tr>
</thead>
<tbody>
<tr>
<td>Andy</td>
<td>10,10</td>
<td>2,0</td>
</tr>
<tr>
<td>Bob</td>
<td>0,2</td>
<td>2,2</td>
</tr>
</tbody>
</table>
This game has two pure strategy Nash equilibria: (Stag, Stag) and (Hare, Hare). There is also a mixed strategy Nash equilibrium “between” (Stag, Stag) and (Hare, Hare), i.e., an equilibrium in which both players randomize between Stag and Hare. This property is shared by all generic $2 \times 2$ games: if a generic $2 \times 2$ game has two pure strategy Nash equilibria, then there is a unique mixed strategy Nash equilibrium between them. This is not necessarily the case, though, when each player has three or more pure strategies. For example, consider the following two extended Stag Hunt games:

**Example 2 Stag-Hare-Boar Games**

<table>
<thead>
<tr>
<th>Stag-Hare-Boar I</th>
<th>Bob</th>
<th>Stag-Hare-Boar II</th>
<th>Bob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stag</td>
<td>10, 10</td>
<td>Stag</td>
<td>10, 10</td>
</tr>
<tr>
<td>Hare</td>
<td>2, 0</td>
<td>Hare</td>
<td>0, 2</td>
</tr>
<tr>
<td>Boar</td>
<td>7, 3</td>
<td>Boar</td>
<td>2, 1</td>
</tr>
</tbody>
</table>

Now each player can choose to hunt a stag, a hare, or a boar. There are three pure strategy Nash equilibria in both games: (Stag, Stag), (Hare, Hare) and (Boar, Boar). The only difference between these two games is the payoff each player can get when he chooses Boar and the other player chooses Stag. In Stag-Hare-Boar I, however, there is no mixed strategy Nash equilibrium between (Stag, Stag) and (Hare, Hare). The mixed strategy equilibrium in the Stag Hunt game, $((1/5, 4/5), (1/5, 4/5))$, is no longer an equilibrium in this extended game. The reason is that given Bob playing $(1/5, 4/5, 0)$, Andy’s best response is to play Boar. Hence the newly available strategy to each player destroys the stability of the old equilibrium. In Stag-Hare-Boar II, on the other hand, not only is there a mixed strategy Nash equilibrium between (Stag, Stag) and (Hare, Hare), but there are also a mixed strategy Nash equilibrium between (Stag, Stag) and (Boar, Boar), and a mixed strategy Nash equilibrium between (Hare, Hare) and (Boar, Boar). Moreover, there is a mixed strategy Nash equilibrium in which players playing each pure strategy with positive probability. In other words, in Stag-Hare-Boar II, for any collection of pure strategy Nash equilibria, there is a mixed strategy Nash equilibrium among them.

What distinguishes the underlying strategic situation in Stag-Hare-Boar II from that in Stag-Hare-Boar I? In particular, we are interested in characterizing strategic situations whose set of Nash equilibria has the following property:

**Definition 1** The set of Nash equilibria of $G$, $N_G$, is said to to have a sandwich structure if for every subset $M \subset N_G$ of pure strategy Nash equilibria with $|M| \geq 2$, there is a unique mixed strategy Nash equilibrium $\sigma \in N_G$ “spanned by $M$”; that is, the set of pure strategies used in $\sigma$ is the set of pure strategies in $M$. 


A sandwich structure is a natural extension of the “betweenness” property in generic $2 \times 2$ games. When a game has a sandwich structure in Nash equilibria, the information on the number of pure strategy Nash equilibria can be used to infer the (minimum) number of mixed strategy Nash equilibria in the game. For example, a quick observation is that if a game $G$ has three or more pure strategy Nash equilibria and $N_G$ has a sandwich structure, then there are more mixed strategy Nash equilibria than pure strategy Nash equilibria in $G$.

Now we provide a sufficient condition for generic $n \times n$ games to have a sandwich structure in Nash equilibria. The condition provided here is based solely on the ordinal structure of the game for the following two reasons. First, while cardinal payoffs provide more information on the underlying strategic situation than ordinal payoffs do, the ordinal structure could reveal regular patterns that cannot be readily seen when the game is presented in cardinal payoffs. Second, since each ordinal payoff matrix is a compact representation of uncountably many cardinal payoff matrices, a condition based solely on the ordinal structure of the game significantly expands the class of games our theorem can be applied to.

Define an order transformation $O^1 : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ as $O^1_{jk}(G^1) = m$ if $G^1(s^1_j, s^2_k)$ is the $m$th largest payoff for player 1 given $s^2_k$. Similarly, $O^2_{jk}(G^2) = m$ if $G^2(s^1_j, s^2_k)$ is the $m$th largest payoff for player 2 given $s^1_j$. The order bimatrix of a game $G$ replaces players’ cardinal payoffs by their ordinal payoffs and is denote by $O(G)$ such that $O(G)_{jk} = (O^1_{jk}(G^1), O^2_{jk}(G^2))$. Clearly $O(G)$ is not uniquely defined, as two strategies may give a player the same payoff given the other player’s strategy. Throughout the paper, when there is more than one order representation of a game; we only require one of them to meet the premise of a theorem for the conclusion to hold.

For $3 \times 3$ games, it turns out that the information on the positions of the second highest payoffs is all we need. Define a bimatrix $T(G) = (T^1(G^1), T^2(G^2))$ of a game $G$ to be such that

$$T^i_{jk}(G^i) = \begin{cases} 1 & \text{if } O^i_{jk}(G^i) = 2 \\ 0 & \text{if } O^i_{jk}(G^i) \neq 2 \end{cases}.$$  

**Proposition 1** Let $G$ be a generic $3 \times 3$ game. If $T^1(G^1)$ and $T^2(G^2)$ have full rank, then $N_G$ has a sandwich structure.$^3$

Proposition 1 states that a generic $3 \times 3$ game has a sandwich structure in Nash equilibria if no pure strategy is a player’s second best response against more than one pure strategy of the other player. The intuition behind this result is provided in the next two examples.

As a generic $3 \times 3$ game has no more than 7 Nash equilibria (Quint and Shubik (1997)), Proposition 1 directly implies the following.$^4$

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$^3$The proof is straightforward and hence omitted.

$^4$Note that our definition of generic games is stronger than that in Quint and Shubik (1997), as differentiability is required here.
Corollary 1 Let $G$ be a generic $3 \times 3$ game. If $T^1(G^1)$ and $T^2(G^2)$ have full rank and $G$ has three pure strategy Nash equilibria, then $G$ has exactly four mixed strategy Nash equilibria.

Example 3 For the Stag-Hare-Boar I game, we have

$$T(G) = \begin{bmatrix}
0,0 & 0,0 & 1,1 \\
0,0 & 0,0 & 0,1 \\
1,1 & 1,0 & 0,0
\end{bmatrix}.$$  

We observe that neither $T^1(G^1)$ nor $T^2(G^2)$ has full rank, as each player’s third strategy, Boar, is a second best response against the other player’s Stag and Hare strategies. This gives either player an incentive to deviate from the equilibrium mixed strategy in the Stag Hunt game, $(1/5, 4/5)$, and hence this game does not have a sandwich structure in Nash equilibria.

Example 4 For the Stag-Hare-Boar II game, we have

$$T(G) = \begin{bmatrix}
0,0 & 0,1 & 1,0 \\
1,0 & 0,0 & 0,1 \\
0,1 & 1,0 & 0,0
\end{bmatrix}.$$  

As $T^1(G^1)$ and $T^2(G^2)$ have full rank, $N_G$ has a sandwich structure. Moreover, as the game has three pure strategy Nash equilibria, by Corollary 1, it has four mixed strategy Nash equilibria. In particular, $((1/5, 4/5), (1/5, 4/5))$ constitutes a Nash equilibrium in this game as Boar is not a profitable deviation. Indeed, the full rank criterion, which requires that each pure strategy cannot be a second best response against more than one pure strategy of the other player, makes sure that each strategy is not “powerful enough” to destroy a mixed strategy Nash equilibrium.

For $n \times n$ games with $n > 3$, the full rank criterion is not sufficient to guarantee that the set of Nash equilibria has a sandwich structure. Consider the following example:

Example 5 Suppose the game $G$ is such that

\[
\begin{array}{c|cccc}
 & b_1 & b_2 & b_3 & b_4 \\
\hline
\text{Player 1} & a_1 & 8,10 & 7,7 & 4,4 & -3,1 \\
 & a_2 & 5,0 & 9,2 & 0,-5 & -1,-2 \\
 & a_3 & 1,3 & 5,4 & 6,8 & 0,5 \\
 & a_4 & 2,0 & 3,-3 & 5,1 & 4,3 \\
\end{array}
\]

Then

\[
O(G) = \begin{bmatrix}
1,1 & 2,2 & 3,3 & 4,4 \\
2,2 & 1,1 & 4,4 & 3,3 \\
4,4 & 3,3 & 1,1 & 2,2 \\
3,3 & 4,4 & 2,2 & 1,1
\end{bmatrix}.
\]
We observe that each player’s \( j \)'th highest payoff matrix has full rank, \( j = 1, \ldots, 4 \), but there does not exist a mixed strategy Nash equilibrium between the two pure strategy Nash equilibria \((a_2, b_2)\) and \((a_3, b_3)\), as \( a_1 \) is a profitable deviation for player 1.

Thus, to ensure that a generic \( n \times n \) game with \( n > 3 \) has a sandwich structure in Nash equilibria, we need something stronger than full rank. For a given matrix \( A \), denote by \( A_{i,*} \) the \( i \)th row of \( A \) and \( A_{*,j} \) the \( j \)th column of \( A \).

**Definition 2** A generic \( n \times n \) game \( G \) is said to have an even order distribution if the following two conditions hold:

(i) **\( G^1 \) has an even order distribution**

There exists a permutation \( P : \{1, \ldots, n\} \to \{1, \ldots, n\} \) such that \( O_{*,P(k)}^1(G^1) + 1 \pmod n = O_{*,P(k+1)}^1(G^1), \ k = 1, \ldots, n - 1. \)

(ii) **\( G^2 \) has an even order distribution**

There exists a permutation \( P : \{1, \ldots, n\} \to \{1, \ldots, n\} \) such that \( O_{P(k),*}^2(G^2) + 1 \pmod n = O_{P(k+1),*}^2(G^2), \ k = 1, \ldots, n - 1. \)

As suggested by its name, the notion of even order distribution captures the idea that the ordinal payoff structure for each player is balanced: each order vector is a rotation of another order vector. It is a generalization of the order structure of generic \( 2 \times 2 \) games with two pure strategy Nash equilibria. Moreover, it can be readily seen that even order distribution implies full rank, but not vice versa (Example 5 is an example where a matrix has full rank but does not have an even order distribution).\(^*\) However, in a generic \( 3 \times 3 \) game with three pure strategy Nash equilibria, it is not difficult to check that full rank and even order distribution are equivalent.

Even order distribution has a nice property that plays a crucial role in establishing our main result (Theorem 1). The property is that if a matrix has an even order distribution, then any submatrix of it with ones on the diagonal also has an even order distribution.

**Example 6** Consider a \( 5 \times 5 \) game \( G \) in which player 1’s order payoff matrix \( O^1(G^1) \) is such that

\[
O^1(G^1) = \begin{bmatrix}
1 & 3 & 2 & 5 & 4 \\
4 & 1 & 5 & 3 & 2 \\
5 & 2 & 1 & 4 & 3 \\
2 & 4 & 3 & 1 & 5 \\
3 & 5 & 4 & 2 & 1
\end{bmatrix}
\]

Clearly \( O^1(G^1) \) has an even order distribution. Let \( G^1_{3\times3} \) denote the \( 3 \times 3 \) submatrix formed from the top left block of \( G^1 \). Then we have

\[
O^1(G^1_{3\times3}) = \begin{bmatrix}
1 & 3 & 2 \\
2 & 1 & 3 \\
3 & 2 & 1
\end{bmatrix},
\]

\(^*\)Note that the interior of the set of generic games with an even order distribution is non-empty.
which again has an even order distribution.

Recall that a circulant matrix of order $n$ is a square matrix of the form (Davis (1970))

$$C = \begin{bmatrix}
c_1 & c_2 & \cdots & c_n \\
c_n & c_1 & \cdots & c_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
c_2 & c_3 & \cdots & c_1
\end{bmatrix}.$$

The connection between even order distribution and circulant matrix can be easily seen:

**Proposition 2** Let $G$ be a two-person $n \times n$ game. The following are equivalent:

(i) $O(G)$ has an even order distribution

(ii) There exist permutation matrices $\{P^i, Q^i\}_{i=1}^2$ such that $P^1O^1(G^1)Q^1$ and $P^2O^2(G^2)Q^2$ are circulant.

Therefore, every ordinal payoff matrix with an even order distribution is a permuted circulant matrix.

Now we state our main result. The proof is relegated to the Appendix.

**Theorem 1** Let $G$ be a generic $n \times n$ game with an even order distribution, $n \geq 2$. Then

(i) $N_G$ has a sandwich structure.

(ii) If $G$ has $n$ pure strategy Nash equilibria, then

(a) $G$ has exactly $2^n - n - 1$ mixed strategy Nash equilibria.

(b) A mixed strategy Nash equilibrium in which each player randomizes among $k$ pure strategies has index $+1$ if and only if $k$ is odd.

Whenever there are multiple pure strategy Nash equilibria, as each player’s best response varies with the other player’s strategy, players properly randomize among those pure strategies constitutes an equilibrium candidate. Having an even order distribution in ordinal payoffs makes sure that this randomization is not dominated by any pure strategy outside the support, for each player and given the other player’s mixed strategy, and hence it is indeed an equilibrium.

The results are established by inductively applying the Lefschetz-Hopf theorem. The mathematical induction is made possible through two important properties of a matrix with an even order distribution. The first property is that if a matrix has an even order distribution, then any submatrix of it with ones on the diagonal also has an even order distribution. The second property is that even order distribution ensures that any strategy outside the support of a mixed strategy equilibrium cannot be a profitable deviation.

Theorem 1 shows that we can fully characterize the structure of Nash equilibria when there are $n$ pure strategy Nash equilibria in a generic $n \times n$ game with an even
order distribution.\textsuperscript{6} In particular, we show the the conjecture by Quint and Shubik (1997) is correct in games with an even order distribution: there are at most $2^n - 1$ Nash equilibria in generic $n \times n$ games with an even order distribution. Moreover, a mixed strategy Nash equilibrium with support of odd (even) number of pure strategies has index $+1 (-1)$. Thus, every mixed strategy with support of even number of pure strategies is unstable under any Nash dynamics (Demichelis and Germano (2002)).

**Example 7** Suppose the game $G$ is such that

<table>
<thead>
<tr>
<th>Player 1</th>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>9,7</td>
<td>$b_1$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>2,0</td>
<td>$b_2$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>0,-4</td>
<td>$b_3$</td>
</tr>
<tr>
<td>$a_4$</td>
<td>5,-10</td>
<td>$b_4$</td>
</tr>
</tbody>
</table>

Then

$$O(G) = \begin{bmatrix} 1,1 & 3,4 & 2,3 & 4,2 \\ 3,2 & 1,1 & 4,4 & 2,3 \\ 4,3 & 2,2 & 1,1 & 3,4 \\ 2,4 & 4,3 & 3,2 & 1,1 \end{bmatrix}.$$  

Thus $G$ has an even order distribution and four pure strategy Nash equilibria. By Theorem 1, $G$ has 15 Nash equilibria in total.

The assumption of even order distribution can be weakened when the number of pure strategy Nash equilibria is smaller than the number of strategies available to each player. Consider a generic $n \times n$ game $G$ with $m < n$ pure strategy Nash equilibria. Without lost of generality, assume $M = \{(s_1^1, s_1^2), ..., (s_m^1, s_m^2)\}$ is the set of pure strategy Nash equilibria. Denote by

$$\tilde{G}_{n \times m}^1 = [G_{s,1}^1 ... G_{s,m}^1], \text{ and } \tilde{G}_{m \times n}^2 = \begin{bmatrix} G_{1,s}^2 \\ \vdots \\ G_{m,s}^2 \end{bmatrix}.$$  

**Definition 3** $G$ is said to have a restricted even order distribution if the following two conditions hold:

1. There exists a $n \times (n - m)$ matrix $H_{n \times (n - m)}$ such that $[\tilde{G}_{n \times m}^1 H_{n \times (n - m)}]$ has an even order distribution

2. There exists a $(n - m) \times n$ matrix $\widehat{H}_{(n - m) \times n}$ such that $[\tilde{G}_{m \times n}^2 \widehat{H}_{(n - m) \times n}]$ has an even order distribution

\textsuperscript{6}For zero-sum games, the set of Nash equilibria is convex, hence there is either a unique Nash equilibrium or there are infinitely many. In other words, a zero-sum game is either non-generic or if it is generic, the Nash equilibrium is unique. In either case, even order distribution has no bite.
This definition is equivalent to that of even order distribution when \( m = n \), but the condition is weaker when \( m < n \). In particular, note that a matrix with a restricted even order distribution need not be a permutated circulant. We observe that the payoffs outside \( G_{n \times m}^1 \) in game \( G \) are irrelevant to player 1 when player 2 assigns zero probabilities on \( \{s_{m+1}^2, \ldots, s_n^2\} \), and the payoffs outside \( G_{n \times n}^2 \) are irrelevant to player 2 when player 1 assigns zero probabilities on \( \{s_{m+1}^2, \ldots, s_n^2\} \). Thus, the proof of Theorem 1 goes through with this weaker assumption and we obtain the following corollary:

**Corollary 2** Suppose \( G \) is a generic \( n \times n \) game with \( m < n \) pure strategy Nash equilibria. If \( G \) has a restricted even order distribution, then \( N_G \) has a sandwich structure.

If a game \( G \) has \( m \) pure strategy Nash equilibria and \( N_G \) has a sandwich structure, then \( G \) has at least \( 2^m - m - 1 \) mixed strategy Nash equilibria. Hence, the following corollary is immediate:

**Corollary 3** Let \( G \) be a generic \( n \times n \) game with a restricted even order distribution. If \( G \) has \( m \) pure strategy Nash equilibria, then \( G \) has at least \( 2^m - m - 1 \) mixed strategy Nash equilibria.

Corollary 2 provides a lower bound on the number of mixed strategy Nash equilibria relative to the number of pure strategy Nash equilibria. We compare our result to that of Gul et al. (1993). Recall that Gul et al. (1993) show that a generic game with \( m \) pure strategy Nash equilibria has at least \( m - 1 \) mixed strategy Nash equilibria. Their proof is based on the observation that a pure strategy Nash equilibrium has index +1 and the sum of indices at Nash equilibria is +1. Therefore the lower bound on the number of mixed strategy Nash equilibria provided by them is linearly increasing with the number of pure strategy Nash equilibria.

Within the class of games with a restricted even order distribution, the lower bound on the number of mixed strategy Nash equilibria provided in Corollary 2 is exponentially increasing with the number of pure strategy Nash equilibria. The exponential growth in mixed strategy Nash equilibria is due to the fact that every single collection of pure strategy Nash equilibria provides an opportunity for players randomizing in their actions to be a stable play. Accordingly, the number of mixed strategy Nash equilibria exceeds the number of pure strategy Nash equilibria whenever there are more than two pure strategy Nash equilibria, and the proportion of pure strategy Nash equilibria converges to zero as the number of pure strategy Nash equilibria increases.

Clearly Gul et al. (1993)’s result is much more general than ours, as it applies to all generic games. However, their lower bound on the number of mixed strategy Nash equilibria, \( m - 1 \) when there are \( m \) pure strategy Nash equilibria, may not be binding for most of the games. On the other hand, our result provides a significantly higher lower bound: \( 2^m - m - 1 \) when there are \( m \) pure strategy Nash equilibria. The price we pay for this stronger result is that it can only be applied to a subclass of games.
4 Concluding Remarks

In this paper, we study the structure of Nash equilibria in generic $n \times n$ games. The main finding is that a generic $n \times n$ game with a restricted even order distribution has a sandwich structure in Nash equilibria, which allows us to provide a lower bound on the number of mixed strategy Nash equilibria relative to the number of pure strategy Nash equilibria in a game. In contrast to Gul et al. (1993), our result applies only to a subclass of generic games, but the lower bound provided here is significantly sharper. The notion of restricted even order distribution is intuitively appealing and provides a partial understanding between the underlying strategic situation and the (exact) structure of Nash equilibria.

5 Appendix

Proof of Theorem 1. We prove it by induction. The statement holds trivially for generic $2 \times 2$ games. Suppose the statement holds for generic $k \times k$ games, $k = 2, ..., n - 1$. We show that it holds for generic $n \times n$ games. Let $G$ be a generic $n \times n$ game with an even order distribution. Fix a subset $M \subset N_G$ of pure strategy Nash equilibria with $|M| \geq 2$. We first show that there exists a unique mixed strategy Nash equilibrium $\sigma \in N_G$ spanned by $M$, which establishes property (i) that $N_G$ has a sandwich structure. Property (ii)(a) then follows immediately from (i). Consider two subcases:

Case 1. $|M| = m < n$. Without loss of generality assume $M = \{(s_1^1, s_1^2), ..., (s_m^1, s_m^2)\}$. Consider now the generic $m \times m$ game $\tilde{G}$ by deleting strategies $\{s_k^i\}_{k=m+1, ..., n; i=1, 2}$ from $G$. As $G$ has an even order distribution, there exists a permutation $P : \{1, ..., n\} \rightarrow \{1, ..., n\}$ such that $O_{*, P(k)}(G^1) + 1 \mod n = O_{*, P(k+1)}(G^1)$, $k = 1, ..., n - 1$. Denote by $k_i$ the $i$th smallest number in $\{1, ..., n\}$ such that $P(k_i) \in \{1, ..., m\}$. We need the following lemma:

**Lemma 1** $\tilde{G}$ has an even order distribution.

Proof of Lemma 1. We show that $\tilde{G}^1$ has an even order distribution. $\tilde{G}^2$ has an even order distribution can be established analogously. As $M = \{((s_1^1, s_1^2), ..., (s_m^1, s_m^2))\} \subset N_G$, $s_k^i$ is player 1’s best response against $s_k^i$ for $k = 1, ..., m, O_{kk}^1(\tilde{G}^1) = O_{kk}^1(G^1) = 1, k = 1, ..., m$. Since $P(k_i) \in \{1, ..., m\}$, $O_{P(k_i)P(k_i)}^1(\tilde{G}^1) = 1, i = 1, ..., m$. As $O_{*, P(k_i)}(G^1) = O_{*, P(k_i)}(G^1) + k_i - k_1 \mod n, O_{P(k_i)P(k_i)}^1(G^1) = 1$ implies $O_{P(k_i)P(k_i)}^1(G^1) = n + 1 - (k_i - k_1), i = 2, ..., m$. We then have

$$O_{P(k_2)P(k_1)}^1(G^1) > O_{P(k_3)P(k_1)}^1(G^1) > ... > O_{P(k_m)P(k_1)}^1(G^1).$$

Note that not all “subgames” of a game with an even order distribution have an even order distribution.
In other words, in the game $G$, among the $m$ strategies $\{s_1^1, ..., s_m^1\}$, $s_{P(k_1)}^1$ gives player 1 the lowest payoff against $s_{P(k_1)}^1$, and $s_{P(k_m)}^1$ gives player 1 the highest payoff against $s_{P(k_1)}^1$. Accordingly, in the game $\tilde{G}$ where the strategies $\{s_k^1\}_{k=m+1,...,n}$ are removed from $G$, player 1’s order vector against $s_{P(k_1)}^2$ can be determined as

$$O_{P(k_1)P(k_1)}^1(\tilde{G}^1) = m + 2 - i, \ i = 2, ..., m.$$ 

Following the equation $O_{P(k_1)P(k_1)}^1(\tilde{G}^1) = O_{P(k_1)P(k_1)}^1(G^1) + k_i - k_1 \ (\text{mod} \ n)$ and the fact that $O_{P(k_1)P(k_1)}^1(G^1) = n + 1 - (k_i - k_1), \ i = 2, ..., m$, $O_{P(k_1)P(k_1)}^1(G^1)$ can be constructed from $O_{P(k_1)P(k_1)}^1(G^1)$ as

$$O_{P(k_1)P(k_1)}^1(G^1) = \begin{cases} 
1 + k_2 - k_1 & i = 1 \\
n + 1 - (k_i - k_1) + k_2 - k_1 & i = 2 \\
n + 1 + k_2 - k_i & i = 3, ..., m.
\end{cases}$$

We observe that $n + 1 + k_2 - k_i - (1 + k_2 - k_1) = n - k_i + k_1 > n - m + 1 > 0$. Therefore in the game $\tilde{G}$,

$$O_{P(k_1)P(k_1)}^1(\tilde{G}^1) = \begin{cases} 
2 & i = 1 \\
1 & i = 2 \\
m + 3 - i & i = 3, ..., m
\end{cases} = O_{P(k_1)P(k_1)}^1(\tilde{G}^1) + 1 \ (\text{mod} \ m).$$

Continuing in this fashion, we conclude that $O_{P(k_1)P(k_1)}^1(\tilde{G}^1) + 1 \ (\text{mod} \ m) = O_{P(k_1)P(k_1)}^1(\tilde{G}^1)$, $i = 1, ..., m - 1$, and hence $\tilde{G}^1$ has an even order distribution. \(\square\)

Given that $\tilde{G}$ is a generic $m \times m$ game with an even order distribution and $m < n$, there exists a unique mixed strategy Nash equilibrium $\sigma \in N_{\tilde{G}}$ spanned by $M$. We show that $\sigma \in N_G$. Pick any strategy $s_j^1$ with $j \in \{m + 1, ..., n\}$. Denote by $\tilde{i} = \min_{i=1,...,m}O_{jP(k_1)}^1(G^1)$. Given $\sigma^2$, $s_j^1$ is strictly dominated by $s_{P(k_1)}^1$ (and hence strictly dominated by $\sigma^1$), as for $i = 1, ..., m$,

$$O_{P(k_1)P(k_1)}^1(G^1) = O_{P(k_1)P(k_1)}^1(G^1) + \begin{cases} 
1 & \text{if } k_i \geq k_1 \\
n + k_i - k_1 & \text{if } k_i < k_1.
\end{cases}$$

Accordingly, $\sigma^1$ is a best response against $\sigma^2$. Similarly, given $\sigma^1$, it can be readily verified that every strategy $s_j^1$ with $j \in \{m + 1, ..., n\}$ is strictly dominated by $\sigma^2$. Therefore $\sigma \in N_G$ and the game has a sandwich structure in Nash equilibria in this subcase.

Let $h$ be the associated Nash map of $G$, and $\tilde{h}$ be the associated Nash map of $\tilde{G}$. We show that the two fixed-point indices in these two games are equivalent:
$i(h, \sigma) = i(\tilde{h}, \sigma)$. By definition, $i(h, \sigma) = \text{sign } H$, where

$$H = \det \begin{bmatrix}
1 - \frac{\partial h_{s_k}^i}{\partial \sigma_j^i} & \cdots & -\frac{\partial h_{s_k}^i}{\partial \sigma_j^m} & e_1^T & 0 \\
\vdots & \ddots & \vdots & \vdots \\
-\frac{\partial h_{s_k}^l}{\partial \sigma_j^i} & \cdots & 1 - \frac{\partial h_{s_k}^l}{\partial \sigma_j^m} & e_1^T & 0 \\
e_1 & \cdots & e_1 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0
\end{bmatrix}.$$  

Permuting the rows and columns gives us

$$H = \det \begin{bmatrix}
\Lambda_{2 \times 2}^{-1} & \Lambda_{2 \times 2}^{-1} \\
-\Lambda_{2 \times 2}^{-1} & 0 \\
0 & 0
\end{bmatrix}.$$  

Since every $s_j^1$ with $j \in \{m+1, \ldots, n\}$ is strictly dominated by $\sigma^1$ given $\sigma^2$, and every $s_j^2$ with $j \in \{m+1, \ldots, n\}$ is strictly dominated by $\sigma^2$ given $\sigma^1$, at $\sigma$ we have

$$\frac{\partial h_{s_k}^l}{\partial \sigma_j^i} = \begin{cases}
\frac{\partial h_{s_k}^l}{\partial \sigma_j^i}, & i, l \in \{1, 2\}; j, k \in \{1, \ldots, m\} \\
0, & i, l \in \{1, 2\}; j \in \{1, \ldots, n\}; k \in \{m+1, \ldots, n\}.
\end{cases}$$  

Accordingly,

$$H = \det \begin{bmatrix}
\Lambda_{2 \times 2}^{-1} & \Lambda_{2 \times 2}^{-1} \\
-\Lambda_{2 \times 2}^{-1} & 0 \\
0 & 0
\end{bmatrix}.$$  

As $\sum_{k=1}^n h_{s_k}^l \equiv 1$, $\sum_{k=1}^n \frac{\partial h_{s_k}^l}{\partial \sigma_j^i} = 0 \forall i, l \in \{1, 2\}$ and $j \in \{1, \ldots, n\}$. Subtracting $\frac{1}{m}(\sum_{i=1}^m H_{i, \ast} + H_{2m+1, \ast})$ from $H_{2(m+1)+j}$, and $\frac{1}{m}(\sum_{i=m+1}^{2m} H_{i, \ast} + H_{2(m+1), \ast})$ from $H_{2(m+1)+n-m+j}$, gives us

$$H = \det \begin{bmatrix}
\Lambda_{2 \times 2}^{-1} & \Lambda_{2 \times 2}^{-1} \\
-\Lambda_{2 \times 2}^{-1} & 0 \\
0 & 0
\end{bmatrix} \text{ det } \tilde{I},$$  

where

$$\tilde{I} = \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} \text{ and } W = \begin{bmatrix} 1 + \frac{1}{m} & \cdots & \frac{1}{m} \\ \vdots & \ddots & \vdots \\ \frac{1}{m} & \cdots & 1 + \frac{1}{m} \end{bmatrix}.$$
Hence we have

\[ i(h, \sigma) = \text{sign} \ H = \text{sign} \ \det \left[ \begin{array}{cc}
I_{2m \times 2m} - \left[ \frac{\partial^2 h}{\partial \sigma \partial s} \right]_{i,l=1,2; \ j,k=1,\ldots,m} & \Lambda^T_{2 \times 2m} \\
-\Lambda_{2 \times 2m} & 0
\end{array} \right] \ \det \tilde{I}
\]

\[ = \text{sign} \ \det \left[ I_{2m \times 2m} - \left[ \frac{\partial^2 h}{\partial \sigma \partial s} \right]_{i,l=1,2; \ j,k=1,\ldots,m} \ \Lambda^T_{2 \times 2m} \\
-\Lambda_{2 \times 2m} & 0
\right] (\det W)^2
\]

\[ = i(\tilde{h}, \sigma).
\]

Case 2. \(|M| = n\). In this case, \(M\) is the set of all pure strategy Nash equilibria in \(G\). Without loss of generality assume \(M = \{(s^1_1, s^2_1), \ldots, (s^1_n, s^2_n)\}\). From Case 1, we conclude that there exists a unique mixed strategy Nash equilibrium \(\sigma_{M'} \in N_G\) spanned by \(M'\) for every subset \(M' \subset M\) such that \(2 \leq |M'| \leq n - 1\). Let \(\Phi = \{\sigma_{M'}| M' \subset M \text{ with } 2 \leq |M'| \leq n - 1\} \) denote the collection of all such equilibria. We claim that \(\Phi\) is the set of all mixed strategy Nash equilibria in \(G\) in which each player randomizes among 2 to \(n - 1\) pure strategies. Let \(\sigma \in N_G\) be a mixed strategy Nash equilibrium in which player 1 assigns positive probability to \(k\) pure strategies, \(2 \leq k \leq n-1\). We show that \(\sigma \in \Phi\). Without loss of generality assume player 1 assigns positive probability to \(s^1_j, i = 1, \ldots, k\). Given \(\sigma^1\), as it has been shown in Case 1, \(G\) has an even order distribution implies that every strategy \(s^2_j \) with \(j \in \{k + 1, \ldots, n\}\) is strictly dominated by \(s^2_l\) for some \(l \in \{1, \ldots, k\}\). Thus \(\sigma^2\) assigns zero probability to \(s^2_j, j = k + 1, \ldots, n\). We argue that \(\sigma^2\) assigns positive probability to \(s^2_j\) for every \(j \in \{1, \ldots, k\}\). Suppose to the contrary that \(\sigma^2\) assigns zero probability to \(s^2_j\) for some \(j \in \{1, \ldots, k\}\). \(G\) has an even order distribution implies that given \(\sigma^2\), \(s^1_j\) is strictly dominated by \(s^1_i\) for some \(i \in \{1, \ldots, k\}\) \{\(j\)\}, which contradicts the premise that \(\sigma^1\) assigns positive probability to \(s^1_i, i = 1, \ldots, k\). Accordingly, the support of \(\sigma\) is spanned by \(\{(s^1_1, s^2_1), \ldots, (s^1_k, s^2_k)\} \subset M\), and hence \(\sigma \in \Phi\).

We are now able to show that there exists a unique mixed strategy Nash equilibrium spanned by \(M\). The invariance property of the fixed-point index established in Case 1, \(i(h, \sigma) = i(\tilde{h}, \sigma)\), allows us to determine the index for each and every mixed strategy \(\sigma\) in \(N_G\) inductively. The index of \(\sigma \in \Phi\) spanned by \(M' \subset M\) with \(|M'| = 2\) is \(-1\), as the unique mixed strategy Nash equilibrium in a generic \(2 \times 2\) game with two pure strategy Nash equilibria has index \(-1\). The index of \(\sigma \in \Phi\) spanned by \(M' \subset M\) with \(|M'| = 3\) is \(+1\). The reason is that according to Case 1, there are \(\binom{3}{2} = 3\) mixed strategy Nash equilibria in which each equilibrium is spanned by two elements in \(M'\) (and hence each has index \(-1\)), and one mixed strategy Nash equilibrium spanned by \(M'\), which is \(\sigma\). Applying the Lefschetz-Hopf theorem to the generic \(3 \times 3\) game spanned by \(M'\), we get \(+1 = +3\) (three pure strategy Nash equilibria) \(-3\) (three mixed strategy Nash equilibria spanned by two elements in \(M'\) + \(i(\tilde{h}, \sigma)\)). Thus, \(i(\tilde{h}, \sigma) = +1\). By the index invariance property, we have \(i(h, \sigma) = i(\tilde{h}, \sigma) = +1\) in game \(G\). Following the same reasoning inductively, we conclude that the index of \(\sigma \in \Phi\) with support \(M' \subset M\) with \(2 \leq |M'| \leq n - 1\) is \(+1\) if and only if \(|M'|\) is odd.
Therefore,

\[ \sum_{\sigma \in \Phi} i(h, \sigma) = \sum_{i=1}^{n-1} \binom{n}{i} (-1)^{i+1} = \begin{cases} 2 & \text{n is even} \\ 0 & \text{n is odd} \end{cases} +1 \] .

The Lefschetz-Hopf theorem then indicates that there exists at least one more equilibrium \( \sigma^* \in N_G \). As \( \sigma^* \notin \Phi \), at \( \sigma^* \) at least one player \( i \) randomizes among all \( n \) pure strategies. Then player \( j \neq i \) must randomize among \( n \) pure strategies as well, for otherwise player \( i \) randomizing among \( n \) pure strategies is not a best response given that \( G \) has an even order distribution. Therefore \( \sigma^* \) is spanned by \( M \). As \( G \) is generic, \( \sigma^* \) is the unique equilibrium spanned by \( M \), and the sandwich structure of \( N_G \) is established. The set of Nash equilibria \( N_G = \Phi \cup \{ \sigma^* \} \). \( i(h, \sigma^*) = +1 \) when \( n \) is odd, and \( -1 \) when \( n \) is even, which establishes (ii)(b) in Theorem 1 and the proof is complete. ■

References


