Realized Beta GARCH:
A Multivariate GARCH Model with Realized Measures of Volatility and CoVolatility

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Abstract

We introduce a multivariate GARCH model that incorporates realized measures of volatility and covolatility. The realized measures extract information about the current level of volatility and covolatility from high-frequency data, which is particularly useful for the modeling of return volatility during periods with rapid changes in volatility and covolatility. When applied to market returns in conjunction with returns on an individual asset, the model yields a dynamic model of the conditional regression coefficient that is known as the beta. We apply the model to a large set of assets and find the conditional betas to be far more variable than is usually found with rolling-window regressions based exclusively on daily returns. In the empirical part of the paper we examine the cross-sectional as well as the time variation of the conditional beta series during the financial crises.

Keywords: Financial Volatility; Beta; Realized GARCH; High Frequency Data.

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1 Introduction

Relatively accurate measurements of volatility and covolatility can be computed from high frequency data, and such statistics are commonly referred to as realized measures. Incorporating realized measures for the modeling the dynamic properties of volatility, such as GARCH models, is very beneficial. The reason is that returns yield very weak signals about latent volatility, whereas realized measures provide accurate measurements. The latter is particularly useful during times with rapid changes volatility and covolatility.

In this paper we propose a multivariate GARCH-type model that utilizes and models realized measures of volatility and covolatility. The model has hierarchical structure where a “market” returns is modeled with a univariate Realized GARCH model, see Hansen and Huang (2012) and Hansen et al. (2010). A multivariate structure is constructed by modeling “individual” returns conditional on the past and contemporary market variables (return and volatility). The resulting model has the structure of a dynamic CAPM model that enables us to extract the “betas” and study their dynamic properties. Moreover, the model is complete in the sense that all observables (returns, realized volatilities and realized correlations) are modeled. The latter enables us to infer the distribution of multi-period returns including the joint distribution of “market” returns and “individual” returns over longer horizons.

The main contributions of our paper are the following. We propose a flexible and tractable framework that enables a modeling of a potentially large set of assets. Unlike conventional multivariate GARCH models, which can suffer from the curse of dimensionality and estimation issues, we avoid such issues by incorporating realized measures and the use of measurement equations. These measurement equations ties realized measures to the latent volatilities that induce a suitable regularization on the model. This particular structure was chosen for a number of reasons. First, the model provides good empirical fit for the wide range of assets used in our empirical study; Second, the structure of the model is amenable to a deeper analysis of secondary quantities such as betas; Third, the model is simple to estimate, which is particularly important when a large set of assets are to be analyzed as is the case in our empirical analysis.

The proposed model structure has a hierarchical structure where the market return and a corresponding realized measure forms the core of the model. The model can be extended to an arbitrary large set of individual returns, by adding a conditional model for an individual return
and two realized measures, one being a realized measure of return volatility, the other being a realized measure of the correlation between the individual return and the market. This yields a flexible model with a dynamic covariance structure that is constantly revised due to the time variation in realized measures.

The concept of realized betas is not new. Bollerslev and Zhang (2003) carry out a large scale estimation of the Fama-French three-factor model using high-frequency (5-minute) data on 6,400 stocks over a period of 7 years. Their analysis showed that high-frequency data can improve the pricing accuracy of asset pricing models. Their approach differs from ours in important ways. For instance, they model raw realized factor loadings and use simple time series processes to forecast these. So there is no explicit link between realized and conditional moments of returns in their framework. Nor do they explicitly account for the measurement error (the sampling error) in the realized quantities. Another related paper is Andersen et al. (2006) who study the time variation in realized variances, covariances, and betas using daily returns to construct quarterly realized measures. They find evidence of long memory in the time series for variance and covariances, while the realized beta time series is less persistent and seemingly a short-memory process, which is indicative of fractional cointegration between realized volatility and realized covariance. Other related studies include: Barndorff-Nielsen and Shephard (2004a) (asymptotic results on realized beta), Patton and Verardo (2009) (impact of news on betas), Todorov and Bollerslev (2010) (continuous and jump component betas), Dovonon et al. (2010) (bootstrap inference on realized betas), Bandi and Russell (2005) (MSE-optimal estimation of realized betas), Tsay and Yeh (2011) (high-frequency betas).

The use of realized volatility measures in this context yields valuable insight about the degree of time-variation in the betas, which has been up for debate in the literature. The studies of Ferson and Harvey (1991, 1993), and Shanken (1990) specify parametric relationships between betas and proxies for the state of the economy and find support for time-varying betas. Gomes et al. (2003) provide a theoretical justification for a time-varying conditional beta specification in the context of a dynamic general equilibrium production economy. Conditional betas have been modeled by means of conventional GARCH models by Braun et al. (1995) and Bekaert and Wu (2000), among others. Lewellen and Nagel (2006) argue that variation in betas would have to be “implausibly large” to explain important asset-pricing anomalies. In our empirical analysis we do find deal of time-variation in the conditional betas, this is particularly the case during the global financial crises period. We find the variation in betas to be substantial, even
over short periods of time, such as a quarter. [MORE DETAILS TBA]

The research devoted to high-frequency volatility measures was spurred by Andersen and Bollerslev (1998), who documented that the sum of squared intraday returns, known as the realized variance, provides an accurate measurement of daily volatility. The theoretical foundation of realized variance was developed in Andersen, Bollerslev, Diebold and Labys (2001) and Barndorff-Nielsen and Shephard (2002). Currently a large number of related estimators, such as realized bipower variation, realized kernels, multiscale estimators, preaveraging estimators and Markov chain estimators have been proposed to deal with issues such as jumps and market microstructure frictions, see Barndorff-Nielsen and Shephard (2004a), Barndorff-Nielsen, Hansen, Lunde and Shephard (2008), Zhang (2006), Jacod et al. (2008), Hansen and Horel (2009) and references therein. The multivariate extensions of the concept of realized volatility is theoretically developed in Barndorff-Nielsen and Shephard (2004a). Estimators that are robust to noise and/or asynchronous observations have been proposed by Hayashi and Yoshida (2005), Voev and Lunde (2007), Griffin and Oomen (2011), Christensen et al. (2010), and Barndorff-Nielsen et al. (2011). In this paper we will rely on the multivariate kernel estimator by Barndorff-Nielsen, Hansen, Lunde and Shephard (2011) that guarantees positive semi-definite estimates of the realized variance-covariance matrices we need.

While volatility is unobservable, the use of realized measures allow us to construct precise ex-post volatility proxies. Currently, a growing body of research investigates the issue to what extend realized measures can be used to specify better models of volatility dynamics and provide more accurate volatility forecasts. Hansen and Lunde (2010) categorize the existing approaches into two broad classes: reduced-form and model-based. Reduced-form volatility forecasts are based on a time series model for the series of realized measures, while a model-based forecast rests on a parametric model for the return distribution. Model-based approaches effectively build on GARCH models in which a realized measure is included as an exogenous variable in the GARCH equation, see e.g. Engle (2002). A complete framework that jointly specifies models for returns and realized measures of volatility was first proposed by Engle and Gallo (2006), who refer to their model as the Multiplicative Error Model (MEM). A simplified MEM structure was proposed in Shephard and Sheppard (2010), who estimated their referred to this model as the HEAVY model. The realized GARCH model by Hansen et al. (2010) involves a different approach to the joint modeling of returns and realized volatility measures. A key component of the Realized GARCH model is a measurement equation that links the realized
measure with the underlying conditional variance. This idea is generalized to the multivariate framework in this paper, where we introduce measurement equations for the realized measures of correlations.

The rest of the paper is structured as follows. The theory of the model and its estimation are presented in sections 2 and 3. Section 4 contains the empirical application of the model, and Section 6 concludes.

2 A Hierarchical Realized GARCH Framework

Our objective is to a large extent the same as that of existing multivariate GARCH models, which is to model the conditional distribution of a vector of returns. But unlike conventional GARCH models we also model the realized measures of volatility and covolatility and make extensive use of these in the modeling of returns. The realized measures are highly informative about local (in time) levels of volatility and covolatility which is the main reason these variables are so valuable in this context. By tying all individual return series to the market return, we are implicitly imposing a particular factor structure on the volatility, where the variation in the correlation structure is entirely driven by time-variation in the correlations between the market return and the individual assets. This keeps the model relatively simple and parsimonious, and the structure facilitates a direct mapping of key model variables into betas.

Our model has a hierarchical structure. The core of our framework is a marginal model for the market return and its realized measure of volatility. Individual returns, their realized measures of volatility and covolatility (with the market) are then modeled separately with a conditional model (where concurrent market variables are being conditioned on).

The modeling strategy we propose combines a marginal model for market returns and the corresponding realized measure of volatility, with a conditional model for the asset-specific return, its realized volatility and the covolatility between the asset and the market (the factor).

The marginal model we use for the market-specific time series is a variant of the Realized GARCH model discussed in Hansen et al. (2010), section 6.3. This model is called the Realized EGARCH model because it shares certain features with the EGARCH model by Nelson (1991). The conditional realized EGARCH model that is used to build a multivariate model is new.

Initially, we present the Realized Beta GARCH model in the simplest situation with a bivariate vector of returns (the market return and an individual asset return) and the corresponding 2x2 matrix realized volatility measures. Subsequently we discuss the straightforward extension
to an arbitrary number of individual assets.

2.1 Notation and Modeling Strategy

Let $r_{0,t}$ and $x_{0,t}$ denote the market return and a corresponding realized measure of volatility, respectively. Similarly, we use the notation $r_{1,t}$ and $x_{1,t}$ for the same variables associated with an individual asset return, and use $y_{it}$ to denote a realized measure of correlation, where $y_{it} \in (-1, 1)$.

In this context with two returns, two realized measures of volatility, and a realized measure of covariance we have five observable variables to model. The natural filtration is given by

$$F_t = \sigma(\mathcal{X}_t, \mathcal{X}_{t-1}, \ldots) \quad \text{with} \quad \mathcal{X}_t = (r_{0,t}, r_{1,t}, x_{0,t}, x_{1,t}, y_{1,t})'. $$

The structure of our model will take advantage of the simple decomposition of the conditional density,

$$f(r_{0,t}, x_{0,t}, r_{1,t}, x_{1,t}, y_{1,t}|F_{t-1}) = f(r_{0,t}, x_{0,t}|F_{t-1})f(r_{1,t}, x_{1,t}, y_{1,t}|r_{0,t}, x_{0,t}, F_{t-1}), \quad (1)$$

which serves to illustrate the hierarchical structure of our model. We will adopt the Realized EGARCH model as our specification of the first term, $f(r_{0,t}, x_{0,t}|F_{t-1})$. The individual asset variables $(r_{1t}, x_{1t}, y_{1t})$ will be modeled with novel structure that conditions on contemporary market variables. This defines a structure that is straightforward to extend to an arbitrarily large number of individual assets. The specification for the second conditional density, $f(r_{1,t}, x_{1,t}, y_{1,t}|r_{0,t}, x_{0,t}, F_{t-1})$, defines how the the time series associated with the individual asset evolves conditional on contemporary market variables. Our specification of this conditional density has a structure that is similar to that of the univariate Realized GARCH model, but has some important adaptations for the modeling of the correlation structure. This structure is very convenient because it avoids the need for introducing realized measures of the correlation measures between the individual assets, it is implicitly assumed that these correlation are characterized through the correlations between the individual returns and the market return.

In practice, the estimation proceeds by first estimating the model for the market data $(r_{0,t}, x_{0,t})$ and then estimating each conditional model for $(r_{i,t}, x_{i,t}, y_{i,t})$ separately for $i = 1, 2, \ldots, n$, where $n$ is the number of assets. In the empirical section, we report results for the assets that were the constituents in the S&P 500 over the sample period 01.01.2002-31.12.2009.
Due to turnover in the constituents this adds up to almost 600 assets we model.

### 2.2 Realized EGARCH Model for Market Returns

The Realized EGARCH model for market returns and realized measures of volatility is given by the following three equations

\[
    r_{0,t} = \mu_0 + \sqrt{h_{0,t}} z_{0,t}, \quad (2)
\]

\[
    \log h_{0,t} = a_0 + b_0 \log h_{0,t-1} + c_0 \log x_{0,t-1} + \tau_0 (z_{0,t-1}) \quad (3)
\]

\[
    \log x_{0,t} = \xi_0 + \varphi_0 \log h_{0,t} + \delta_0 (z_{0,t}) + u_{0,t}, \quad (4)
\]

where we model \( z_{0,t} \sim \text{iid} \mathcal{N}(0,1) \), \( u_{0,t} \sim \text{iid} \mathcal{N}(0,\sigma_{u_0}^2) \). As is the case in conventional GARCH models, \( h_{0,t} \), denotes a conditional variance, \( h_{0,t} = \text{var}(r_{0,t}|\mathcal{F}_{t-1}) \), the key difference being that the information set, \( \mathcal{F}_t \), is richer than in the conventional framework. The normality of \( u_{0,t} \) is motivated by findings in Andersen, Bollerslev, Diebold and Labys (2001), Andersen, Bollerslev, Diebold and Ebens (2001) and Andersen et al. (2003), who document that realized volatility is approximately log-normal. Furthermore, Andersen, Bollerslev, Diebold and Ebens (2001) find that returns standardized by realized volatility are approximately normally distributed.

The functions \( \tau(z) \) and \( \delta(z) \) are called leverage functions because they model aspects related to the leverage effect, which refers to the dependence between returns and volatility. Hansen et al. (2010) found that a simple second-order polynomial form provides a good empirical fit. We will adopt this structure in our framework, and set \( \tau(z) = \tau_1 z + \tau_2 (z^2 - 1) \) and \( \delta(z) = \delta_1 z + \delta_2 (z^2 - 1) \). This leads to a GARCH equation that is somewhat similar to that of an EGARCH model. An important difference is that we also utilize the realized measure \( x_{t-1} \) in this equation to model the dynamic variation in volatility.

We refer to the first two equations, (2) and (3), as the return equation and the GARCH equation, respectively. These two equations define a GARCH-X model, similar to those that were estimated by Engle (2002), Barndorff-Nielsen and Shephard (2007), and Visser (2010). See also Chen et al. (2009) for additional variants of the GARCH-X model and some related models.

The third equation, (4) called the measurement equation, completes the model. Tying the realized measure, \( x_t \), to the conditional variance, \( h_t \), is motivated by the fact that the GARCH equation trivially implies that

\[
    \log(r_t - \mu)^2 = \log h_t + \log z_t^2.
\]
Since the realized measure, \( x_t \), is similar to \( r^2_t \) in the sense of being a measurement of volatility (just far more accurate), it is natural to expect that \( \log x_t \approx \log h_t + f(z_t) + \text{error}_t \). Because we may compute realized measures of volatility over a shorter period of time than the one spanned by the return (e.g., if we use only data from the trading session, which often excludes the overnight period), some flexibility in the specification may be required motivating the “intercept” \( \xi_0 \) and the “slope” \( \varphi_0 \). So long as \( x_{0,t} \) is roughly proportional to \( h_{0,t} \), we should expect \( \varphi_0 \approx 1 \), and \( \xi_0 < 0 \), which is always the case empirically.

Note that we do not follow the conventional GARCH notation, because we want to reserve the notation “\( \beta \)” for
\[
\beta_{1,t} = \text{cov}(r_{1,t}, r_{0,t} | \mathcal{F}_{t-1}) / \text{var}(r_{0,t} | \mathcal{F}_{t-1}),
\]

(5)

We are particularly interested in the dynamic properties and the cross-sectional variation of \( \beta_{i,t} \), where \( i = 1, \ldots, N \) with \( N \) being the number of individual assets in our analysis.

### 2.3 Conditional Model for Individual Asset Returns, Volatility, and Co-volatility

To extend the framework to a joint model for the market returns/volatility and another asset’s return/volatility and their covolatility, we shall formulate a model for the time series associated with the individual asset, conditional on contemporaneous “market” variables, i.e., a specification for

\[
f(r_{1,t}, x_{1,t}, y_{1,t} | r_{0,t}, x_{0,t}, \mathcal{F}_{t-1}).
\]

We utilize a further decomposition of this conditional density, specifically

\[
f(r_{1,t}, x_{1,t}, y_{1,t} | r_{0,t}, x_{0,t}, \mathcal{F}_{t-1}) = f(r_{1,t} | r_{0,t}, x_{0,t}, \mathcal{F}_{t-1}) f(x_{1,t}, y_{1,t} | r_{1,t}, r_{0,t}, x_{0,t}, \mathcal{F}_{t-1}).
\]

The first part, \( f(r_{1,t} | r_{0,t}, x_{0,t}, \mathcal{F}_{t-1}) \), is modeled with three equations. The first two have the Realized EGARCH structure as above,

\[
r_{1,t} = \mu_1 + \sqrt{h_{1,t}} z_{1,t},
\]

(6)

\[
\log h_{1,t} = a_1 + b_1 \log h_{1,t-1} + c_1 \log x_{1,t-1} + d_1 \log h_{0,t} + \tau_1(z_{1,t-1}).
\]

(7)
The difference between the two GARCH equations, (3) for the market return and (7) for the asset return, is the presence of the term, \( d_1 \log h_{0,t} \). This term relates the market conditional variance to the conditional variance of an individual asset under consideration. Note that since \( h_{0,t} \) is \( F_{t-1} \)-measurable, \( h_{1,t} \) may still be considered as the conditional variance of \( r_{1,t} \). The parameter \( d_1 \) can be interpreted as a spillover effect that measures the extend to which the market’s volatility affects the volatility of the individual asset after having accounted for the asset-specific volatility dynamics.

To capture the dependence between market returns and individual returns, we introduce the conditional covariance

\[
\rho_{1,t} = \text{cov}(z_{0,t}, z_{1,t} | F_{t-1}),
\]

and it follows directly that \( \rho_{1,t} \) is the conditional correlation between \( r_{0,t} \) and \( r_{1,t} \), so that the beta for asset 1 is given by

\[
\beta_{1,t} = \rho_{1,t} \sqrt{h_{0,t} h_{1,t}} = \rho_{1,t} \sqrt{h_{0,t} / h_{1,t}}.
\]

To complete this part of the model we need to specify the dynamic properties of \( \rho_t \), and this is where we introduce the realized correlation measure, \( y_{1,t} \). For this purpose we make use of the Fisher transformation (also known as the inverse hyperbolic tangent, \( \text{arctanh} \)), \( \rho \mapsto F(\rho) \equiv \frac{1}{2} \log \frac{1+\rho}{1-\rho} \), which is a one-to-one mapping from \((-1, 1)\) into \( \mathbb{R} \). The GARCH equation for the Fisher transformation of the correlation variables is thus given by

\[
F(\rho_{i,t}) = a_{i0} + b_{i0} F(\rho_{i,t-1}) + c_{i0} F(y_{i,t-1}).
\]

Finally, the model is completed by specifying the following two measurement equations:

\[
\log x_{1,t} = \xi_1 + \varphi_1 \log h_{1,t} + \delta_1 (z_{1,t}) + u_{1,t}, \tag{8}
\]

and

\[
F(y_{i,t}) = \xi_{i0} + \varphi_{i0} F(\rho_{i,t}) + v_{i,t}. \tag{9}
\]

Conditional on contemporaneous market variables, the covariance structure for the error terms in the three measurement equations (4), (8) and (9), \((u_{0,t}, u_{1,t}, v_{1,t})\), is explicitly specified as
\[ \Sigma = \text{var} \begin{pmatrix} u_{0,t} \\ u_{1,t} \\ v_{1,t} \end{pmatrix} = \begin{bmatrix} \sigma_{u0}^2 & \sigma_{u0,u1} & \sigma_{u0,v1} \\ \cdots & \sigma_{u1}^2 & \sigma_{u1,v1} \\ \cdots & \cdots & \sigma_{v1}^2 \end{bmatrix}, \]

so we allow the error terms in the three measurement equations to be correlated, which is important in practice.

2.4 The Extensions to Multiple Individual Assets

We have specified the model structure for a market return and a single individual assets (along with their corresponding realized volatility variables). Next we discuss the extension to multiple individual assets. Fortunately, the existing structure is highly amendable to such an extension, however, some additional assumptions are needed before certain interpretations carry over to the general context. First we need to redefine the natural filtration, \( F_t = \sigma(\mathcal{X}_t, \mathcal{X}_{t-1}, \ldots) \), to be defined by the full set of variables,

\[ \mathcal{X}_t = (r_{0,t}, r_{1,t}, \ldots, r_{N,t}, x_{0,t}, x_{1,t}, \ldots, x_{N,t}, y_{1,t}, \ldots, y_{N,t})'. \]

This induces additional moment conditions, because the structure assumed about the individual assets is assumed to be invariant to the enhancement of the information set. A tacit assumption is that the dynamic properties of the conditional covariance structure (which is given by \( N + 1 \times N + 1 \) matrix) is fully described by the \( N + 1 \) conditional variances and the \( N \) conditional correlations.

2.4.1 Model diagnostics

Diagnosing the model and in particular the validity of the single-factor structure can be carried out by analyzing the correlation structure of the model residuals

\[ \hat{w}_{it} = \frac{\hat{z}_{it} - \hat{\rho}_{it} \hat{z}_{0t}}{\sqrt{1 - \hat{\rho}_{it}^2}}. \]

So far the model structure has been silent about the dependence structure across the population equivalents of these residuals,

\[ w_{it} = \frac{z_{it} - \rho_{it} z_{0t}}{\sqrt{1 - \rho_{it}^2}}, \]
and the same is true for the conditional variabes, \((u_{it}, v_{it}|u_{0t})\), that are associated with the errors in the measurement equations. We cast light on this dependence structure in our empirical section.

3 Estimation

In this section, we define the quasi log-likelihood function and exploit its structure to simplify the estimation problem. We have five observed variables, \((r_{0,t}, x_{0,t}, r_{1,t}, x_{1,t}, y_{1,t})\), and we consider their joint density conditional on past information, \(F_{t-1}\). Without loss of generality we can decompose this “joint” density as stated in (1), and for the estimation we need to be specific about the “marginal” density, \(f(r_{0,t}, x_{0,t}|F_{t-1})\), and the “conditional” densities, \(f(r_{i,t}, x_{i,t}, y_{i,t}|r_{0,t}, x_{0,t}, F_{t-1}), i = 1, \ldots, N\).

For this purpose we will adopt a Gaussian specification, where the studentized returns, \((z_{0,t}, z_{1,t})\), are taken to be independent of the error terms in the three measurement equations, \((u_{0,t}, u_{1,t}, v_{1,t})\). This enables us to decompose the quasi log likelihood function into four terms as we discuss below.

3.1 The Marginal Model for Market Variables

The marginal model is essentially that of Hansen et al. (2010), which implicitly entails a further decomposition of the conditional density,

\[
f(r_{0,t}, x_{0,t}|F_{t-1}) = f_{r_0}(r_{0,t}|F_{t-1}) f_{x_0}(x_{0,t}|r_{0,t}, F_{t-1}).
\]

The two densities are given from \(r_{0,t} \sim N(\mu_0, h_{0,t})\) and \(\log x_{0,t} \sim N(\xi_0 + \phi_0 \log h_{0,t} + \tau_0(z_{0,t}), \sigma^2_{u_0})\), which leads to the following two contribution to the log-likelihood function,

\[
\ell_{z_0} = \sum_{t=1}^{T} \log h_{0,t} + \frac{(r_{0,t} - \mu_0)^2}{h_{0,t}} = \sum_{t=1}^{T} \log h_{0,t} + z_{0,t}^2.
\]

\[
\ell_{u_0} = \sum_{t=1}^{T} \log \sigma^2_{u_0} + \frac{\left(\log x_{0,t} - \xi_0 - \phi_0 \log h_{0,t} - \tau_0(z_{0,t})\right)^2}{\sigma^2_{u_0}} = \sum_{t=1}^{T} \log \sigma^2_{u_0} + \frac{u_{0,t}^2}{\sigma^2_{u_0}}.
\]
3.2 The Conditional Model for Individual Assets

Next we consider the likelihood contributions from the conditional model. The conditional model also permits a further decomposition of the conditional density,

$$ f(r_{1,t}, x_{1,t}, y_{1,t} | r_{0,t}, x_{0,t}, F_{t-1}) = f_{r_{1}|r_{0},x_{0}}(r_{1,t}|r_{0,t}, x_{0,t}, F_{t-1}) \times f_{x_{1},y_{1}|r_{1},r_{0},x_{0}}(x_{1,t}, y_{1,t} | r_{1,t}, r_{0,t}, x_{0,t}, F_{t-1}). $$

The first term is the density of the individual asset return conditional on the contemporaneous market variables (and the past). Due to the Gaussian specification we only need to derive the conditional mean and variance of $r_{1,t}$ in order to compute the appropriate likelihood term. The assumed independence between $(z_{0,t}, z_{1,t})$ and $u_{0,t}$ and the iid assumptions imply that

$$ E[g(r_{1,t}) | r_{0,t}, x_{0,t}, F_{t-1}] = E[g(r_{1,t}) | z_{0,t}, u_{0,t}, F_{t-1}] = E[g(r_{1,t}) | r_{0,t}, F_{t-1}], $$

for any function $g$ for which the conditional mean is well defined. Hence,

$$ \text{var}(r_{1,t} | r_{0,t}, x_{0,t}, F_{t-1}) = \text{var}(r_{1,t} | r_{0,t}, F_{t-1}) = h_{1,t} - (\rho_{1,t}\sqrt{h_{0,t}h_{1,t}})^2/h_{0,t} = (1 - \rho_{1,t}^2)h_{1,t}, $$

since $\text{cov}(r_{1,t}, r_{0,t} | F_{t-1}) = \rho_{1,t}\sqrt{h_{0,t}h_{1,t}}$. Next the conditional mean of $r_{1,t}$ is

$$ E(r_{1,t} | r_{0,t}, x_{0,t}, F_{t-1}) = \mu_{1} + \beta_{1,t}(r_{0,t} - \mu_{0}) = \mu_{1} + \rho_{1,t}\sqrt{h_{0,t}h_{1,t}}(r_{0,t} - \mu_{0}) = \mu_{1} + \rho_{1,t}\sqrt{h_{1,t}}z_{0,t}, $$

So that the contribution to the log-likelihood function from this conditional density is,

$$ \ell_{z_{1}|z_{0}} = \sum_{t=1}^{T} \log[(1 - \rho_{1,t}^2)h_{1,t}] + \frac{(r_{1,t} - \mu_{1} - \rho_{1,t}\sqrt{h_{1,t}}z_{0,t})^2}{(1 - \rho_{1,t}^2)h_{1,t}}. $$

The last likelihood term, $\ell_{u_{1,v_{1}|u_{0}}}$, which relates to the two measurement equations is associated with the conditional density, $f_{x_{1},y_{1}|r_{1},r_{0},x_{0}}(x_{1,t}, y_{1,t} | r_{1,t}, r_{0,t}, x_{0,t}, F_{t-1})$. First we note that the conditional distribution of $(u_{1,t}, v_{1,t})$ given $(u_{0,t}, z_{0,t}, z_{1,t})$ is Gaussian with mean

$$ \begin{pmatrix} \sigma_{u_{1},u_{0}}/\sigma_{u_{0}}^2 \\ \sigma_{v_{1},u_{0}}/\sigma_{u_{0}}^2 \end{pmatrix} u_{0,t}, $$
and variance
\[
\Omega = \begin{bmatrix}
\sigma^2_u & \sigma_{u,v_1} \\
0 & \sigma^2_{v_1}
\end{bmatrix} - \begin{bmatrix}
\sigma_{u,u_0} \\
\sigma_{v_1,u_0}
\end{bmatrix} \frac{1}{\sigma^2_{u_0}} \begin{bmatrix}
\sigma_{u_0,u_1} & \sigma_{u_0,v_1}
\end{bmatrix}.
\]

So it does not depend on \((z_{0,t}, z_{1,t})\) due to the assumed independence. The implication is that
\[
f_{x_1,y_1|r_1,r_0,x_0}(x_{1,t}, y_{1,t}|r_{1,t}, r_{0,t}, x_{0,t}, F_{t-1}) = f_{x_1,y_1|r_1,r_0,x_0}(x_{1,t}, y_{1,t}|u_{0,t}, F_{t-1}),
\]
and that the last term in the log-likelihood is given by
\[
\ell_{u_1,v_1|u_0} = \sum_{t=1}^{T} \log \det \Omega + U_{1,t}^T \Omega^{-1} U_{1,t},
\]
where we have defined
\[
U_{1,t} = \begin{pmatrix}
u_{1,t} \\
v_{1,t}
\end{pmatrix} - \begin{pmatrix}
\sigma_{u_1,u_0}/\sigma^2_{u_0} \\
\sigma_{v_1,u_0}/\sigma^2_{u_0}
\end{pmatrix} u_{0,t}.
\]

### 3.3 Simplification in Estimation

To simplify the estimation we can concentrate the likelihood function with respect to the covariance matrix of \((u_{0,t}, u_{1,t}, v_{1,t})\). Let \(\hat{u}_{0,t}, \hat{u}_{1,t}\) and \(\hat{v}_{1,t}\) be the residuals of the three measurement equations. The Gaussian likelihood implies that the maximum likelihood estimators of the variance-covariance parameters are given by
\[
\hat{\sigma}^2_{u_0} = \frac{1}{T} \sum_{t=1}^{T} \hat{u}_{0,t}^2, \quad \hat{\sigma}_{u_1,u_0} = \frac{1}{T} \sum_{t=1}^{T} \hat{u}_{1,t} \hat{u}_{0,t}, \quad \hat{\sigma}_{v_1,u_0} = \frac{1}{T} \sum_{t=1}^{T} \hat{v}_{1,t} \hat{u}_{0,t},
\]
and
\[
\hat{\Omega} = \frac{1}{T} \sum_{t=1}^{T} \hat{U}_{1,t} \hat{U}_{1,t}' , \quad \text{where} \quad \hat{U}_{1,t} = \begin{pmatrix}
\hat{u}_{1,t} \\
\hat{v}_{1,t}
\end{pmatrix} - \begin{pmatrix}
\hat{\sigma}_{u_1,u_0}/\hat{\sigma}^2_{u_0} \\
\hat{\sigma}_{v_1,u_0}/\hat{\sigma}^2_{u_0}
\end{pmatrix} \hat{u}_{0,t}.
\]

The reduces the number of free parameters that the likelihood has to be maximized over, to \(\theta = (\theta_0', \theta_1')'\), where
\[
\theta_0 = (\mu_0, \omega_0, a_0, b_0, c_0, \tau_{01}, \tau_{02}, \xi_0, \varphi_0, \delta_{01}, \delta_{02}, h_{0,1})',
\]
is the vector of (remaining) parameters in the market model, and
\[
\theta_1 = (\mu_1, \omega_1, a_1, b_1, c_1, d_1, \tau_{11}, \tau_{12}, \xi_1, \varphi_1, \delta_{11}, \delta_{12}, a_{10}, b_{10}, c_{10}, \xi_{10}, \varphi_{10}, h_{1,1}, \rho_{1,1})',
\]
is the vector of (remaining) parameters in the conditional model. Here we follow the convention and threat the initial values for the latent variables, \( h_{0,1}, h_{1,1}, \) and \( \rho_{1,1} \), as were they unknown parameters.

These parameters are now estimated by maximizing

\[
\ell(\theta) = -\frac{1}{2} \left( \ell_{z_0}(\theta_0) + \ell_{u_0}(\theta_0) + \ell_{z_1|z_0}(\theta_1) + \ell_{u_1,u_1|u_0}(\theta_1) \right),
\]

where \( \ell_{z_0}(\theta_0) = \left( \sum_{t=1}^{T} \log h_{0,t}(\theta_0) + z_{0,t}^2(\theta_0) \right) \), \( \ell_{u_0}(\theta_0) = T(\log \hat{\sigma}_{u_0}^2(\theta_0) + 1) \), and

\[
\ell_{z_1|z_0}(\theta_1) = \left( \sum_{t=1}^{T} \log \left\{ 1 - \rho_{1,t}^2 \right\} h_{1,t}(\theta_1) \right) + \frac{(z_{1,t}(\theta) - \rho_{1,t}(\theta) z_{0,t}(\theta))^2}{1 - \rho_{1,t}^2(\theta)}.
\]

In practice this amounts to the following procedure:

1. Given initial values for \( \theta_0 \), the time series for \( z_{0,t} \) and \( h_{0,t} \) are computed iteratively. First,

\[
z_{0,1} = (r_{0,1} - \mu_0) / \sqrt{h_{0,1}},
\]

then for \( t = 2, \ldots, T \) we compute

\[
h_{0,t}(\theta_0) = \exp \{ a_0 + b_0 \log h_{0,t-1} + c_0 \log x_{0,t-1} + \tau_0(z_{0,t-1}) \},
\]

and \( z_{0,t}(\theta_0) = \frac{r_{0,t} - \mu_0}{\sqrt{h_{0,t}(\theta_0)}} \). This produces the first term of the log-likelihood function,

\[
\ell_{z_0}(\theta_0) = \sum_{t=1}^{T} \log h_{0,t}(\theta_0) + z_{0,t}^2(\theta_0).
\]

2. Next, we compute \( u_{0,t}(\theta_0) = \log x_{0,t} - \xi_0 - \varphi_0 \log h_{0,t} - \tau_0(z_{0,t}) \) for \( t = 1, \ldots, T \), which yields the second term of the log-likelihood function, \( \ell_{u_0}(\theta_0) = T \left[ \log \sigma_{u_0}^2(\theta_0) + 1 \right] \), where

\[
\sigma_{u_0}^2(\theta_0) = \frac{1}{T} \sum_{t=1}^{T} u_{0,t}^2(\theta_0).
\]

3. Now we turn to the conditional model. We compute \( z_{1,1}(\theta_1) = (r_{1,1} - \mu_1) / \sqrt{h_{1,1}} \) and then for \( t = 2, \ldots, T \), we proceed with

\[
h_{1,t}(\theta_1) = \exp \{ a_1 + b_1 \log h_{1,t-1} + c_1 \log x_{1,t-1} + d_1 \log h_{0,t} + \tau_1(z_{1,t-1}) \},
\]

and \( z_{1,t}(\theta_1) = \frac{r_{1,t} - \mu_1}{\sqrt{h_{1,t}}} \). The notation above suppress that \( h_{1,t} \), and hence \( z_{1,t} \), depend on the market parameters, \( \theta_0 \) (unless \( d_1 = 0 \)). This is implicit since \( h_{0,t} = h_{0,t}(\theta_0) \) depends on \( \theta_0 \), and a similar dependence on \( \theta_0 \) arises below through \( z_{0,t} \) and \( u_{0,t} \). To make this dependence explicit we shall add the argument, \( \theta_0 \), to the likelihood terms below, which is short for the market variables, \( \{ z_{0,t}(\theta_0), h_{0,t}(\theta_0), u_{0,t}(\theta_0) \} \).
Independently of $h_{1,t}$ and $z_{1,t}$, we can compute $\rho_{1,t}(\theta_1) = F^{-1}\{a_{10} + b_{10}F(\rho_{1,t-1}) + c_{10}F(y_{1,t-1})\}$ recursively, for $t = 2, \ldots, T$. The yields the third likelihood term, which is given by

$$
\ell_{z_1|z_0}(\theta_1; m_{0_0}) = \sum_{t=1}^{T} \log\{(1 - \rho_{1,t}(\theta_1))h_{1,t}(\theta_1)\} + \frac{(z_{1,t}(\theta_1) - \rho_{1,t}(\theta_1)z_{0,t})^2}{1 - \rho_{1,t}(\theta_1)}.
$$

4. The last step involves the two measurement equations in the conditional model, whose residuals are computed by

$$
u_{1,t}(\theta_1) = \log x_{1,t} - \xi_1 - \varphi_1 \log h_{1,t} - \delta_1(z_{1,t}),$$

$$v_{1,t}(\theta_1) = F(y_{1,t}) - \xi_{0,0} - \varphi_{1,0}F(\rho_{1,t}).$$

With these residuals in place, we compute the sample covariances, $\sigma_{u_{1,u_0}}(\theta_1) = T^{-1} \sum_{t=1}^{T} u_{1,t}(\theta_1)u_{0,t}$ and $\sigma_{v_{1,u_0}}(\theta_1) = T^{-1} \sum_{t=1}^{T} v_{1,t}(\theta_1)u_{0,t}$, (that also depend on $\theta_0$ through $u_{0,t} = u_{0,t}(\theta_0)$).

The leads to the last likelihood term, $\ell_{u_{1,v_1}|u_0}(\theta_1; m_{0_0}) = T(\log \det \Omega(\theta_1; m_{0_0}) + 2)$, where

$$
\Omega(\theta_1; m_{0_0}) = \frac{1}{T} \sum_{t=1}^{T} U_{1,t}U'_{1,t},
$$

with

$$
U_{1,t} = U_{1,t}(\theta_1; m_{0_0}) = \begin{pmatrix} u_{1,t}(\theta_1) \\ v_{1,t}(\theta_1) \end{pmatrix} - \begin{pmatrix} \sigma_{u_{1,u_0}}(\theta_1)/\sigma_{u_0}^2 \\ \sigma_{v_{1,u_0}}(\theta_1)/\sigma_{u_0}^2 \end{pmatrix} u_{0,t}.
$$

### 3.4 Estimation of Large Systems

When estimating a large system, it is advantages to use a two-step procedure that the hierarchi-
cal structure is well suited for. First we estimate the market model by maximizing

$$
-\frac{1}{2} \left\{ \sum_{t=1}^{T} \left[ \log h_{0,t}(\theta_0) + z_{0,t}^2(\theta_0) \right] + T \left[ \log \left( T^{-1} \sum_{t=1}^{T} u_{0,t}^2(\theta_0) \right) + 1 \right] \right\}.
$$

Then in a second step, where we take $\{h_{0,t}, z_{0,t}, u_{0,t}\}$ as given, which amount to dropping the argument $m_{0_0}$ in the expressions of the previous section (steps 3 and 4). So with the two-step procedure, we estimate $\theta_i$ by maximizing

$$
-\frac{1}{2} \sum_{t=1}^{T} \log\{(1 - \rho_{i,t}^2(\theta_i))h_{i,t}(\theta_i)\} + \frac{(z_{i,t}(\theta_i) - \rho_{i,t}(\theta_i)z_{0,t})^2}{1 - \rho_{i,t}^2(\theta_i)} + T(\log \det \Omega(\theta_i) + 2),
$$

for each of the individual assets, $i = 1, \ldots, N$.  

15
4 Empirical Analysis

4.1 Data Description

4.1.1 Sample Period and Universe of Assets

The model is estimated for a large cross section of assets. The assets were selected as the union of the ticker symbols of the S&P 500 constituents on January 19, 2006 and June 25, 2010. Some assets were dropped because the sample size was deemed to sort. So we retained all tickers that have more than 1000 trading days at some point during our sample period from January 3, 2002 to the end of 2009. Thus our sample period spans a total of 2,008 trading days, although the actual number of days with observations may be less for some individual stock, due to missing observations, some of which arose due to our cleaning of extreme outliers.

4.1.2 Data Sources and Merging of Data

Two sources were used to construct the data set, the TAQ dataset and the CRSP daily stock files, and these were accessed via the WRDS research service. The TAQ dataset is used to obtain high-frequency transaction data from which daily realized volatility measures are computed. The high-frequency transaction data were cleaned according to the trade data filtering algorithm described in Barndorff-Nielsen et al. (2009). The TAQ database uses ticker symbols as stock identifiers which can be problematic for a comprehensive analysis such as this one, because a single company may trade under different ticker symbols over different parts of the sample. As mentioned below, we find a substantial number of companies which have multiple tickers over our sample period. The CRSP dataset is one of the most popular data sources for empirical finance. An advantage of the CRSP data is the availability of the so-called CRSP Permanent Company Numbers (PERMNOs) which identify a given company throughout its existence. To match the two datasets we proceed as follows. First, we match the ticker symbols of the S&P 500 constituents in the CRSP dataset and obtain their PERMNOs. Second, we extract the (possibly multiple) ticker symbols that were associated with each PERMNO over the sample period. Last, we “fill in” the ticker symbol list with any tickers that were detected in the second step and were not in the initial ticker symbol list. The resulting ticker list is then used to extract high-frequency data from the TAQ. Finally, the daily data from the CRSP is matched to the high-frequency based realized measures using the PERMNOs as a company identifier resulting in a total of 743 assets. Our dataset is thus sorted by PERMNOs and contains for each company
the daily open and close price, split and dividend adjustment factors and the realized volatility measures (realized volatility of the stock and the market and their covolatility). We use the SPRD exchange traded fund as the market index. The realized measures are computed using the multivariate kernel methodology developed in Barndorff-Nielsen et al. (2011).

The availability of adjustment factors in the CRSP dataset allows us to compute split- and dividend-adjusted returns. Using the PERMNO as a company identifier ensures that we track a stock issue even when the ticker symbol changes. Over the sample period we consider, around 10% of the companies had 2 or more ticker symbols which highlights the importance of using the PERMNO instead of the ticker as a company identifier. An even more serious problem is that a particular ticker symbol can be used over time for more than one company. An interesting example is the ticker “T” (AT&T) which was associated with two PERMNOs and thus with two companies according to the CRSP classification. Within our sample, “T” is associated with the PERMNO 10,401 for the period January 3, 2002 to November 18, 2005 and with the PERMNO 66,093 for the period December 1, 2005 to December 31, 2009. Similar occurrences are not uncommon and imply that using tickers as company identifiers can result in mixing up of data for 2 or more companies over time. The final sample after we .... contains 594 stocks.

4.2 Empirical Results

We summarize the estimation results in Table 1 and Figure 1. The top row presents estimates for the marginal model for the market return. The following rows of the table reports cross-sectional statistics of the parameters estimates corresponding to equations (6)-(9). Note that no estimates of $\phi_0$ and $\phi_{i0}$ are presented in Table 1. The reason is that we found that no significant reduction in the log-likelihood appeared when imposing the restriction $\phi_0 = \phi_{i0} = 1$. Also, to preserve space the cross-sectional statistics of the individual estimates of $\Sigma$ are omitted. The parameter $c_0$ which captures the effect of the lagged realized measure on the conditional variance is large and significant while the GARCH parameter $b_0$ is much smaller in magnitude from what we see in standard GARCH models. This is a typical finding in the literature on GARCH-X models, the consequence of which is that these models react and adapt much faster to abrupt changes in the volatility compared to standard GARCH models improving empirical fit and forecasting (at least at short horizons) significantly. The negative $\tau_{01}$ and positive $\tau_{02}$ indicate the presence of a leverage effect, see Hansen et al. (2010) for the relation of these leverage

---

1 The initial values for the latent variables, $h_{0,t}$, $h_{i,t}$ and $\rho_{i,t}$, are treated as parameters, and their estimated values are reported as $h_{0,1}$, $h_{i,1}$ and $\rho_{i,1}$, respectively.
functions to the news impact curve. Examining the parameters of the measurement equation, we find that $\xi_0$ is negative as expected reflecting the fact that the realized measures computed over the open-to-close period capture only a fraction of the close-to-close return variance. There is also a clear indication of a leverage effect in the realized volatility measure. The conditional model for the individual stocks has an extra parameter $d_1$ that measures the spillover effect of market to individual stock volatility. Both the mean and the median of this coefficient is positive, indicating that generally market volatility has a positive contemporaneous effect on individual asset volatility (commonality in volatility). The distribution of selected model parameters are presented in the histogram plots in Figure 1.

Figure 1: Histograms of selected model parameters. Each plot presents the empirical distribution of the particular parameter in the cross-section of stocks.

Note that the rows with statistics of parameters of the volatility model for the individual stocks do not correspond to one particular stock. For example, the 1% quantile of $b_0$ and $c_0$ are not values for one particular stock.
Table 1: Parameter estimates of the Realized GARCH Beta model.

<table>
<thead>
<tr>
<th>Volatility parameters</th>
<th>Correlation Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_{0,1}$, $\mu_0$, $a_0$, $b_0$, $c_0$</td>
<td>$\tau_1$, $\tau_2$, $\xi_0$, $\delta_{01}$, $\delta_{02}$</td>
</tr>
<tr>
<td>$h_{i,1}$, $\mu_i$, $a_i$, $b_i$, $c_i$, $d_i$</td>
<td>$\tau_{i1}$, $\tau_{i2}$, $\xi_i$, $\delta_{i1}$, $\delta_{i2}$</td>
</tr>
<tr>
<td>Mean 8.157, 0.023, 0.212, 0.584, 0.326</td>
<td>0.055, -0.037, 0.011, -0.332, -0.031</td>
</tr>
<tr>
<td>Median 3.461, 0.019, 0.197, 0.588, 0.329</td>
<td>0.042, -0.038, 0.010, -0.339, -0.032</td>
</tr>
<tr>
<td>Min 0.025, -0.132, -0.023, 0.291, 0.152</td>
<td>-0.021, -0.086, -0.022, -0.895, -0.099</td>
</tr>
<tr>
<td>Min(1) 0.110, -0.123, -0.011, 0.380, 0.168</td>
<td>-0.011, -0.081, -0.017, -0.882, -0.099</td>
</tr>
<tr>
<td>Min(2) 0.138, -0.103, 0.004, 0.382, 0.173</td>
<td>-0.009, -0.079, -0.013, -0.822, -0.085</td>
</tr>
<tr>
<td>Min(3) 0.169, -0.083, 0.005, 0.399, 0.176</td>
<td>-0.009, -0.077, -0.013, -0.810, -0.083</td>
</tr>
<tr>
<td>Min(4) 0.218, -0.082, 0.015, 0.404, 0.190</td>
<td>-0.009, -0.074, -0.011, -0.759, -0.079</td>
</tr>
<tr>
<td>Min(5) 0.244, -0.074, 0.022, 0.418, 0.208</td>
<td>-0.007, -0.073, -0.010, -0.757, -0.073</td>
</tr>
<tr>
<td>1% 0.252, -0.070, 0.022, 0.422, 0.213</td>
<td>-0.007, -0.072, -0.009, -0.737, -0.069</td>
</tr>
<tr>
<td>5% 0.738, -0.044, 0.064, 0.461, 0.251</td>
<td>0.002, -0.059, -0.004, -0.558, -0.060</td>
</tr>
<tr>
<td>95% 27.54, 0.102, 0.419, 0.701, 0.396</td>
<td>0.144, -0.012, 0.028, -0.080, -0.000</td>
</tr>
<tr>
<td>99% 58.54, 0.148, 0.521, 0.752, 0.425</td>
<td>0.183, -0.000, 0.036, -0.005, 0.012</td>
</tr>
<tr>
<td>Max(-5) 60.66, 0.153, 0.549, 0.754, 0.427</td>
<td>0.184, 0.000, 0.038, -0.004, 0.012</td>
</tr>
<tr>
<td>Max(-4) 75.92, 0.156, 0.590, 0.755, 0.433</td>
<td>0.185, 0.002, 0.038, 0.008, 0.014</td>
</tr>
<tr>
<td>Max(-3) 78.16, 0.160, 0.592, 0.762, 0.433</td>
<td>0.192, 0.003, 0.042, 0.010, 0.016</td>
</tr>
<tr>
<td>Max(-2) 103.5, 0.169, 0.645, 0.763, 0.458</td>
<td>0.206, 0.004, 0.042, 0.011, 0.019</td>
</tr>
<tr>
<td>Max(-1) 154.6, 0.219, 0.679, 0.797, 0.464</td>
<td>0.212, 0.010, 0.045, 0.054, 0.025</td>
</tr>
<tr>
<td>Max 323.9, 0.230, 0.679, 0.805, 0.473</td>
<td>0.345, 0.014, 0.048, 0.086, 0.026</td>
</tr>
</tbody>
</table>

The table reports cross-sectional statistics of estimates of the parameters of the Realized GARCH Beta model described in Equations (2)-(9). Initializing variables, $h_{0,1}$, $h_{i,1}$ and $\rho_{i,1}$, are treated as parameters.
In Figure 2 we present the realized variance of CVX and SPY against the model-implied conditional variance.\(^3\) Clearly, the conditional variance tracks the realized series closely but exhibits smaller variation (note that the measurement equation implies that the conditional variance is equal to a deterministic function of the realized measure multiplied by a noise term). The apparent downward bias of the realized measure is due to the fact that it is computed over the trading period only and is related to the coefficients \(\xi_0\) and \(\xi_1\) being negative.

![Figure 2: Realized kernel (RK) variance and conditional variance of CVX (upper panel) and SPY (lower panel) over the period 2007 – 2009.](image)

We turn next to the model-implied betas, given by

\[
\hat{\beta}_t = \hat{\rho}_t \sqrt{\hat{h}_{1,t}/\hat{h}_{0,t}}, \tag{10}
\]

where \(\hat{\rho}_t = \rho_t(\hat{\theta})\) and \(\hat{h}_{i,t} = h_{i,t}(\hat{\theta}), i = 0, 1\), denote the estimated quantities. The time series can be contrasted to the realized betas

\[
\tilde{\beta}_t = y_{1,t} \sqrt{x_{1,t}/x_{0,t}}, \tag{11}
\]

that are computed exclusively from high-frequency data on day \(t\).

\(^3\)The plots for all stocks are available from the authors upon request.
The model-implied betas take into account the presence of measurement error in the realized quantities as well as the dynamic linkages between realized measures and conditional moments. To get an idea of the time variation of $\hat{\beta}_t$ in our model compared to its raw realized counterpart, we continue with our previous example, and present graphic results for the realized and the conditional beta and correlation of CVX over the last three years in our sample in Figure 3.

In the example above, correlation changes rapidly during the sample period, which is reflected in the systematic risk of CVX represented by its beta ranging from close to zero to over one. Variation of this magnitude would be close to impossible to obtain with standard approaches using rolling window OLS techniques based on daily returns. The question which arises is whether short-term variations in the systematic risk of a company of such magnitude are plausible and can be rationalized from an asset-pricing perspective. In light of the findings of Lewellen and Nagel (2006), large time variation in betas can help explain asset pricing “anomalies”. We plan to address this issue in future research.

In Figures 4 and 5 we present quantile time series plots of the cross sectional variation in the conditional correlation and beta for the initial crisis period June – December 2008 around events such as the collapse of Lehman Brothers.
Figure 4: Quantile time series plot of conditional realized GARCH correlations for the period 06.2008 – 12.2008.

An event that may have triggered a lot of variation in asset risk is the SEC announcement on July 15th that temporarily prohibited naked short selling in the securities of Fannie Mae and Freddie Mac. This date was followed by a period of decreasing and left-skewed correlations indicating that for a short period of time stocks became somewhat more susceptible to idiosyncratic shocks rather than market-wide shocks. To the contrary, the distribution of conditional betas became more right-skewed and increased on average. The downwards shift in correlations along with the upwards shift in upper tail of betas, suggest that individual asset volatility increased more than market volatility to an extent that outweighed the decreased correlation, this follows from the identity $\beta_t = \rho_{t,t} \sqrt{h_{i,t}/h_{0,t}}$. After this initial chaotic period correlations
started to increase and the variation in betas decreased. Eventually, correlations peaked around mid-November with a median value of over 70% well above the 55% value at the beginning of June.

### 4.3 Discussion of Time Variation in Beta

Conditional betas are at the heart of asset pricing models. The implications of our model for the ability of the CAPM (and extended factor models) to price the cross section of assets is an interesting research topic which we plan to address in a subsequent study. Following the influential work of Fama and French (1992, 1993), the overwhelming body of literature on the subject has addressed the issue in the context of pricing particular portfolio sorts (e.g., momentum, size, book-to-market, etc). The main source of data in these empirical studies is the CRSP and the analyzed frequency is monthly.

![Quantile time series plot of conditional realized GARCH betas for the period 06.2008 – 12.2008.](image)

It is important to understand that the high degree of variation in the beta cannot simply be attributed to variation in the realized quantities. In fact, the main source of this variation is driven by daily returns. The reason is simply that it is time variation in the dependence
structure in daily returns that causes the realized measures to be found to be useful predictors in the GARCH equations.

4.4 Residual Correlations and Test for Constant Correlations

The Realized Beta GARCH model implies that the correlation between the individual studentized returns, \( z_{it} \) and \( z_{jt} \), is time varying. Recall the decomposition

\[
z_{i,t} = \rho_{i,t}z_{0,t} + z_{i,t} - \rho_{i,t}z_{0,t} = \rho_{i,t}z_{0,t} + \sqrt{1 - \rho_{i,t}^2}w_{i,t},
\]

where \( w_{i,t} \) and \( z_{0,t} \) are uncorrelated, both have mean zero and unit variance, and in the likelihood analysis we modeled both at standard Gaussian random variables. It follows that

\[
corr(z_{i,t}, z_{j,t}) = \rho_{i,t}\rho_{j,t} + \sqrt{(1 - \rho_{i,t}^2)(1 - \rho_{j,t}^2)}E(w_{i,t}w_{j,t}),
\]

which is time varying unless \( E(w_{i,t}w_{j,t}) \) behaves in a rather unlikely way that offsets the variation in \( \rho_{i,t} \) and \( \rho_{j,t} \). We have not stated explicit assumptions about the correlation, \( E(w_{i,t}w_{j,t}) \), which induces additional dependence between \( z_{i,t} \) and \( z_{j,t} \), beyond that inherited from their correlations with the market return. This additional channel for dependence is ignored in our estimation (in order to make the estimation of large systems feasible). A non-zero correlation between \( w_{i,t} \) and \( w_{j,t} \) is evidence that the Realized Beta GARCH model does not fully characterize the complete system, so that the estimated model will need to be enhanced to capture such effects. It would also suggest that the estimation is inefficient to some extent, albeit this is to be expected with a relatively simple estimation procedure in a highly complex model.

In this section we study the magnitude of \( E(w_{i,t}w_{j,t}) \) and the potential evidence of time-variation in this correlations. Since our model implies time variation in the correlation between \( z_{i,t} \) and \( z_{j,t} \) we shall evaluate the empirical evidence of this.

First we consider test for constant correlation that is based on the general theory by Nyblom (1989). This is the underlying framework of several test for parameter constancy including that of Hansen (1992) (linear regression models) and that of Hansen and Johansen (1999) (cointegration VAR).

Consider a bivariate process \((x_t, y_t)\) of studentized variables, \( E(x_t) = E(y_t) = 0 \) and \( E(x_t^2) = \)
E(y^2) = 1; So that the correlation is given by
\[ \rho_t = E(x_t y_t). \]

We are to construct tests for constant correlation and zero correlation. The maintained hypothesis is that the partial sum
\[ W_T(u) = T^{-1/2} \sum_{s=1}^{\lfloor uT \rfloor} (x_s y_s - \rho_s), \quad u \in [0, 1], \]
satisfies a functional central limit theorem, so that \( W_T(u) \Rightarrow \sigma_W B(u) \) where \( B(u) \) is a standard Brownian motion, and \( \sigma_W^2 \) is the long-run variance of \( x_t y_t - \rho_t \).

Under the null hypothesis, \( H_0: \rho_t = \rho \) (constant correlation) it follows that
\[
\text{NB}_c = \frac{T^{-1} \sum_{t=1}^{T} (T^{-1/2} \sum_{s=1}^{t} (x_s y_s - \bar{\rho}))^2}{\hat{\sigma}_W^2} \Rightarrow \int_0^1 B_b(u)^2 du,
\]
where \( B_b(u) = B(u) - uB(1) \) is a standard Brownian bridge, \( \bar{\rho} = T^{-1} \sum_{t=1}^{T} x_t y_t \) and \( \hat{\sigma}_W^2 \) is some consistent estimator of \( \sigma_W^2 \). Under the null hypothesis \( H_0: \rho_t = 0 \) (zero correlation) we have
\[
\text{NB}_0 = \frac{T^{-1} \sum_{t=1}^{T} (T^{-1/2} \sum_{s=1}^{t} x_s y_s)^2}{\hat{\sigma}_W^2} \Rightarrow \int_0^1 B(u)^2 du,
\]
where \( \hat{\sigma}_W^2 \xrightarrow{p} \sigma_W^2 \). In the absence of serial dependence we can use the estimator \( \hat{\sigma}_W^2 = T^{-1} \sum_{t=1}^{T} (x_t y_t - \bar{\rho})^2 \), which is consistent for \( \sigma_W^2 \) under both null hypotheses. The 5\% critical values of these limit distributions are 0.462 and 1.656, respectively, see Nyblom (1989).

In our application we shall apply the test for constant correlation to \( z_{i,t} z_{j,t} \) and \( w_{i,t} w_{j,t} \), and we apply the test for zero correlation to \( w_{i,t} w_{j,t} \).

### 4.4.1 Empirical Results Concerning Residual Correlation Structure

With 594 stocks in our cross section there are 176,121 distinct correlation series to look at. To handle this we aggregate the correlation estimation and test results by industrial segmentation. We employ the sector definition given by the Global Industry Classification Standard (GICS) that is the industry taxonomy developed by MSCI and Standard & Poor’s. The GICS structure consists of 10 sectors, 24 industry groups, 68 industries and 154 sub-industries and it is used as a basis for S&P and MSCI financial market indexes. To make our analysis as clear as possible.
we aggregate to sector level.

To match our stocks to the teen GICS sector we pair TAQ with Standard & Poor’s CapitalIQ database that contains continuously updated GICS classifications for a large set of publicly listed companies assigned by S&P’s analysts. These GICS classifications reflect those used by many wealth and investment managers and financial institutions. To match CUSIP and Ticker identifiers from TAQ with the GICS identifiers of the TAQ stock identifiers are first matched by CUSIP, and double checked for a matched with company names from CapitalIQ. In cases without a match from this procedure, Ticker’s are used. If this procedure does not provide a match, CapitalIQ Equity Listings report is used to check for inactive listings and these are again matched according to exchange tickers. If none of the above procedures achieve a positive match, CapitalIQ’s business description is used to identify company name changes and a final match is reattempted. The above series of matching procedures match all considered TAQ identifiers with available GICS classifications. The 10 sector names and company counts for each sector are presented in Table 2.

Table 2: Sector Statistics.

<table>
<thead>
<tr>
<th>Sector</th>
<th>Company Counts</th>
<th>Min beta</th>
<th>Median</th>
<th>Max Beta</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy</td>
<td>46</td>
<td>0.298</td>
<td>1.008</td>
<td>2.374</td>
</tr>
<tr>
<td>Materials</td>
<td>37</td>
<td>0.383</td>
<td>0.869</td>
<td>2.335</td>
</tr>
<tr>
<td>Industrials</td>
<td>63</td>
<td>0.325</td>
<td>0.931</td>
<td>2.264</td>
</tr>
<tr>
<td>Consumer Discretionary</td>
<td>103</td>
<td>0.170</td>
<td>0.912</td>
<td>2.373</td>
</tr>
<tr>
<td>Consumer Staples</td>
<td>46</td>
<td>0.339</td>
<td>0.966</td>
<td>2.176</td>
</tr>
<tr>
<td>Healthcare</td>
<td>64</td>
<td>0.358</td>
<td>1.070</td>
<td>2.563</td>
</tr>
<tr>
<td>Financials</td>
<td>101</td>
<td>0.306</td>
<td>1.034</td>
<td>2.601</td>
</tr>
<tr>
<td>Information Technology</td>
<td>86</td>
<td>0.339</td>
<td>0.940</td>
<td>2.466</td>
</tr>
<tr>
<td>Telecommunication Services</td>
<td>9</td>
<td>0.728</td>
<td>1.078</td>
<td>1.608</td>
</tr>
<tr>
<td>Utilities</td>
<td>40</td>
<td>0.284</td>
<td>0.985</td>
<td>2.299</td>
</tr>
</tbody>
</table>

The table gives summary statistics of the sectorial aggregation.

Tables 3-5 gives the sector aggregated results of estimating the residual correlations and testing these for constant correlations.
Table 3: Unconditional Correlations (Grouped by GICS)

<table>
<thead>
<tr>
<th>Energy</th>
<th>Materials</th>
<th>Industrials</th>
<th>Consumer Discretionary</th>
<th>Consumer Staples</th>
<th>Healthcare</th>
<th>Financials</th>
<th>Information Technology</th>
<th>Telecommun. Services</th>
<th>Utilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy</td>
<td>0.599</td>
<td>0.324</td>
<td>0.275</td>
<td>0.202</td>
<td>0.160</td>
<td>0.180</td>
<td>0.229</td>
<td>0.217</td>
<td>0.197</td>
</tr>
<tr>
<td>Materials</td>
<td>0.418</td>
<td>0.377</td>
<td>0.309</td>
<td>0.243</td>
<td>0.244</td>
<td>0.342</td>
<td>0.299</td>
<td>0.273</td>
<td>0.285</td>
</tr>
<tr>
<td>Industrials</td>
<td>0.402</td>
<td>0.331</td>
<td>0.261</td>
<td>0.264</td>
<td>0.357</td>
<td>0.323</td>
<td>0.288</td>
<td>0.286</td>
<td></td>
</tr>
<tr>
<td>Consumer Discretionary</td>
<td>0.323</td>
<td>0.238</td>
<td>0.234</td>
<td>0.327</td>
<td>0.283</td>
<td>0.252</td>
<td>0.238</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Consumer Staples</td>
<td>0.266</td>
<td>0.211</td>
<td>0.263</td>
<td>0.207</td>
<td>0.216</td>
<td>0.235</td>
<td>0.235</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Healthcare</td>
<td>0.263</td>
<td>0.260</td>
<td>0.231</td>
<td>0.218</td>
<td>0.223</td>
<td>0.223</td>
<td>0.223</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Financials</td>
<td>0.429</td>
<td>0.302</td>
<td>0.290</td>
<td>0.300</td>
<td>0.360</td>
<td>0.268</td>
<td>0.230</td>
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<td></td>
</tr>
<tr>
<td>Information Technology</td>
<td>0.360</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Telecommun. Services</td>
<td></td>
<td>0.368</td>
<td>0.250</td>
<td>0.250</td>
<td>0.487</td>
<td>0.487</td>
<td>0.487</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Average sample correlations for residuals grouped by industry classification (GICS). Upper panel is for $z_{i,t}$ and $z_{j,t}$ and the numbers in the lower panel are based on $w_{i,t}$ and $w_{j,t}$. 
Table 4: Testing for Constant Correlations (Grouped by GICS)

<table>
<thead>
<tr>
<th></th>
<th>Energy</th>
<th>Materials</th>
<th>Industrials</th>
<th>Consumer Discretionary</th>
<th>Consumer Staples</th>
<th>Healthcare</th>
<th>Financials</th>
<th>Information Technology</th>
<th>Telecommun. Services</th>
<th>Utilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy</td>
<td>0.648</td>
<td>0.766</td>
<td>0.839</td>
<td>0.610</td>
<td>0.516</td>
<td>0.396</td>
<td>0.693</td>
<td>0.784</td>
<td>0.616</td>
<td>0.544</td>
</tr>
<tr>
<td>Materials</td>
<td>0.592</td>
<td>0.648</td>
<td>0.601</td>
<td>0.438</td>
<td>0.323</td>
<td>0.592</td>
<td>0.592</td>
<td>0.602</td>
<td>0.612</td>
<td>0.657</td>
</tr>
<tr>
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<td>0.573</td>
<td>0.398</td>
<td>0.599</td>
<td>0.602</td>
<td>0.612</td>
<td>0.570</td>
<td>0.511</td>
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</tr>
<tr>
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<td>0.634</td>
<td>0.447</td>
<td>0.372</td>
<td>0.632</td>
<td>0.556</td>
<td>0.560</td>
<td>0.640</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Consumer Staples</td>
<td>0.456</td>
<td>0.421</td>
<td>0.488</td>
<td>0.517</td>
<td>0.560</td>
<td>0.640</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Healthcare</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.359</td>
<td>0.359</td>
<td>0.387</td>
<td>0.458</td>
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</tr>
<tr>
<td>Financials</td>
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<td>0.547</td>
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<td>Information Technology</td>
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<td>0.722</td>
<td>0.533</td>
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</table>

<table>
<thead>
<tr>
<th></th>
<th>Energy</th>
<th>Materials</th>
<th>Industrials</th>
<th>Consumer Discretionary</th>
<th>Consumer Staples</th>
<th>Healthcare</th>
<th>Financials</th>
<th>Information Technology</th>
<th>Telecommun. Services</th>
<th>Utilities</th>
</tr>
</thead>
<tbody>
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<td>Energy</td>
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<td>0.434</td>
<td>0.198</td>
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<td>0.144</td>
<td>0.123</td>
<td>0.251</td>
<td>0.132</td>
<td>0.075</td>
<td>0.128</td>
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</tr>
<tr>
<td>Industrials</td>
<td>0.140</td>
<td>0.158</td>
<td>0.114</td>
<td>0.093</td>
<td>0.166</td>
<td>0.130</td>
<td>0.071</td>
<td>0.138</td>
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</tr>
<tr>
<td>Consumer Discretionary</td>
<td>0.185</td>
<td>0.105</td>
<td>0.080</td>
<td>0.133</td>
<td>0.125</td>
<td>0.124</td>
<td>0.143</td>
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<tr>
<td>Consumer Staples</td>
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<td>0.209</td>
<td>0.138</td>
<td>0.174</td>
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<td>0.417</td>
<td>0.114</td>
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<td>Utilities</td>
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<td></td>
<td></td>
<td>0.273</td>
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<td></td>
</tr>
</tbody>
</table>

Rejection frequencies for the $NB_c$ test for constant correlation using a 5% significance level. The upper table are the results for $\hat{z}_{i,t}\hat{z}_{j,t}$ and the lower table are those for $\hat{w}_{i,t}\hat{w}_{j,t}$. 
Table 5: Testing for Zero Correlations (Grouped by GICS)

<table>
<thead>
<tr>
<th></th>
<th>Energy</th>
<th>Materials</th>
<th>Industrials</th>
<th>Consumer Discretionary</th>
<th>Consumer Staples</th>
<th>Healthcare</th>
<th>Financials</th>
<th>Information Technology</th>
<th>Telecomm. Services</th>
<th>Utilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy</td>
<td>1.000</td>
<td>0.599</td>
<td>0.307</td>
<td>0.346</td>
<td>0.378</td>
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<td>0.489</td>
<td>0.284</td>
<td>0.331</td>
<td>0.868</td>
</tr>
<tr>
<td>Materials</td>
<td>0.941</td>
<td>0.749</td>
<td>0.459</td>
<td>0.218</td>
<td>0.186</td>
<td>0.347</td>
<td>0.184</td>
<td>0.183</td>
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<td></td>
</tr>
<tr>
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<td>0.752</td>
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<td>0.242</td>
<td>0.212</td>
<td>0.298</td>
<td>0.343</td>
<td>0.143</td>
<td>0.147</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Consumer Discretionary</td>
<td>0.694</td>
<td>0.297</td>
<td>0.191</td>
<td>0.427</td>
<td>0.265</td>
<td></td>
<td>0.145</td>
<td>0.176</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Consumer Staples</td>
<td>0.719</td>
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<td>0.432</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Healthcare</td>
<td>0.713</td>
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<tr>
<td>Financials</td>
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<td>0.446</td>
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</tr>
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<td>Information Technology</td>
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<td>0.247</td>
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</tr>
<tr>
<td>Telecomm. Services</td>
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<td></td>
<td></td>
<td>1.000</td>
<td></td>
<td></td>
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<tr>
<td>Utilities</td>
<td>0.369</td>
<td>0.207</td>
<td>0.272</td>
<td>0.189</td>
<td>0.325</td>
<td>0.247</td>
<td>0.157</td>
<td>0.176</td>
<td>0.228</td>
<td></td>
</tr>
</tbody>
</table>

Rejection frequencies for the NB$_0$ test for zero correlation applied to $w_{i,t}$ and $w_{j,t}$.

There is overwhelming evidence of residual correlation in across the $w_{i,t}$ variables, in particular within the industry sectors. [MORE DETAILS TBA]

5 Forecasting

In this section we discuss how multistep predictions of volatilities and correlations as well as return density forecasts can be obtained with our model. Denote $\tilde{h}_{0,t} \equiv \log h_{0,t}$, $\tilde{h}_{i,t} \equiv \log h_{i,t}$ and $\tilde{\rho}_{i,t} \equiv F(\rho_{i,t})$. Point forecasts turn out to be very easy to obtain owing to the fact that the vector $(\tilde{h}_{0,t}, \tilde{h}_{i,t}, \tilde{\rho}_{i,t})$ can be represented as a VARMA(1,1) system. Substituting each of the measurement equations (4), (8) and (9) into the equations for the corresponding conditional moments one obtains

\begin{align*}
\tilde{h}_{0,t+1} &= a_0 + c_0 \xi_0 + (b_0 + c_0 \varphi_0) \tilde{h}_{0,t} + c_0 \delta_0 (z_{0,t}) + \tau_0 (z_{0,t}) + c_0 u_{0,t} \\
\tilde{h}_{i,t+1} &= a_i + c_i \xi_i + (b_i + c_i \varphi_i) \tilde{h}_{i,t} + d_i \tilde{h}_{0,t+1} + c_i \delta_i (z_{i,t}) + \tau_i (z_{i,t}) + c_i u_{i,t} \\
\tilde{\rho}_{i,t+1} &= a_{\rho_0} + c_{\rho_0} \xi_{\rho_0} + (b_{\rho_0} + c_{\rho_0} \varphi_{\rho_0}) \tilde{\rho}_{i,t} + c_{\rho_0} v_{i,t}
\end{align*}

(12)
Let $V_t = (\tilde{h}_{0,t}, \tilde{h}_{i,t}, \tilde{\rho}_{i,t})'$, then by substituting the equation for $\tilde{h}_{0,t+1}$ into that for $\tilde{h}_{i,t+1}$, one can show that

$$V_{t+1} = C + AV_t + B\varepsilon_t,$$

where $\varepsilon_t = (\delta_0(z_{0,t}), \tau_0(z_{0,t}), \delta_i(z_{i,t}), \tau_i(z_{i,t}), u_{0,t}, u_{i,t}, v_{i,t})'$ and $A$, $B$, and $C$ are given in the appendix.

The innovation process, $\varepsilon_t$, is a martingale difference sequence but is slightly heterogeneous. Time-variation in the distribution of $\varepsilon_t$ arises from (and is fully described by) $\rho_{i,t} = \text{corr}(z_{0,t}, z_{i,t} | \mathcal{F}_{t-1})$.

If follows that $E(V_{t+k} | V_t) = A^k V_t + \sum_{j=0}^{k-1} A^j C$ which can be used to produce a $k$-step ahead forecast of $V_{t+k}$. Forecast of the conditional distribution of $V_{t+k} | \mathcal{F}_t$, which can be used to deduce unbiased forecasts of the non-transformed variables, e.g., $h_{0,t} = \exp(\tilde{h}_{0,t})$, can be obtained by simulation methods or the bootstrap. In the simulation approach, we first generate

$$\eta_t = \begin{pmatrix} z_{0,t} \\ \tilde{z}_{i,t} \\ u_{0,t} \\ u_{i,t} \\ v_{i,t} \end{pmatrix} \sim N_5 \left( 0, \begin{bmatrix} I_2 & 0 \\ 0 & \Sigma \end{bmatrix} \right), \quad t = 1, \ldots, n.$$

Given an initialization for $\rho_{i,0}$, one can produce the entire time series $\{\tilde{\rho}_{i,t}\}$ from $\{v_{i,t}\}$ using (12). Next one can define $z_{i,t} = \rho_{i,t} z_{0,t} + \sqrt{1 - \rho_{i,t}^2} w_{i,t}$, which has the proper correlation with $z_{0,t}$, and thus finally $\varepsilon_t$ can be computed.

Alternatively, a bootstrap approach can be preferable if the Gaussian assumption concerning the distribution of $\eta_t$ is questionable. From the estimated model we can obtain residuals, $(\hat{\eta}_1, \ldots, \hat{\eta}_n)$, from which we can draw resamples instead of sampling from the Gaussian distribution. Time series for $V_t$ can now be generated from the bootstrapped residuals $\{\hat{\eta}_t^*\}$ in the same manner as with simulated $\{\eta_t\}$.

To simulate the time series for larger systems is straightforward using the bootstrap of the residuals from the estimated structure. Simulations would require one to take an explicit stand about the correlation structure of $w_{i,t}$-variables and the correlation structure of the errors in the various measurement equations.
6 Conclusion

In this paper we propose a multivariate GARCH model that utilizes realized measures of volatility and correlation, and includes a modeling of their dynamic properties. The model builds on a self-contained system of equations that link realized measures to the appropriate conditional quantities of volatility and covolatility. This implies a dynamic model of the conditional betas, that are are popular measures of risk in finance. The approach is easy to apply to multiple assets and amenable to a number of extensions. We provide a detailed roadmap of the estimation procedure in the case of single-factor structure and an outlook into the multi-factor extension. The specification we employ allows for leverage effects and spillover effects between the assets’ and the market volatility. In this respect the model combines the flexibility of the GARCH modeling framework with the statistical precision in volatility measurement resulting from the use of high-frequency data.

The empirical study we undertake reveals some interesting features of the cross-sectional variation of the conditional betas, as well as their time-series variation. In particular, we find that the betas exhibit far more variation at a daily frequency – variation that is largely concealed in the rolling-window estimates of $\beta$ that one can obtain with regression methods using daily returns.

A Appendix

$$C = \begin{bmatrix} a_0 + c_0 \xi_0 \\ a_1 + c_i \xi_i + d_i(a_0 + c_0 \xi_0) \\ a_i \xi_i + c_{i0} \xi_i \xi_i \end{bmatrix}, \quad A = \begin{bmatrix} b_0 + c_0 \varphi_0 & 0 & 0 \\ d_i(b_0 + c_0) & b_i + c_i \varphi_i & 0 \\ 0 & 0 & b_{i0} + c_{i0} \varphi_{i0} \end{bmatrix},$$

$$B = \begin{bmatrix} c_0 & 1 & 0 & 0 & c_0 & 0 & 0 \\ d_i c_0 & d_i & c_i & 1 & d_i c_0 & c_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{i0} \end{bmatrix}$$
References


Hansen, P. R. and Huang, Z. (2012), ‘Exponential garch modeling with realized measures of volatility’, *working paper*.


