CRITICAL TYPES
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Abstract. Economic models employ assumptions about agents’ infinite hierarchies of belief. We might hope to achieve reasonable approximations by specifying only finitely many levels in the hierarchy. However, it is well known since Rubinstein (1989) that the behaviors of some fully specified hierarchies can be very different from the behavior of such finite approximations. Examples and earlier results in the literature suggest that these critical types are characterized by some strong assumptions on higher-order beliefs. We formalize this connection. We define a critical type to be any hierarchy at which the rationalizable correspondence exhibits a discontinuity. We show that critical types are precisely those types for which there is common belief in a certain class of event. All types from finite type spaces and almost all types in common prior type spaces are critical. On the other hand, we show that regular types, i.e. types which exhibit no discontinuities, are generic. In particular they form a residual set in the product topology. This second result strengthens a previous one due to Weinstein and Yildiz (2006) in two ways. First, while Weinstein and Yildiz (2006) considered a fixed game, our regular types have continuous behavior across all games. Second, our result applies to an arbitrary space of basic uncertainty and does not require the rich-fundamentals assumption employed by Weinstein and Yildiz (2006). Our proofs involve a novel characterization of the strategic topology first introduced by Dekel, Fudenberg, and Morris (2006a).

1. Introduction

Economic models are simplified approximations of complicated environments. We can have confidence in simplifying assumptions when we know that they deliver conclusions which also approximate the outcomes of richer models. In game-theoretic models, to completely describe the environment requires specifying agents’ infinite hierarchies of belief. In practice, researchers typically pay close attention only to beliefs of finite order and the model is closed by imposing assumptions about higher-order beliefs made mostly for convenience. More typically, the model is specified in reduced form using types in a Harsanyi type space, so that assumptions about hierarchies are made only implicitly. However, it is well-known since the example of Rubinstein (1989) that the details of higher-order beliefs can matter for predicted outcomes. Some critical types give rise to hierarchies whose behavior is far from the behavior of approximating hierarchies which coincide on finitely (but arbitrarily) many levels. Formally, a critical type is a point of discontinuity in the solution mapping relative to
the standard product topology on hierarchies of belief. We characterize critical types. Our characterization is based on new results concerning the structure of the universal type space. First, we observe that the universal type space admits a non-trivial partial order $\succeq$, where $u_i \succeq v_i$ iff in every game the set of rationalizable actions for hierarchy $u_i$ includes the set of rationalizable actions for hierarchy $v_i$. Second, we present new results on the topological structure of the universal type space. We show that the metric topology introduced by Dekel, Fudenberg, and Morris (2006a) is in fact the weakest topology such that the $\varepsilon$-rationalizable correspondences are both upper and lower-hemicontinuous.\footnote{Dekel, Fudenberg, and Morris (2006a) define the strategic topology by a metric and show that convergence of a sequence in this topology is equivalent to convergence of rationalizable behavior. Because convergent sequences do not uniquely define a topology, their analysis leaves open the question of whether there exists a weaker, non-metric topology which characterizes continuity of rationalizable behavior.} It follows, that a type $u_i$ is a critical type if and only if the product topology is strictly weaker than the strategic topology at the point $u_i$.

Our characterization of critical types establishes the connection between continuity of behavior and common-belief in events, as defined by Monderer and Samet (1989). Say that a subset $W_i$ of hierarchies for $i$ is an upper-contour set if it includes all hierarchies that are larger under $\succeq$ than the hierarchies in $W_i$. Our first main theorem shows that $u_i$ is a critical type if and only if there exists an upper contour set $W_i$ that is a closed proper subset of the universal type space for $i$ such that for some $p > 0$, $u_i$ exhibits common $p$-belief in the set $W_i$. We show that nearly all types ever employed in applied models satisfy this condition and are therefore critical types.\footnote{We know of no example of a type space used in applications which does not consist entirely of critical types.} In particular, all types from finite type spaces and almost all types in common prior type spaces are critical.

For applied analysis, our results have the following implications. When we use common-priors, finite type spaces or other simple models, we are implicitly imposing some common-belief assumptions. Often these type spaces are used for convenience and these implicit assumptions are unintended consequences, indeed they are typically too subtle to even see or state explicitly. But they bring with them behavioral consequences that may just as well be unintended. Indeed, for any critical type there are types which coincide with these up to arbitrarily high orders of the belief hierarchy but do not satisfy any common-belief assumption. These types are usually harder to describe and therefore less convenient, but they may do just as well as models for the informational assumptions that the analyst has explicitly in mind. Our results show that there always exist games in which the behavior of these types is very different than the behavior of the critical type. The conclusions we draw from simple type spaces must therefore be interpreted with caution.
Nevertheless, our second main result shows that critical types are rare. Indeed, their complement in the universal type space (which we call the set of regular types) is a residual set: generic in the product topology. This result strengthens the earlier result due to Weinstein and Yildiz (2006) in two ways. First, in Weinstein and Yildiz (2006) a specific game was held fixed so that the regular types they found were only guaranteed to exhibit continuous behavior in that given game. We find a generic set of regular types which by construction have continuous behavior across all games. Second the results in Weinstein and Yildiz (2006) require an assumption that the space of payoff-relevant events is sufficiently rich that all actions are dominant actions for some types. Our results require no assumption about the space of payoff-relevant events apart from the standard technical assumption that it is a compact Polish space. On the other hand, Weinstein and Yildiz (2006) obtain generic uniqueness of rationalizable actions, a property that would not hold without some assumption of this type.

1.1. Overview of Results. In this subsection we briefly preview some of our main results. The exposition here is informal and precise definitions are deferred to the main body of the paper.

Throughout, an underlying space $\Omega$ of basic uncertainty is held fixed. The elements of $\Omega$ are called states of nature. We consider the universal type space $U_i(\Delta \Omega)$ introduced in Ely and Peski (2006). The elements of $U_i(\Delta \Omega)$ represent players’ hierarchies of beliefs as originally modeled by Mertens and Zamir (1985), but suitably enriched in order to capture the determinants of interim rationalizable behavior. The space $U_i(\Delta \Omega)$ has a natural product topology which captures convergence of higher-order beliefs.

A two-player game $G$ of incomplete information is defined by, for each $i$, a set of actions $A_i$, and a payoff function $g_i : A_1 \times A_2 \times \Omega \rightarrow \mathbb{R}$ depending on the pair of actions chosen and the state. For any game $G$ and any type $u_i \in U_i(\Delta \Omega)$, we can identify the set of interim rationalizable actions for type $u_i$, denoted $R(u_i|G)$. Similarly, given $\varepsilon > 0$, the set of interim $\varepsilon$-rationalizable actions is denoted $R_i(u_i|G,\varepsilon)$.

We can partially-order the types in $U_i(\Delta \Omega)$ according to their rationalizable behaviors. Take two hierarchies $u_i$ and $u'_i$. We say that $u_i \succeq u'_i$ if $u_i$ has a weakly larger set of $\varepsilon$-rationalizable actions than $u'_i$ in all games, for all $\varepsilon \geq 0$, i.e. $R_i(u'_i|G,\varepsilon) \subseteq R_i(u_i|G,\varepsilon)$. A set

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3We showed in Ely and Peski (2006) that the space $U_i(\Delta \Omega)$ was the minimal type space for characterizing interim rationalizability. All of the results in this paper have counterparts for the alternative concept of interim correlated rationalizability introduced by Dekel, Fudenberg, and Morris (2007).

4To be precise, the solution concept of interim rationalizability is not defined directly on hierarchies, but rather on Harsanyi type spaces which are implicit models of hierarchies. But any two types with the same hierarchies in $U_i(\Delta \Omega)$ will have the same rationalizable actions in all games, regardless of the type space to which they belong. We may thus think of the rationalizable correspondence as depending only on hierarchies. See Ely and Peski (2006) and also Theorem 1 below.
$V \subset U_i(\Delta\Omega)$ is called an upper-contour set if $V$ includes all hierarchies that are larger than those in $V$ under the relation $\succeq$. Formally $V = \bigcup_{v_i \in V} \{u_i : u_i \succeq v_i\}$.

Monderer and Samet (1989) introduced the concept of common $p$-belief. A hierarchy $u_i$ exhibits common $p$-belief in a subset $V \subset U_i(\Delta\Omega)$ if $u_i \in V$, and $u_i$ assigns probability at least $p$ to the set of hierarchies for $-i$ who assign probability at least $p$ to $V$, etc. A formal definition is given in Section 4.2 below. We let $C^p_i(V)$ denote the set of $i$’s hierarchies for which it is common $p$-belief that $i$’s hierarchy is in $V$. Our characterization of critical types is based on the following two lemmas.

**Lemma 1.** Let $V \subset U_i(\Delta\Omega)$ (strict inclusion.) For any $p > 0$, the complement of $C^p_i(V)$ is dense in the product topology. If $V$ is closed, then the complement of $C^p_i(V)$ is open.

Thus, given any type $u_i$, there is a sequence of types which do not have common $p$-belief in $V$ but whose higher-order beliefs converge to those of $u_i$. In fact, our proof of Lemma 1 shows that for every $k$, there is a hierarchy in the complement of $C^p_i(V)$ which is identical to $u_i$ up to order $k$.

**Lemma 2.** Let $W \subset U_i(\Delta\Omega)$ (strict inclusion) be an upper-contour set which is closed in the product topology. For $\varepsilon > 0$ small enough and $p > 0$ there exists $V \subset U_i(\Delta\Omega)$, a game $G$, and an action $a_i$ such that $W \subset V$ and

1. If $u_i \in C^p(W)$ then $a_i$ is interim-rationalizable for $u_i$.
2. If $u_i \notin C^{p-2\varepsilon(1-p)}(V)$ then $a_i$ is not interim-$\varepsilon$-rationalizable for $u_i$.

This lemma says that we can always find a game with an action whose rationalizability hinges on whether there is common $p$-belief in certain events. There is a simple intuition behind the Lemma. We construct a coordination game with a pair of actions $(a_i, a_{-i})$ such that player $i$ plays $a$ only if her hierarchy of beliefs belongs to $V$ and she believes with probability at least $p$ that the opponent plays $a_{-i}$ On the other hand, player $-i$ plays $a_{-i}$ only if she believes with probability at least $p$ that player $i$ plays $a_i$.

Putting these lemmas together provides a sufficient condition for a type to be critical. Suppose that type $t_i$ has a hierarchy $u_i$ which exhibits common $p$-belief in some closed, upper-contour, proper subset of $U(\Delta\Omega)$. Then by the first lemma, for any set $V$, there is a sequence of hierarchies which do not have common $p$-belief in $V$ and whose higher-order beliefs converge to those of $u_i$. By the second lemma, there exists a game $G$ and an action $a_i$ such that $a_i$ is rationalizable for $t_i$ but not $\varepsilon$-rationalizable for any type along the sequence. Thus there is a discontinuity in rationalizable behavior at the type $t_i$, i.e. $t_i$ is a critical type.

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5Precisely, there is a failure of lower hemi-continuity.
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In fact we show the converse: critical types are exactly the types whose hierarchies satisfy common \( p \)-belief of this form for some \( p > 0 \). The necessity part is a consequence of the following lemma:

**Lemma 3.** For any game \( G \), any player \( i \), there is a closed, upper contour, proper subset \( W_i \subset U_i(\Delta \Omega) \) and \( p^* > 0 \), such that for any \( p \leq p^* \), for any type with hierarchy \( u_i^* \not\in C^p(W_i) \), there is an open neighborhood \( V \ni u_i^* \) such that if action \( a_i \) is interim rationalizable for any type with a hierarchy \( u_i^* \), then, it is \( 6p \)-interim rationalizable for any type with a hierarchy \( u_i \in V \).

The proof of Lemma 3 shows that there exists a set of actions \( A_i \) and an open, lower contour set of hierarchies \( U \subseteq U_i(\Delta \Omega) \), such that \( a_i \in A \) is a rationalizable action for any type with a hierarchy in \( U \). Suppose that \( p > 0 \) is small and find any type with hierarchy \( u_i^* \not\in C^p(U_i(\Delta \Omega) \setminus U) \). Such a type believes with a probability at least \( 1 - p \) that player \(-i\) believes with a probability at least \( 1 - p \) that hierarchy of beliefs of \( i \) belongs to set \( U \), or that \( i \) believes with a probability at least \( 1 - p \) that ... (at most, finitely many times) that player \( i \)'s hierarchy of beliefs belongs to \( U \). We use the fact that \( U \) is open to show that any action that is rationalizable for a type with hierarchy \( u_i^* \) is also \( 6p \)-rationalizable for types with hierarchies sufficiently close to \( u_i^* \). (Note that set \( A \) plays a similar role to the dominant action in Weinstein and Yildiz (2006). This is why we do not need their assumption.)

Suppose now that \( u_i \) does not satisfy any non-trivial common \( p \)-belief statement:

\[
\bigcup_{p > 0} \bigcup_{W \subseteq U_i(\Delta \Omega), \text{W is closed and upper contour}} C^p(W).
\]

It follows from Lemma 3 that \( u_i \) is not critical.

Finally, we show that regular types, i.e. those that are not critical, are generic. For any \( p > 0 \) and closed \( W \), the set of types which do not have common \( p \)-belief in \( W \) is open and dense. By the previous results, the set of regular types is the intersection of all such sets. We show there is a countable collection of closed upper-contour sets \( W^k \) such that the regular types consist of all hierarchies which do not have common \( p \) belief in \( W^k \) for all \( k \) and for all rational \( p \). Thus, the set of regular types is a countable intersection of open and dense sets, hence residual in the product topology.

1.2. Examples. In this subsection, we present some examples to illustrate critical and regular types and other aspects of the results.

1.2.1. Partial Order. An important ingredient in our characterization is a partial order on types. We say that \( u_i \succeq u_i' \) if \( u_i \) has a weakly larger set of \( \varepsilon \)-rationalizable actions than \( u_i' \) in
all games, for all $\varepsilon \geq 0$, i.e., $R_i(u'_i|G, \varepsilon) \subset R_i(u_i|G, \varepsilon)$. To see that this is indeed a non-trivial order, recall the example from Ely and Peski (2006), reproduced below.

\[
\begin{array}{c|ccc}
-1 & 0 & 1/4 & -1 \\
+1 & 1/4 & 0 & +1 \\
\end{array}
\]

$\omega = -1$ $\omega = +1$

**Figure 1. A type space**

The figure illustrates a type space over a space of basic uncertainty containing two elements, $\omega \in \{-1, +1\}$. There are two players, each with two types, also labeled $\{-1, +1\}$. The type space has a common prior and the tables show the probabilities of various type-profile/state combinations. We can compare this type to a simpler type space in which each player has exactly one type, labeled $*$ and the common-prior attaches equal probability to the two states.

Let us first compare, for player 1 say, type $*$ with any of the types from Figure 1, say $+1$. There is a close connection between their best-reply correspondences. Recall that a game is an action set and a payoff function mapping action profiles and states into utilities. For any game, take any action $a$ played by type $*$ of player 2, and consider the set of best-replies for type $*$ of player 1. This is exactly the set of best-replies for type $+1$, to the strategy of player 2 which plays $a$ irrespective of type. The same argument applies to $\varepsilon$-best-replies. Thus, any action which can be a best-reply for $*$ is also a best-reply for type $+1$. It follows that (the hierarchy represented by) $+1$ is weakly larger than (the hierarchy represented by) $*$. Indeed the ordering is strict. As demonstrated by the example in Ely and Peski (2006), there are games in which the set of rationalizable actions for $+1$ strictly includes the set of rationalizable actions for $*$. On the other hand, the types $+1$ and $-1$ have the same best-response correspondences and therefore their hierarchies are equivalent under the ordering.

This example illustrates a partial characterization of the ordering $\succeq$ that can be stated directly in terms of belief-hierarchies. First $u_i \sim u'_i$ if and only if $u_i = u'_i$. That is, two types have the same rationalizable actions in all games if and only if they have the same $\Delta$-hierarchies. This was the main result in Ely and Peski (2006) and the result is restated below in Theorem 1. Second, $u_i \succeq u'_i$ only if $u_i$ and $u'_i$ represent the same Mertens-Zamir hierarchies of belief. This follows from a result in Dekel, Fudenberg, and Morris (2006a), namely that for any two types with distinct Mertens-Zamir hierarchies there is a game in which they have mutually disjoint rationalizable sets.
1.2.2. Finite Types. We now turn to some examples illustrating our characterization of critical types. Critical types are pervasive. In most models of incomplete information used in applied analysis, simple type spaces are used for convenience. For example, finite type spaces are often adopted for analytical simplicity. All types in finite type spaces are critical types. We can derive this result from our characterization as follows. Let $T$ be a finite type space. First, consider the finite subset $W$ of the universal type space corresponding to the hierarchies represented by the types in $T_i$. The set $W$ is common-knowledge for all of the hierarchies in $W$. Next, for each of the hierarchies $u_i$ in $W$, consider the corresponding upper-contour set $\{v_i : v_i \succeq u_i\}$. This is a closed, proper subset of the universal type space. The fact that it is closed follows from the result that rationalizable behavior is upper hemi-continuous in the product topology (see Dekel, Fudenberg, and Morris (2006a).) Because it contains only hierarchies that represent the same Mertens-Zamir hierarchy, it is a proper subset. It follows that the set $V := \cup_{u_i \in W} \{v_i : v_i \succeq u_i\}$ is an upper-contour proper subset that is closed (as the union of finitely many closed sets) and includes $W$. Since $W$ is common-knowledge among all the hierarchies in $W$, it follows that $V$ is also common-knowledge among these hierarchies. We have found a closed, upper-contour proper subset that is common-knowledge, hence all of the types in $T_i$ are critical.

Rubinstein first illustrated the special nature of finite types in the example of the e-mail game. While the example is now familiar, we will revisit it here as it will be a useful starting point for discussing critical types in more complicated cases.

Two players play a game of incomplete information in which there are two payoff-relevant states $\omega \in \{-1, +1\}$. We first consider the simplest type space in which it is common-knowledge that each player has exactly one type and with probability 1 the state of the world is +1. A game is now described by a set of actions for each player and a payoff function giving the players’ utilities as a function of the action profile and the state. The Rubinstein game has two actions $A$ and $B$ for each player and the payoffs are given in the following tables.

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>3,3</td>
<td>0,0</td>
</tr>
<tr>
<td>$B$</td>
<td>0,0</td>
<td>2,2</td>
</tr>
</tbody>
</table>

$\omega = +1$

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$B$</th>
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</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0,3</td>
<td>0,0</td>
</tr>
<tr>
<td>$B$</td>
<td>1,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

$\omega = -1$

**Figure 2.** The E-mail game

When it is common-knowledge that the state is +1, the game is effectively one of complete information and both actions are rationalizable for both players. In particular, the action
profile \((A, A)\) is rationalizable and indeed constitutes a (Bayesian) Nash equilibrium. Next we consider a type space in which there is some non-trivial incomplete information. Each player \(i\) has a countable set of types \(T_i = \{t^k_i\}_{k=0}^\infty\). There is a common prior \(\rho \in \Delta((-1, +1) \times T_1 \times T_2)\) from which the state and the type profile is drawn. The prior \(\rho\) is defined by \(\rho(-1, 0, 0) = 1/2\) and for some \(0 < \beta < 1\),

\[
\rho(+1, t^k_1, t^l_2) = \begin{cases} 
\frac{(1-\beta)\beta^{k+l}}{2} & \text{if } k = l \text{ or } k = l + 1 \\
0 & \text{otherwise}
\end{cases}
\]

For each player, we can view the sequence of types \((t^k_i)\) as approximating the complete-information type from the first type space. Indeed, for every \(k\), the first \(k\) orders of belief for the type \(t^k_i\) are identical to the first \(k\) orders of belief of the complete-information type, namely mutual knowledge of order \(k\) (1 knows that 2 knows that \ldots, \(k\) times) that the state is \(-1\). Formally, this sequence of hierarchies converges in the product topology to the hierarchy representing complete information.

Rubinstein showed that \(B\) is the unique rationalizable action for every type of both players. In fact the action \(A\) is not even approximately rationalizable for any type. Formally, for \(\varepsilon > 0\), an action is an \(\varepsilon\)-best-reply to some strategy for the opponent, if it earns an expected payoff within \(\varepsilon\) of the payoff to a best-reply. An action is \(\varepsilon\)-rationalizable if it survives an iterative procedure of elimination of actions which are not \(\varepsilon\)-best-replies to the surviving strategies. Action \(A\) is not \(\varepsilon\)-rationalizable for any type of either player for any \(\varepsilon < 1/2\). Here is the discontinuity. The minimum \(\varepsilon\) such that \(A\) is \(\varepsilon\)-rationalizable for the approximating types does not converge to zero, which is the corresponding value for the complete-information type.

1.2.3. Common Prior Types. In the Rubinstein example, the “limit” type is a critical type because it exhibits common-knowledge in a single payoff-relevant state. On the other hand, the types in the email information structure do not have common-knowledge in any proper subset of states. In fact, there is no \(p > 0\) such that these types have common \(p\)-belief in any proper subset of states. They would therefore seem to be immune to the type of construction we applied to the complete-information type. Nevertheless, we can apply our characterization and show that all of the e-mail types are critical types as well. This is because our characterization of critical types is in terms of common belief in subsets of hierarchies, not states. Common belief in a proper subset of hierarchies is a strictly weaker requirement than common-belief in a subset of payoff-relevant states.

Indeed, we show below in Theorem 5 that in any type space with a common prior, almost all types have common belief in a proper subset of hierarchies, and indeed almost all types are critical types. (Formally, critical types have probability 1 under the common prior.)
We can illustrate this result using the email types from the Rubinstein example. To demonstrate that these are all critical types, we will show that for each of these types, rationalizable behavior is “discontinuous” as we approximate their hierarchies to higher and higher orders. Indeed, for this example we are able to construct a single game to exhibit the discontinuity for all of the email types. The game is presented below.

\[
\begin{array}{c|ccccc}
 & A_1 & B_1 & \emptyset & A_2 & B_2 \\
\hline
U & 2, 2 + \omega & 0, 0 & 0, 5/2 & 2, 2 - \omega & 0, 0 \\
D & 1, 0 & 1, 2 + \omega & 2, 5/2 & 1, 0 & 1, 2 - \omega \\
\end{array}
\]

**Figure 3.** A game showing that the Rubinstein types are critical

This game, together with the Rubinstein information structure, has a Bayesian Nash equilibrium in pure strategies as follows. All types of player 1 play $U$. All types of player 2 play $A_1$ with the exception of type 0 who plays $A_2$. This is clearly a best-reply for player 1, whose payoffs do not depend on the state. For player 2, the best reply to $U$ is either $A_1$ or $A_2$ depending on the probabilities attached to the two states. Type 0 who is certain of state $-1$ optimally chooses $A_2$, and all other types, who are certain of state 1 optimally choose $A_1$.

We will now show how to construct for each player $i$, each type $t^k_i$ and for every $n$, a hierarchy which coincides exactly with the hierarchy represented by $t^k_i$ up to order $n$. The actions $A_1$ and $A_2$ will not be rationalizable for any of the hierarchies we construct for player 2 and the action $U$ will not be rationalizable for any of the hierarchies for 1. To begin with we consider any hierarchy $u^0$ for player 2 whose first-order belief assigns equal probability to the two states. For any such hierarchy, the action $\emptyset$ is (interim) strongly dominant and hence the unique rationalizable action. Now consider any of the Rubinstein types $t^k_i$ and any $n$. Consider a hierarchy which coincides with that of $t^k_i$ up to order $n$ and for all orders $m > n$, exhibits mutual knowledge of order $m - n$ that player 1 has hierarchy $u^0$. It is straightforward but notationally demanding to show the existence of such a hierarchy, either implicitly through a type space or by explicit construction. The formal construction appears below in the proof of Lemma 1.\(^6\)

When player 2 knows that player 1 has hierarchy $u^0$, player 2 knows that player 1 will play $\emptyset$. The unique best-reply for player 2, regardless of the state, is to play $D$. Thus, $D$ is the unique rationalizable action for any such hierarchy for 2. When player 1 knows that player 2 knows that 1 has hierarchy $u^0$, player 1 knows that 2 will play $D$. Neither action $A_1$ nor $A_2$ is a best-reply to $D$, regardless of the state. Thus, neither $A_1$ nor $A_2$ can be rationalizable for any of the hierarchies we construct for player 2 and the action $U$ will not be rationalizable for any of the hierarchies for 1.

\(^6\)The difficulty in explicitly constructing the hierarchy is in ensuring that it satisfies the requirement of coherency which is a necessary condition for a hierarchy to be derivable from some type space.
rationalizable for such a hierarchy for 1. We can continue the argument to show that for any of the hierarchies we have constructed will $U$, $A_1$ or $A_2$ be rationalizable. In fact, for no $\varepsilon$ smaller than $1/2$ will any of these actions be even $\varepsilon$-rationalizable.

We were able to create a discontinuity in rationalizable behavior because for all of the Rubinstein types, it is common-knowledge that the first-order beliefs of both players belong to a proper subset, indeed a finite subset, of the set of all first-order beliefs. This is necessary to construct the approximating hierarchies that do not have common-knowledge of this set which in turn generates a discontinuity in rationalizable behavior. However, a similar construction can be applied to types which have only common $p$-belief for some $p < 1$ in a proper subset of first-order beliefs. Indeed, even types for which there is no common belief in any proper subset of first-order beliefs can be critical types. Our characterization shows that any type which has common $p$-belief for some $p > 0$ in some proper subset of hierarchies (that is closed and upper-contour) is a critical type.

1.2.4. **Regular Types.** While critical types are pervasive in applications, they are in a formal sense very rare: they form a residual subset of the universal type space relative to the natural product topology on higher-order beliefs. The regular types, those with no discontinuities in behavior, are the typical ones. Nevertheless, they are in a certain sense elusive: actually describing a regular type is a serious challenge in its own right. It is thus not surprising that they do not appear in applied analysis. Indeed, without the simplifying tools of either finite or common-prior type spaces to implicitly describe hierarchies, we are not well-equipped to describe them at all. In Appendix A we provide a non-constructive description of a regular hierarchy via a type space. Here we give an informal sketch.

The universal type space has a countable, dense subset $Q$. We consider the set of all finite truncations of the hierarchies in $Q$. Now we construct a single hierarchy by “stacking” these finite truncations. To see how this is done, take the first $k$-order belief truncation of a hierarchy $u$, and the first $l$-order belief truncation of a hierarchy $v$. Now construct a $k + l$-order finite hierarchy. We first copying the initial $k$-orders of belief from $u$. Then, the orders $k + m$ ranging between $k + 1$ and $k + l$ are taken to be $k$-order mutual certainty of the $m$-th order belief of $v$. We continue in this way, interweaving all of the finite truncations of all of the hierarchies in $Q$. The resulting type $u^*$ is regular. This follows from two observations. First, any set $W$ which is common belief for $u^*$ must include $Q$. This is because every hierarchy in $Q$ is believed with probability 1 at some level of the hierarchy. Next, since $Q$ is dense, there is no closed, proper subset which includes $Q$, thus by our characterization $u^*$ is regular.
If $X$ is a measurable space, then $\Delta X$ refers to the space of Borel probability measures on $X$. When $X$ is a topological space, we treat $X$ as a measurable space equipped with the Borel $\sigma$-algebra. If $f : X \to Y$ is a mapping between two measurable spaces, then we write $\Delta f : \Delta X \to \Delta Y$ for the induced mapping between the corresponding spaces of measures.

We consider two-player games with incomplete information. We fix throughout a space of basic uncertainty $\Omega$. In a game with incomplete information, payoffs depend on action choices as well as the realization of $\Omega$. We assume that $\Omega$ is a compact Polish space.

The players’ uncertainty is modeled by a Harsanyi type space over $\Omega$. A type space over $\Omega$, denoted $T = (T_i, \mu_i)_{i=1,2}$ consists of a pair of measurable spaces $T_i$ and two belief mappings $\mu_i : T_i \to \Delta (\Omega \times T_{-i})$. The probability measure $\mu_i(t_i) \in \Delta(\Omega \times T_{-i})$ indicates the belief of type $t_i$ about the basic uncertainty and the type of the opponent. Throughout, we use the notation $C_\Omega \mu_i(t_i)(\cdot) : T_{-i} \to \Delta(\Omega)$ to represent a version of conditional probability over $\Omega$ as a function of the opponent’s type. We assume that there exist jointly measurable functions $\beta_i : T_i \times T_{-i} \to \Delta \Omega$, such that

$$\beta_i(t_i, t_{-i}) = C_\Omega \mu_i(t_i) (t_{-i})^7.$$

Let $T(\Omega)$ be the collection of all type spaces over $\Omega$.

A game form (or simply game) over $\Omega$ is a tuple $G = (A_i, g_i)_{i=1,2}$, where for each $i$, $A_i$ is a finite set of actions and $g_i : A_i \times A_{-i} \times \Omega \to \mathbb{R}$ is a payoff function.

2.1. **Interim Rationalizability.** We base our analysis on the concept of interim rationalizability. An early definition was given in Morris and Skiadis (2000) for games with finitely many types. Our definition of interim rationalizability generalizes to games with infinitely many types. An alternative concept, interim correlated rationalizability has been proposed by Dekel, Fudenberg, and Morris (2007). We discuss in Section 6 how to extend the results in this paper to interim correlated rationalizability.

Fix a type space $T \in T(\Omega)$, and a game $G = (A_i, g_i)$. An assessment is a pair of subsets $\alpha = (\alpha_1, \alpha_2)$ where $\alpha_i \subset T_i \times A_i$. Alternatively an assessment can be defined by the pair of correspondences $\alpha_i : T_i \rightrightarrows A_i$, with $\alpha_i(t_i) := \{a_i : (t_i, a_i) \in \alpha_i\}$. The image $\alpha_i(t_i)$ is interpreted as the set of actions that player $i$ of type $t_i$ could conceivably play.

A behavioral strategy for player $i$ is a measurable function $\sigma_i : T_i \to \Delta A_i$. The expected payoff to type $t_i$ of player $i$ from choosing action $a_i$ when the opponent’s strategy is $\sigma_{-i}$ is

---

7Thus, we consider type spaces with strongly measurable beliefs as in Ely and Peski (2006).

8See Ely and Peski (2006) for additional details.
given by

\[
\pi_i(a_i, \sigma_{-i}|t_i) = \int_{T_i \times \Omega} \int_{A_{-i}} g_i(a_i, a_{-i}, \omega) d\sigma_{-i}(t_{-i}) d\mu_i^T(t_i). \tag{2.1}
\]

The strategy \( \sigma_i \) is a selection from the assessment \( \alpha \) if for each \( i \), \( \sigma_i(t_i) \in \Delta \alpha_i(t_i) \) for all \( t_i \in T_i \). Let \( \Sigma_i(\alpha_i) \) be the set of all strategies for \( i \) that are selections from \( \alpha \).

For any \( \varepsilon \geq 0 \), an action \( a_i \) is an interim \( \varepsilon \)-best-response for \( t_i \) against \( \sigma_{-i} \) if \( \pi_i(a_i, \sigma_{-i}|t_i) \geq \pi_i(a'_i, \sigma_{-i}|t_i) - \varepsilon \) for all \( a'_i \in A_{-i} \). Let \( B_i(\sigma_{-i}|t_i, \varepsilon) \) denote the set of all interim \( \varepsilon \)-best-responses for \( t_i \) to \( \sigma_{-i} \). If \( \alpha \) is an assessment, then \( B_i(\alpha_{-i}|t_i; \varepsilon) \) is the set of all \( \varepsilon \)-best-responses to strategies in \( \Sigma_{-i}(\alpha_{-i}) \).

An assessment \( \alpha \) has the \( \varepsilon \)-best-response property if every action attributed to player \( i \) is an interim \( \varepsilon \)-best-reply to some selection from \( \alpha_{-i} \), i.e.,

\[
\alpha_i \subset \{ (t_i, a_i) : a_i \in B_i(\alpha_{-i}|t_i; \varepsilon) \}
\]

If the above is satisfied with equality, then we say that \( \alpha \) has the \( \varepsilon \) fixed-point property.

**Definition 1.** Given a type space \( T \), and a game \( G \), the interim \( \varepsilon \)-rationalizable correspondence is the maximal assessment with the \( \varepsilon \)-fixed-point property, denoted \( R(\cdot|G, \varepsilon) \). We say that \( a_i \) is interim \( \varepsilon \)-rationalizable for type \( t_i \) if \( a_i \in R_i(t_i|G, \varepsilon) \).

Ely and Peski (2006) show that the interim 0-rationalizable correspondence exists and can be understood as a measurable function from \( T_i \) to \( 2^{A_i} \). The same is true of the \( \varepsilon \)-rationalizable correspondence. The next Lemma makes a convenient connection between \( \varepsilon \)-rationalizability and \( \varepsilon' \)-rationalizability:

**Lemma 4.** For each game \( G = (A_j, g_i) \) and each \( \varepsilon \geq 0 \), there is a game \( G' = (A'_j, g'_j) \), where \( A'_j = A_i \times A_{-i} \), such that for any \( t_i \in T_i \), any \( \varepsilon' \geq 0 \)

\[
a_i \in R_i(t_i|G, \varepsilon + \varepsilon') \quad \text{if and only if} \quad (a_i, a_{-i}) \in R_i(t_i|G', \varepsilon').
\]

**Proof.** Define payoffs

\[
g'_i \left((a_i^i, a_{-i}^i) : (a_{-i}^i, a_{-i}^i) \right) := g'_i \left(a_i^i, a_{-i}^i \right) + \varepsilon \cdot 1 \{ a_i^i = a_{-i}^i \}.
\]

Then, for any player \( i \), her payoffs do not depend on the choice of action \( a_{-i}^i \). They depend on the opponent’s choice of action \( a_{-i}^i \) - and in particular, player \( i \) may get \( \varepsilon \) payoff additionally if the opponent’s choice is equal to \( i \)'s choice \( a_i^i = a_{-i}^i \).

We will also consider the strict \( \varepsilon \)-rationalizable correspondence, denoted

\[
R_0(\cdot|G, \varepsilon) = \bigcup_{\varepsilon' < \varepsilon} R(\cdot|G, \varepsilon) \tag{2.2}
\]
Thus, an action is strictly $\varepsilon$-rationalizable if it is $\varepsilon'$-rationalizable for some $\varepsilon' < \varepsilon$. In particular, no action is strict 0-rationalizable. \(^9\) We conclude this subsection with the following lemma whose proof is in Appendix C.

**Lemma 5.** For any $\varepsilon \geq 0$, any game $G$,

$$R_i (\cdot|G, \varepsilon) = \bigcap_{\varepsilon' > \varepsilon} R_i (\cdot|G, \varepsilon').$$

### 2.2. The Universal Type Space.

A type space is an implicit description of a player’s higher-order beliefs. Our characterization of critical types will be in terms of their hierarchies of beliefs, explicitly described. This ensures that our classification is not dependent on any particular choice of type space.

Let $X$ be a Polish space. Let $U^0_i (X) = \{ * \}$ and define by induction on $k \geq 1$

$$U^k_i (X) := \Delta (X \times U^{k-1}_{-i} (X)).$$

$U^k_i$ is the space of all $k$-th order beliefs for $i$. There are natural projections $proj_{k+1}^{k+1} : U^{k+1} \rightarrow U^k$ for $k \geq 0$ defined inductively by

$$proj_{k+1}^{k+1} = \Delta [id_X \times proj_k^{k}]$$

A hierarchy of beliefs for $i$ over the space $X$ is a sequence $(u_i^k)_{k=1}^{\infty}$ such that $u_i^k \in U^k_i$ and for all $k$

$$u_i^k = proj_{k+1}^{k+1} (u_i^{k+1}).$$

The latter condition is the requirement that the sequence be coherent. The set of all such hierarchies of belief is denoted $U_i (X)$.

Mertens and Zamir (1985) consider the space $U (\Omega) = (U_i (\Omega), \mu_i)_i$, where the belief mapping $\mu_i : U_i (\Omega) \rightarrow \Delta (\Omega \times U_{-i} (\Omega))$ makes $U (\Omega)$ a type space over $\Omega$. Ely and Peski (2006) consider the universal type space over $X = \Delta \Omega$, called the space of $\Delta$-hierarchies. The higher-order beliefs of a given type implicitly described within a given type space associates this type with a unique element of $U_i (\Delta \Omega)$. Each hierarchy $u_i \in U_i (\Delta \Omega)$ is uniquely associated with a belief which is a probability measure in $\Delta (\Delta \Omega \times U_{-i} (\Delta \Omega))$. We use the symbol $u_i$ interchangeably to refer to the hierarchy or the belief. It turns out that rationalizable behavior of any type is determined by this type’s $\Delta$-hierarchy:

**Theorem 1** (Ely and Peski (2006)). For any type space $T = (T_i, \mu_i)_i$, there exist measurable mappings $u_i^T : T_i \rightarrow U_i (\Delta \Omega)$, such that for any player $i$, any types $t_i, t'_i \in T_i$, any game $G$ over $\Omega$, any $\varepsilon > 0$

$$R_i (t_i|G, \varepsilon) = R_i (t'_i|G, \varepsilon) \iff u_i^T (t_i) = u_i^T (t'_i).$$

\(^9\)Note that this is distinct from iterated elimination of actions which are not $\varepsilon$-strict best responses.
The mappings \( u^T_i \) preserve beliefs. In particular, \( u^T_i(t_i) = u_i \) implies that for any measurable \( F \subset U_{-i}(\Delta \Omega) \),
\[
u_i(F) = \mu_i(t_i)(\phi^{-1}(F)).
\]
Also, there exists type space over \( \Omega \), denoted \( L = (L_i, \mu^L_i) \), such that \( \mu^L_i \) is continuous in the weak*-topology on \( \Delta(\Omega \times L_{-i}) \), \( u^L_i \) is continuous and onto, the inverse correspondence \( (u^L_i)^{-1} : U_i(\Delta \Omega) \Rightarrow L_i \) is continuous and the correspondence \( R(\cdot | G, \varepsilon) : L_i \Rightarrow A_i \) is upper hemi-continuous.

Proof. Ely and Peski (2006) show the Theorem when \( \varepsilon = 0 \). For \( \varepsilon > 0 \), apply Lemma 4. \( \square \)

Theorem 1 allows us to consider \( R_i \) as a correspondence defined directly on \( U_i(\Delta \Omega) \), i.e. \( R_i(u_i|G, \varepsilon) \) is the set of \( \varepsilon \)-rationalizable actions for any type \( t_i \) whose \( \Delta \)-hierarchy is \( u_i \), independent of the type space to which \( t_i \) belongs. It follows from the last part of the theorem that this correspondence is upper hemi-continuous.

### 2.3. Structure of the Universal Type Space.

The space \( U_i(\Delta \Omega) \) has some natural topological and order structure.

**Definition 2.** The product topology on \( U_i(\Delta \Omega) \) is the Tichonoff topology inherited from the infinite product \( \prod_{k=1}^{\infty} U^k_i(\Delta \Omega) \).

By standard results, this topology is separable and metrizable.

We can partially order the hierarchies in \( U_i(\Delta \Omega) \) according to the partial ordering of their rationalizable actions across games. Two hierarchies are ordered if the first has a smaller set of rationalizable actions in every game.

**Definition 3.** For any \( u_i, v_i \in U_i(\Delta \Omega) \), we write \( v_i \preceq u_i \) iff for any game \( G \) and any \( \varepsilon \geq 0 \), \( R(v_i|G, \varepsilon) \subseteq R(u_i'|G, \varepsilon) \).

One can show that this relation is closed in the sense that for any convergent sequences \( u^n_i, v^n_i, \) s.t. \( v^n_i \preceq u^n_i \) for all \( n \), \( \lim v^n_i \leq \lim u^n_i \).\(^{10}\)

**Definition 4.** Set \( A \subseteq U_i(\Delta \Omega) \) is a lower contour set if
\[
A = \{ v_i : there \ is \ u_i \in A, \ s.t. \ v_i \preceq u_i \}.
\]

Set \( A \subseteq U_i(\Delta \Omega) \) is an upper contour set if
\[
A = \{ v_i : there \ is \ u_i \in A, \ s.t. \ u_i \preceq v_i \}.
\]

Hence, set \( A \) is lower contour if it contains all hierarchies that are smaller with respect to the relation ”\( \preceq \)” from all hierarchies in set \( A \). Observe that if \( A \) is a lower contour set, then \( U_i(\Delta \Omega) \setminus A \) is an upper contour set.

\(^{10}\)This follows from the fact that rationalizable correspondence is u.h.c.
3. Strategic topologies on hierarchies of beliefs

We are interested in a topology on types which is derived from strategic behavior. Dekel, Fudenberg, and Morris (2006a) introduced one such topology connected to correlated interim rationalizability. Here, for both solution concepts, we will investigate several natural alternative definitions and show that they are all equivalent.

3.1. Strategic topology. Perhaps the most natural approach is to derive the topology from the rationalizable correspondences. It is known from more familiar contexts, such as complete information games, that the \( \varepsilon \)-rationalizable correspondence is upper hemi-continuous and the strict \( \varepsilon \)-rationalizable correspondence is lower hemi-continuous. Recall that if \( Y \) and \( Z \) are two topological spaces, then a correspondence \( F : Y \to Z \) is upper hemi-continuous if for every open \( U \subset Z \), the strong inverse image \( \{ y \in Y : F(y) \subset U \} \) is open, and lower hemi-continuous if the weak inverse image \( \{ y \in Y : F(y) \cap U \neq \emptyset \} \) is open. This leads to our first definition.

**Definition 5** (Strategic Topology). The strategic topology on \( U_i(X) \) is the coarsest topology such that for every \( \varepsilon \geq 0 \), the \( \varepsilon \)-rationalizable correspondence is upper hemi-continuous and the strict \( \varepsilon \)-rationalizable correspondence is lower hemi-continuous.

A sub-basis\(^{11}\) for the strategic topology thus consists of sets of the form

\[
\{ u_i : R(u_i|G,\varepsilon) \subset B \}
\]
\[
\{ u_i : R^c(u_i|G,\varepsilon) \cap B \neq \emptyset \}
\]

for all \( \varepsilon > 0 \) games \( G \) and subsets \( B \) of actions in \( G \). Note that upper hemi-continuity obtains if and only if the first class of subsets is included and lower hemi-continuity if and only if the second class of subsets is included. Since rationalizable correspondence is upper hemi-continuous in product topology, the sets of the first class are open in product topology. Sets from the second class are not necessarily open.

3.2. \( h \)-topology. Another familiar property of these correspondences leads to an alternative definition. For any game \( G \) and action \( a_i \), define

\[
h(a_i, G, u_i) = \inf \{ \varepsilon : a_i \in R(u_i|G,\varepsilon) \}
\]

In games with complete information, where there is a single state and hence \( u_i \) is fixed, the function \( h \) is a continuous function of the payoffs. It is therefore natural to seek a topology on types which reproduces this continuity.

\(^{11}\)Recall that a collection of sets \( B \) is a basis for a topology if every open set is a union of sets in \( B \). A collection \( B \) is a sub-basis if \( B \) together with all of its finite intersections forms a basis. The resulting topology is said to be generated by \( B \).
Definition 6 \((h\text{-topology})\). The \(h\text{-topology} \) is the coarsest topology on \(U_i(X)\) such that for each fixed \(G\) and \(a_i\), the function \(h(a_i,G,\cdot)\) is continuous.

A sub-basis for the \(h\)-topology is thus the following collection of open sets for \(\varepsilon > 0\)
\[
\{u_i : h(a_i,G,u_i) > \varepsilon\},
\]
\[
\{u_i : h(a_i,G,u_i) < \varepsilon\}.
\]
Again, the two subcollections separately correspond to upper and lower hemi-continuity respectively.

3.3. \(C\text{-topology}\). Next, we can motivate a definition by considering sequences that “should” converge.

Definition 7. A sequence \(u^k_i\) lower-converges to a limit type \(u_i\) if for every action \(a_i\) that is 0-rationalizable for \(u_i\), there is a sequence \(\tilde{\varepsilon}^k \downarrow 0\) such that \(a_i\) is \(\tilde{\varepsilon}^k\)-rationalizable for \(u^k_i\). The sequence upper-converges to \(u_i\) if for every action \(a_i\) that is not 0-rationalizable for \(u_i\), there is a \(k'\) such that \(a_i\) is not 0-rationalizable for \(u^k_i\) for all \(k > k'\).

These convergence notions were proposed by Dekel, Fudenberg, and Morris (2006a). While the requirement that these sequences converge does not in general “define” a topology, it suggests a natural topology which we call the \(C\)-topology.

Definition 8 \((C\text{-topology})\). The \(C\)-topology is generated by the following class of sets for \(\varepsilon > 0\)
\[
\{u_i : a_i \notin R(u_i|G,\varepsilon)\},
\]
\[
\{u_i : a_i \in R^c(u_i|G,\varepsilon)\}.
\]

The first class clearly corresponds to upper-convergence. The connection between the second class and lower-convergence deserves some explanation. Note that \(a_i\) is \(\varepsilon\)-rationalizable if and only if it is strictly \(\bar{\varepsilon}\)-rationalizable for all \(\bar{\varepsilon} > \varepsilon\). Thus, the collection \(\{u_i : a_i \in R^c(u_i|G,\bar{\varepsilon})\}\) for \(\bar{\varepsilon} > \varepsilon\) is a family of neighborhoods for the limit type \(u_i\). The sequence \(u^k_i\) enters such a neighborhood iff \(a_i\) is \(\tilde{\varepsilon}^k\)-rationalizable for some \(\tilde{\varepsilon}^k < \bar{\varepsilon}\).

3.4. Metric topology. Finally, Dekel, Fudenberg, and Morris (2006a) define a metric, which we shall call the metric topology. Let \(G_m\) be the subclass of games with at most \(m\) actions whose payoffs are bounded by 1, i.e. \(\max_{a \in A, \omega \in \Omega} |g_i(a,\omega)| \leq 1\). Then, for any \(0 < \beta < 1\), let
\[
d(u_i,u'_i) = \sum_{m=1}^{\infty} \beta^m \sup_{G \in G_m} \sup_{a_i \in A_i} |h(a_i,G,u_i) - h(a_i,G,u'_i)|
\]
Dekel, Fudenberg, and Morris (2006a) show that the above is a proper metric.
**Definition 9** (Metric Topology). The metric topology is the topology generated by the metric (3.1).

3.5. **Equivalence.** We show that all definitions of topology from above are equivalent.

**Theorem 2.** The strategic topology is equivalent to topologies $h$ and $C$ and the metric topology.

The Theorem is proven in the Appendix B.

4. **Critical Types**

4.1. **Regular and critical hierarchies.** We have defined a critical type to be a type with a hierarchy of belief such that changes in beliefs at arbitrarily high order can have a discontinuous effect on rationalizable behavior. Precisely, at a critical type, there is either a failure of upper hemi-continuity of the $\varepsilon$-rationalizable correspondence or there is a failure of lower hemi-continuity of the strict $\varepsilon$-rationalizable correspondence, relative to the product topology on hierarchies of belief. Given our definition/characterization of the strategic topology as the weakest topology yielding continuity of these correspondences, we have the following formal definition of a critical type.

**Definition 10.** We say that hierarchy $u_i \in U_i(\Delta\Omega)$ is regular if for any set $u_i \in V_i \subseteq U_i(\Delta\Omega)$, $V_i$ open in the strategic topology, there is a set $U_i$, open in the product topology, such that $u_i \in U_i \subseteq V_i$. We say that hierarchy is critical if it is not regular.

We are going to say that type is critical (or regular) if it has critical (or regular) hierarchy of beliefs. The critical types are those around which the product topology is strictly weaker than the strategic topology. In the remainder of this section, we will characterize critical types in terms of a version of common belief.

4.2. **Common Belief.** Fix subsets of hierarchies, $W_j \subseteq U_j(\Delta\Omega)$, for $j = 1, 2$. The set of hierarchies for player $j$ that $p$-believe in $W_{-j}$ is given by

$$B^p_j(W_{-j}) = \{u_j \in U_j(\Delta\Omega) : u_j(\Delta\Omega \times W_{-j}) \geq p\}.$$  

For the product event $W = W_1 \times W_2$, we define

$$B^p_j(W) = W_j \cap B^p_j(W_{-j})$$  

$$B^p(W) = B^p_j(W) \times B^p_{-j}(W) \subset U(\Delta\Omega).$$  

Note that $B^p(W) \subset W$. Common $p$-belief in $W$ occurs when both players $p$-believe in $W$, and both players $p$-believe in $B^p(W)$, and .... This concept was introduced by Monderer
and Samet (1989). Formally,
\[ C_p(W) = \bigcap_{k \geq 0} [B_p]^k(W). \]

We have the following version of the original characterization due to Monderer and Samet (1989).

**Lemma 6.** Let \( W \subset U(\Delta \Omega) \) be a product event. Then \( C_p(W) \) is a product event and
\[
C_p(W) = W_j \cap B_j^p C_{p-j}(W) = B_j^p \left( \bigcap_{k \geq 0} [B_p]^k(W) \right).
\]

Abusing notation, when \( W_j \in U_j(\Delta \Omega) \) we can view it implicitly as the product event \( W_j \times U_{-j}(\Delta \Omega) \) and write
\[
B_j^p(W_j) := B_j^p(W_j \times U_{-j}(\Delta \Omega))
\]
\[
C_j^p(W_j) := C_j^p(W_j \times U_{-j}(\Delta \Omega)).
\]

**4.3. Characterization of Critical Types.** The main result of the paper characterizes the set of regular hierarchies.

**Theorem 3.** A hierarchy \( u_i^0 \in U_i(\Delta \Omega) \) is critical if and only if there exists \( p > 0 \) and a closed upper contour proper subset \( W_i \subset U_i(\Delta \Omega) \), such that
\[ u_i^0 \in C_i^p(W_i). \]

**Proof.** The proof applies Lemma 1 and Lemma 2 stated in the introduction. The proofs of these lemmas are in the following subsections. Suppose that \( u_i^0 \in C_i^p(W_i) \) for some closed upper contour proper subset \( W_i \subset U_i(\Delta \Omega) \). Let \( \varepsilon = \frac{p}{4(1-p)} \). Then, by Lemma 2, there exists \( V \subset U_i(\Delta \Omega) \) and a game \( G \) with an action \( a_i \), such that
\[ a_i \in R_i(u_i|G, 0) \]
but
\[ a_i \notin R_i(v_i|G, \varepsilon) \]
for any \( v_i \in C_i^{p/2}(V) \). By Lemma 1, there is a sequence of hierarchies \( u_i^n \to u_i^0 \), and \( u_i^n \notin C_i^{p/2}(V) \). Hence, \( u_i^0 \) is critical.

Now, suppose that \( u_i^0 \notin C_i^p(W_i) \) for every closed upper contour subset \( W_i \subset U_i(\Delta \Omega) \) and every \( p > 0 \). We will show that \( u_i^0 \) is regular, i.e. for any set \( V \ni u_i^0 \), such that \( V \) is open in the strategic topology, there is a set \( O \) with \( u_i^0 \in O \subseteq V \) and \( O \) is open in the product topology.
We can assume w.l.o.g. that $V$ is an element of the subbasis of the strategic topology. By Theorem 2, strategic topology is equivalent to $C$-topology (Definition 8). Suppose that first for some $\varepsilon > 0$, game $G$ and action $a_i$

$$V = \{ u_i : a_i \notin R(u_i|G, \varepsilon) \}.$$ 

Then our result follows from the fact that the rationalizable correspondence is u.h.c. on the universal type space $U_i(\Delta \Omega)$ (see remarks after Theorem 1).

Next, suppose that for some $\varepsilon > 0$, game $G$ and action $a_i \in A_i$

$$V = \{ u_i : a_i \in R^\circ(u_i|G, \varepsilon) \}.$$ 

By the definition of the correspondence $R^\circ(\cdot)$ in equation (2.2), there is $0 < \varepsilon' < \varepsilon$, such that

$$u_0^i \in \{ u_i : a_i \in R(u_i|G, \varepsilon') \} \subseteq V.$$ 

By Lemma 4, there is a game $G'$, such that

$$u_0^i \in \{ u_i : a_i \in R(u_i|G', 0) \} \subseteq \{u_0^i \in \{ u_i : a_i \in R(u_i|G', \varepsilon') \} \subseteq V.}$$ 

Let $W_i \subseteq U_i(\Delta \Omega)$ be the closed upper contour set and $p^* > 0$ be the probability given by Lemma 3, and let $p = \min \left( p^*, \frac{\varepsilon - \varepsilon'}{12} \right) > 0$. Since $u_0^i \notin C_i^p(W_i)$, Lemma 3 implies that there is an open neighborhood $O$ such that

$$u_0^i \in O \subseteq \{ u_i : a_i \in R(u_i|G', \varepsilon - \varepsilon') / 2 \} \subseteq V.$$ 

Recall that subset of topological space is residual if it is a countable intersection of dense and open sets.

**Theorem 4.** The set of regular types forms a residual subset (in the product topology) of $U_i(\Delta \Omega)$.

**Proof.** For any $p > 0$ and nonempty closed $W_i \subseteq U_i(\Delta \Omega)$, the set $C_i^p(W_i)$ is closed as the intersection of closed sets. Hence, $U_i(\Delta \Omega) \setminus C_i^p(W_i)$ is open. By Lemma 1, it is also dense.

Note that for any two open sets $W_i \subseteq W_i'$, any $p' \leq p$,

$$C_i^p(W_i) \subseteq C_i^{p'}(W_i').$$
Find a sequence of open sets $U_1^i, U_2^i, \ldots \subseteq U_i(\Delta\Omega)$ such that for any open lower contour set $U'$, there is $n$, such that $U^1 \subseteq U'$. Such a sequence exists, since the space $U_i(\Delta\Omega)$ is separable and metrizable. The set of regular hierarchies of player $i$ is equal to

$$
\bigcap_{n > 0} \bigcap_{m} U_i(\Delta\Omega) \setminus C_{i}^{n/m} (U_i(\Delta\Omega) \setminus U_i^m)
$$

and is therefore residual as an intersection of a countable family of open and dense sets.

4.4. **Proof of Lemma 1.** Note that when $V$ is closed, the sets $[B^p]^k(V)$ are closed in the product topology. As the intersection of closed sets, $C^p(V)$ is closed, hence $U_i(\Delta\Omega) \setminus C^p_i(V)$ is open.

To show that it is dense, let $u$ be an arbitrary hierarchy for $i$. We will construct a sequence of hierarchies from $U_i(\Delta\Omega) \setminus C^p_i(V)$ which converges to $u$ in the product topology. Let $y \in U_i(\Delta\Omega) \setminus V$ be any hierarchy not in $V$. Write $y = y^1, y^2, \ldots$ where $y^l \in U_i^l(\Delta\Omega)$. We introduce the following sets for every $l = 1, 2, \ldots$ and $k = l + 1, l + 2, \ldots$

$$
Y^l_i = \{y^l\}
$$

$$
Y^k_i = \{u^k \in U^k : u^k(\Delta\Omega \times Y^{k-1}_i) = 1\}.
$$

Note that for any $m$ even, any for any hierarchy $\hat{u}$,

$$
\hat{u}^{l+m} \in Y^{l+m}_i \iff \hat{u} \in \bigcap_{m \text{ times}} B_1^1 B_1^{-1} \ldots B_1^1 B_1^{-1} \left(\{v_i \in U_i(\Delta\Omega) : v^l_i = y^l\}\right) \quad (4.3)
$$

We use the following lemma, proved in the appendix, to construct the convergent sequence. For all $k$, set $Y^k_0 = U^k(\Delta\Omega)$.

**Lemma 7.** There exists a sequence of mappings $\varphi_i^k : Y^k_i \to U^{k+1}(\Delta\Omega)$ for $l \geq 0$ and $k > l$, satisfying

1. $\text{proj}_{l+1}^{k+1} \circ \varphi_i^k = \text{id}_{Y^k_i}$
2. $\varphi_i^k(Y^k_i) \subset Y^{k+1}_{l+1}$.

We define a hierarchy $z$ by iteratively applying the maps $\varphi$. Fix $k$ even. Set $z^m = u^m$ for all $m \leq k$ and for all $m \geq k$,

$$
z^{m+1} = \varphi_{m-k}^m(z^m).
$$

It follows from part 1 of the lemma that

$$
\text{proj}_{m}^{m+1}(z^{m+1}) = z^m
$$

for all $m \geq k$ and this equality is satisfied for $m < k$ by the coherency of $u$. Thus, $z$ is coherent.
Next, by part 2 of the lemma and Equation 4.3,
\[ z \in \bigcap_{l=1}^{\infty} B_{l}^{1} B_{l-1}^{1} \ldots B_{0}^{1} B_{1}^{1} \left( \{ v_{i} \in U_{i}(\Delta \Omega) : v_{i}^{l} = y^{l} \} \right), \]
that is,
\[ z \in B_{l}^{1} B_{l-1}^{1} \ldots B_{0}^{1} B_{1}^{1} \left( \{ y \} \right), \]
and so in particular, \( z \notin C_{i}^{p}(V) \) for any \( p > 0 \) since \( y \notin V \).

For a given \( k \) we have constructed a hierarchy \( z \) from \( U_{i}(\Delta \Omega) \) \( \setminus C_{i}^{p}(V) \) which agrees with \( u \) up to order \( k \). The sequence of such hierarchies \( (z)^{k} \) therefore converges to \( u \) in the product topology.

### 4.5. Proof of Lemma 2.
We begin with the following preliminary result, proved in Section D.5. It provides a game with three important features. First, the action set has a product structure and the first dimension of \( i \)'s action is irrelevant for \( -i \)'s payoffs. Second, the rationalizable correspondence has a product structure. Finally, there is a distinguishing subset of actions for player \( i \) that are rationalizable only for a proper subset of types that includes \( W \).

**Lemma 8.** Fix player \( i \). Let \( W_{i} \subseteq U_{i}(\Delta \Omega) \) be a closed upper contour set. For any type \( u_{i}^{*} \notin W_{i} \), there is \( \varepsilon > 0 \), a game \( G = (A_{j}, g_{j}) \), such that \( A_{i} = A_{i}^{0} \times A_{i}^{1} \) and

1. Payoffs of player \(-i\) do not depend on the \( a_{i}^{0}\)-dimension of player's \( i \) action: for any \( a_{-i} \in A_{-i} \), any \( a_{i}^{1} \in A_{i}^{1} \), any \( a_{i}^{0} \), \( \hat{a}_{i}^{0} \in A_{i}^{0} \), any \( \omega \),
   \[ g_{-i} (a_{-i}, (a_{i}^{0}, a_{i}^{1}), \omega) = g_{-i} (a_{-i}, (\hat{a}_{i}^{0}, a_{i}^{1}), \omega). \]
2. There are correspondences \( A^{0} : U_{i}(\Delta \Omega) \Rightarrow A_{i}^{0}, A^{1} : U_{i}(\Delta \Omega) \Rightarrow A_{i}^{1} \), such that for any \( u_{i} \),
   \[ R_{i} (u_{i}|G, 0) = A^{0} (u_{i}) \times A^{1} (u_{i}). \]
3. There is a subset \( A^{0*} \subseteq A_{i}^{0} \),
   \[ [A^{0*} \times A^{1} (u_{i})] \subseteq R_{i} (u_{i}|G, 0) \text{ for any } u_{i} \in W_{i}, \]
   \[ [A^{0*} \times A^{1} (u_{i}^{*})] \cap R_{i} (u_{i}^{*}|G, \varepsilon) = \emptyset. \]

To prove Lemma 2, consider the following game \( G^{*} \), derived from the game \( G \) given in Lemma 8. The set of actions for \( j \) is \( A_{j} \times \{0, 1\} \), i.e. the product of the actions from \( G \), with a binary coordinate which we denote \( z_{j} \). The payoffs to an action profile
\[ a^{*} = ((a_{i}, z_{i}), (a_{-i}, z_{-i})) \]
are as follows.

\[ g_i^*(a^*, \omega) = g_i(a, \omega) + \begin{cases} 1 & \text{if } z_i = z_{-i} = 1, \\ \frac{-p}{1-p} & \text{if } z_i = 1 \text{ and } z_{-i} = 0, \\ 0 & \text{if } z_i = 0. \end{cases} \]

\[ g_{-i}^*(a^*, \omega) = g_{-i}(a, \omega) + \begin{cases} 1 & \text{if } z_{-i} = z_i = 1 \text{ and } a_i^0 \in A^{0*}, \\ \frac{-p}{1-p} & \text{if } z_{-i} = 1 \text{ and } (a_i^0 \notin A^{0*} \text{ or } z_i = 0), \\ 0 & \text{if } z_{-i} = 0. \end{cases} \]

The game \( G^* \) is simply \( G \) augmented with a binary coordination game where coordination is desirable only if an element of the distinguished set \( A^{0*} \) is chosen by player \( i \).

Define the following sets of hierarchies.

\[
V = \left\{ u_i : A^{0*} \times A^1 (u_i) \subseteq R_i (u_i|G, 0) \right\},
\]

\[
V^\varepsilon = \left\{ u_i : A^{0*} \times A^1 (u_i) \cap R_i (u_i|G, \varepsilon) \neq \emptyset \right\}.
\]

By the third part of Lemma 8, \( W \subseteq V \subseteq V^\varepsilon \) and \( V^\varepsilon \) is a proper subset of \( U_i(\Delta \Omega) \). Let

\[
Z = A^{0*} \times A^1 \times \{1\}
\]

be the set of actions for \( i \) in \( G^* \) whose first coordinate belongs to \( A^{0*} \) and whose last coordinate is \( z_i = 1 \). We show in two steps that:

- If \( u_i \in C_i^p(V) \), then \( Z \cap R_i (u_i|G^*, 0) \neq \emptyset \)
- If \( u_i \notin C_i^p(V^\varepsilon) \), then \( Z \cap R_i (u_i|G^*, \varepsilon) = \emptyset \).

To do this, we fix a type space \((T_j, \mu_j)\). Recall that any type space can be mapped into \( U(\Delta \Omega) \) via the mappings \( u_j : T_j \rightarrow U_j(\Delta \Omega) \). If, for example, \( u_i(t_i) \in B_i^p(W_{-i}) \), for some \( W_{-i} \subseteq U_{-i}(\Delta \Omega) \), then we will conserve on notation and write, e.g. \( t_i \in B_i^p(W_{-i}) \), and we will use repeatedly the fact that the mappings \( u_i(t_i) \) preserve beliefs (see Theorem 1), so that, for example, \( t_i \in B_i^p(W_{-i}) \) is equivalent to \( \mu_i(t_i) \left[ u_{-i}^{-1}(W_{-i}) \right] \geq p \).

**Step 1:** We first show that the assessment \( \alpha \), where

\[
\alpha_j(t_j) = \begin{cases} R_j (t_j|G, 0) \times \{1\} & \text{for } t_j \in C_j^p(V), \\ R_j (t_j|G, 0) \times \{0\} & \text{for } t_j \notin C_j^p(V). \end{cases}
\]

has the best response property. Thus, by Definition 1, for \( t_i \in C_i^p(V) \), \( R_i (u_i|G, 0) \times \{1\} \subset R_i (u_i|G^*, 0) \). Since \( C_i^p(V) \subset V \), the definition of \( V \) then implies \( Z \cap R_i (u_i|G^*, 0) \neq \emptyset \).

Let us start with player \( j = -i \) and type \( t_{-i} \). Take any \( a_{-i} \in R_{-i} (t_{-i}|G, 0) \). By definition, the rationalizable correspondence for \( G \) has the best-response property, so let \( \sigma_i \) be a
behavioral strategy of player $i$ in game $G$ that makes $a_{-i}$ a best response for $t_{-i}$ and that

$$\sigma_i(t_i)(R_i(t_i|G,0)) = \sigma_i(t_i)(A^0(t_i) \times A^1(t_i)) = 1$$

for any $t_i$.

By properties 1 and 2 of Lemma 8 we can choose $\sigma_i$ so that for any type $t_i \in C^p_i(V) \subset V$

$$\sigma_i(t_i)(A^{0*} \times A^1(t_i)) = 1.$$

Now we use $\sigma_i$ to define a behavioral strategy for player $i$ in $G^*$. Let

$$\sigma^*_i(t_i) = \begin{cases} 
(\sigma_i(t_i),1) & \text{for } t_i \in C^p_i(V), \\
(\sigma_i(t_i),0) & \text{for } t_i \notin C^p_i(V).
\end{cases}$$

Let $q$ be the probability type $t_{-i}$ assigns to the event $t_i \in C^p_i(V)$. We will first show that the action $(a_{-i},1)$ is a best-reply to $\sigma^*_i$ for any type $t_{-i} \in C^p_{-i}(V)$. In this case, by Lemma 6, $q \geq p$. Since $a_{-i}$ is a best-reply to $\sigma_i$, for any $a'_{-i} \in A_{-i}$, the difference in payoffs between actions $(a_{-i},1)$ and $(a_{-i},0)$.

$$\pi_{-i}((a_{-i},1),\sigma^*_i|t_{-i}) - \pi_{-i}((a_{-i},0),\sigma^*_i|t_{-i}) \geq q - \frac{p}{1-p}(1-q) \geq 0.$$

Next, consider $t_{-i} \notin C^p_{-i}(V)$. In this case, by Lemma 6, $q \leq p$. We now show that the action $(a_{-i},0)$ is a best-reply to $\sigma^*_i$. For any $a'_{-i} \in A_{-i}$,

$$\pi_{-i}((a_{-i},0),\sigma^*_i|t_{-i}) - \pi_{-i}((a_{-i},1),\sigma^*_i|t_{-i}) \geq \frac{p}{1-p}(1-q) - q \geq 0.$$

The same argument applies for player $j = i$.

**Step 2**: Let $q = p - 2\varepsilon(1-p)$. Observe that

$$U_i(\Delta\Omega) \setminus C^p_i(V^\varepsilon) = U_i(\Delta\Omega) \setminus V^\varepsilon \cup \bigcup_{k \geq 1} U_i(\Delta\Omega) \setminus [B^q_i B^{q-1}_{-i}]^k(V^\varepsilon).$$

By definition, $Z \cap R_i(t_i|G^*,\varepsilon) = \emptyset$ for any type $t_i \in U_i(\Delta\Omega) \setminus V^\varepsilon$. We are going to show by induction on $k$ that no action $(a_j,1)$ is $\varepsilon$-interim rationalizable for any type $t_i \notin [B^q_i B^{q-1}_{-i}]^k(V^\varepsilon)$.

By the induction hypothesis, if $t_{-i} \notin B^q_i [B^q_{-i} B^{q-1}_{-i}]^k(V^\varepsilon)$, then $t_{-i}$ assigns probability at most $q$ to the set of types $t_i$ for $i$ such that the intersection $Z \cap R_i(t_i|G^*,\varepsilon)$ is non-empty. Thus, for any $a_{-i} \in A_{-i}$, for any strategy $\sigma_i$ of player $i$, such that $\sigma_i(t_i) \in \Delta R_i(t_i|G^*,\varepsilon)$,

$$\pi_{-i}((a_{-i},1),\sigma^*_i|t_{-i}) - \pi_{-i}((a_{-i},0),\sigma^*_i|t_{-i}) \leq -\frac{p}{1-p} + q \left(1 + \frac{p}{1-p}\right) = \frac{q-p}{1-p} = -2\varepsilon.$$

Thus, $(a_{-i},1)$ is not $\varepsilon$-interim rationalizable for $t_{-i}$.

Now let $t_i$ be a type for $i$ such that $t_{-i} \notin B^q_i B^{q-1}_{-i} [B^q_i B^{q-1}_{-i}]^k(V^\varepsilon)$. The previous argument implies that $t_{-i}$ assigns probability at most $q$ to types of $-i$ for whom $z_{-i} = 1$ is part of
a $\varepsilon$-rationalizable action. Hence, for any action $a_i \in A_i$, for any strategy $\sigma_{-i}$ such that 

$$\sigma_{-i} (t_{-i}) \in \Delta R_{-i} (t_{-i} | G^* , \varepsilon)$$

$$\pi_i ((a_i, 1), \sigma^*_i | t_i) - \pi_i ((a_i, 0), \sigma^*_i | t_i) \leq - \frac{p}{1 - p} + q \left( 1 + \frac{p}{1 - p} \right) = \frac{q - p}{1 - p} = -2\varepsilon,$$

and we have shown that $Z \cap R_i (t_i | G^*, \varepsilon) = \emptyset$ completing the proof of step 2.

4.6. **Proof of Lemma 3.** Fix player $i$ and a game $G$. Define a collection of subsets of actions

$$\mathcal{A}_\varepsilon = \{ A'_i \subseteq A_i : R_i (u_i | G, \varepsilon) = A'_i \text{ for some } u_i. \}$$

The collection $\mathcal{A}_\varepsilon$ is a non-empty collection of non-empty sets and it is ordered with respect to $\varepsilon$ in the following way: for any $\varepsilon' < \varepsilon$, any $A'_i \in \mathcal{A}_\varepsilon$, there is $A''_i \in \mathcal{A}_{\varepsilon'}$ such that $A''_i \subseteq A'_i$. Since the number of all actions is finite, the number of all subsets of actions is finite and, therefore, there is $\varepsilon^A > 0$, such that for all $0 \leq \varepsilon \leq \varepsilon^A$, we have $\mathcal{A}_\varepsilon = \mathcal{A}_0$ Let $A^*_i$ be a minimal element of $\mathcal{A}_0$, i.e. $A_i \in \mathcal{A}_0$ and there is no $Z_i \subseteq A^*_i$ belonging to $\mathcal{A}_0$. Define

$$U^A = \{ u_i : A^*_i = R_i (u_i | G, \varepsilon^A) \}.$$ 

Then, $U^A$ is open because the correspondence $R_i (\cdot | G, \varepsilon) : U_i (\Delta \Omega) \Rightarrow A_i$ is u.h.c. It is a non-empty lower contour set because of the choice of set $A^*_i$. Finally, for any $0 \leq \varepsilon \leq \varepsilon^A$, for any $u_i \in U^A$, $A^*_i = R (u_i | G, \varepsilon)$.

Take any $p \leq \frac{\varepsilon^A}{6}$. By Lemma 6, if $u^*_i \notin C_i^p (W_i)$, then

$$u^*_i \in U_i (\Delta \Omega) \setminus C_i^p (W_i)$$

$$= U_i (\Delta \Omega) \setminus B_i^p \left( \bigcap_{k \geq 0} [B^p]^k (W_i) \right)$$

$$= U_i (\Delta \Omega) \setminus \bigcap_{k \geq 0} B_i^p \left( [B^p]^k (W_i) \right)$$

$$= \bigcup_{k \geq 0} U_i (\Delta \Omega) \setminus B_i^p \left( [B^p]^k (W_i) \right).$$

Lemma 3 follows from the next result, proven in the appendix.

**Lemma 9.** Consider the closed, upper contour proper subset

$$W_i = U_i (\Delta \Omega) \setminus U^A$$

For any $p < \frac{\varepsilon^A}{6}$, any player $j$, any $k \geq 0$, any

$$u^0_j \notin B_j^p \left( [B^p]^k (W_i) \right),$$

any sequence $u^n_j \rightarrow u^0_j$ convergent in the product topology, and any action $a_j \in A_j$, such that $a_j \in R (u_j | G, 0)$, there is $n^*$ sufficiently high, so that for any $n \geq n^*$, $a_j \in R \left( u^n_j | G, 6p \right)$. 
Thus, for any $u_i^* \notin C_p^2(W_i)$, there is an open neighborhood $V \ni u_i^*$, such that if action $a_i$ is interim rationalizable for any type with a hierarchy $u_i^*$, then, it is $6p$-interim rationalizable for any type with a hierarchy $u_i \in V$.

5. Common prior and critical hierarchies

We will show that almost all types from types spaces with a common prior are critical. Let $T = (T_i, \mu_i)$ be a type space over $X$. Say that $\psi \in \Delta (T_i \times T_{-i})$ is a common prior on $T$ if for any bounded measurable function $f : T_i \times T_{-i}$, any player $i$

$$\psi[f(t_i, t_{-i})] = \int (\mu_i(t_i) [f(t_i, t_{-i})]) d\psi_i(t_i),$$

where $\psi_i = \text{marg}_{T_i} \psi$. This is a non-standard definition. We do not require in particular that the common prior is a measure also over uncertainty $\Omega$. In a sense, ours is a weaker definition and, as a consequence, the subsequent result is stronger.

**Theorem 5.** Suppose that $\psi$ is a common prior on a type space $T = (T_i, \mu_i)$. Then, for each player $i$, $t_i$ has critical hierarchy $\psi_i$-almost surely.

The Theorem is a corollary to three lemmas. The first lemma says that any common prior on type space over $\Omega$ corresponds to a common prior on the universal type space $U(\Delta \Omega)$.

**Lemma 10.** For any common prior $\psi$ on type space $T$, there is a common prior $\psi^*$ on the universal type space $U(\Delta \Omega)$, such that for any measurable $E_i \subseteq U_i(\Delta \Omega)$ for both players $i = 1, 2$,

$$\psi^*(E_1 \times E_2) = \psi\left((u_1^T)^{-1}(E_1) \times (u_2^T)^{-1}(E_2)\right).$$

The second result says that the support of a common prior on $U(\Delta \Omega)$ can be approximated by closed upper contour sets.

**Lemma 11.** For any common prior $\psi^*$ on the universal type space $U(\Delta \Omega)$, any $\epsilon > 0$, there are upper contour closed proper subsets $V_j \subset U_j(\Delta \Omega)$, such that

$$\psi^*(V_1 \times V_2) \geq 1 - \epsilon.$$

The following is a version of one-side of the critical path lemma by Morris-Shin.

**Lemma 12.** Let $\psi^*$ be a common prior on type space $U(\Delta \Omega)$. For any measurable sets $V_i \subseteq U_i(\Delta \Omega)$, there are measurable subsets $S_i \subseteq V_i$, such that

$$\psi(S_1 \times S_2) \geq \frac{3}{2} \psi(V_1 \times V_2) - \frac{1}{2},$$

and for any player $i$, any type $u_i \in S_i$,

$$u_i(\Delta \Omega \times S_1 \times S_2) \geq \frac{1}{4}.$$
We can prove the Theorem:

**Proof of Theorem 5.** Suppose that $\psi^*$ is a common prior on the universal type space $U(\Delta \Omega)$.

By Lemma 11, for any $\varepsilon > 0$, there are closed, upper contour, proper subsets $V_j \subset U_j(\Delta \Omega)$, such that $\psi^*(V_1 \times V_2) \geq 1 - \varepsilon$. Next, Lemma 12 implies that

$$\psi^*(C^{1/4}(V_1 \times V_2)) \geq 1 - \frac{3}{2}\varepsilon.$$ 

Hence, for any player $i$, with $\psi^*$-probability at least $1 - \frac{3}{2}\varepsilon$, player $i$'s hierarchy is critical. Since the latter is true for any $\varepsilon > 0$, it means that $\psi^*$-almost all hierarchies are critical. Therefore, if $\psi$ is a common prior on type space $T$ over $\Omega$, then, Lemma 10 implies that $\psi$-almost all types are critical. \hfill \Box

5.1. **Proof of Lemma 10.** Recall that the types in $T$ are mapped via $u^T$ to the universal type space $U(\Delta \Omega)$. This lemma is an immediate consequence of the fact that $u^T$ preserves beliefs. See Theorem 1.

5.2. **Proof of Lemma 11.** We begin with an observation. Let $u^{MZ,T}_i: T_i \to U_i(\Omega)$ be the Mertens-Zamir mapping assigning types to their hierarchies of beliefs over $\Omega$. Dekel, Fudenberg, and Morris (2006a) show the following Lemma. (To be precise, they state this result for interim correlated rationalizability, but their proof applies unchanged to interim independent rationalizability.)

**Lemma 13** (Dekel, Fudenberg, and Morris (2006a)). For any two types $t_i, t'_i$, if $u^{MZ,T}_i(t_i) \neq u^{MZ,T}_i(t'_i)$, then there is a game $G$ and an action $a_i$ such that $a_i \in R(t_i|G,0)$ and $a_i \notin R(t'_i|G,0)$.

The Lemma has two implications. First, together with Theorem 1, it implies that there is a continuous onto mapping $v_i: U_i(\Delta \Omega) \to U_i(\Omega)$, such that for any type space and a type $t_i \in T$,

$$v_i(u^T_i(t_i)) = u^{MZ,T}_i(t_i).$$

Second, for any hierarchies $u'_i \succeq u_i$, any type space and types $t_i, t'_i \in T_i$, such that $u^T_i(t_i) = u_i$ and $u^T_i(t'_i) = u'_i$, and for any game $G$, it must be that $u^{MZ,T}_i(t_i) = u^{MZ,T}_i(t'_i)$.

For any $\rho > 0$, $u^{MZ}_i \in U_i(\Omega)$, let

$$V_i(u^{MZ}_i, \rho) := U_i(\Delta \Omega) \setminus u^{-1}(B(u^{MZ}_i, \rho)),$$

where $B(u^{MZ}_i, \rho)$ is an open ball in the universal type space $U_i(\Omega)$ with center at $u^{MZ}_i$ and radius at $\rho$. By the above, each $V(u^{MZ}_i, \rho)$ is a nonempty, closed, upper contour of $U_i(\Delta \Omega)$. For $\rho$ small enough, $V(u^{MZ}_i, \rho)$ is a proper subset of $U_i(\Delta \Omega)$.

Finally, since the cardinality of $U_i(\Delta \Omega)$ is infinite, for any $\varepsilon > 0$, there exist $u^{MZ}_j$ and $\rho > 0$ for any $j = 1, 2$, such that $\psi^*(V_i \times V_{-i}) \geq 1 - \varepsilon$. 

5.3. **Proof of Lemma 12.** Define inductively sets: $V_i^{(0)} = V_i$

$$V_i^{(n+1)} = \left\{ t_i : \mu_i(t_i) \left( \Delta \Omega \times V_1^{(n)} \times V_2^{(n)} \right) \geq \frac{1}{4} \right\}. $$

To save on redundant repetitions, in what follows we drop the term "\( \Delta \Omega \times \)" from the product over which beliefs are defined. Let

$$S_i = \bigcap_{n \geq 0} V_i^{(n)}. $$

Since the sequence of sets \( \left( V_i^{(n)} \right) \) is decreasing, for any player \( i \), any \( u_i \in S_i \), (5.2) holds.

Recall that \( \psi_i \) denotes the marginal prior over types of player \( i \). Notice that

$$ \psi \left( V_1^{(n+1)} \times V_2^{(n+1)} \right) \geq \psi \left( V_1^{(n)} \times V_2^{(n)} \right) - \frac{1}{4} \psi_1 \left( V_1^{(n)} \setminus V_1^{(n+1)} \right) - \frac{1}{4} \psi_2 \left( V_2^{(n)} \setminus V_2^{(n+1)} \right). $$

Hence,

$$ \psi \left( S_1 \times S_2 \right) \geq \psi \left( V_1 \times V_2 \right) - \frac{1}{4} \left( \psi_1 \left( V_1 \setminus S_1 \right) + \psi_2 \left( V_2 \setminus S_2 \right) \right). $$

On the other hand, for both players \( i \),

$$ \psi_i \left( V_i \setminus S_i \right) \leq 1 - \psi_i \left( S_i \right) \leq 1 - \psi \left( S_1 \times S_2 \right). $$

Together with the above inequality, this implies that

$$ \frac{1}{4} \left( \psi_1 \left( V_1 \setminus S_1 \right) + \psi_2 \left( V_2 \setminus S_2 \right) \right) \leq \frac{1}{2} \left( 1 - \psi \left( V_1 \times V_2 \right) \right), $$

and Equation 5.1 follows.

6. **Interim Correlated Rationalizability**

All of the results in this paper have counterparts for the solution concept of interim correlated rationalizability introduced by Dekel, Fudenberg, and Morris (2006a).

**Theorem 3** has a simpler statement in this case. Hierarchy \( u_i \in U_i(\Omega) \) is critical iff there exists a \( p > 0 \) and a closed subset \( U \subseteq U_i(\Omega) \) such that \( u_i \in C_p^u(U) \). When we consider correlated rationalizability, then the partial order \( \succeq \) is trivial, and the relevant universal type space is \( U_i(\Omega) \). Also, the statement for correlated rationalizability is true for any number of players.
Appendix A. Example of a Regular Hierarchy

In this Section, we describe an explicit construction of regular hierarchy. The idea is fairly simple. Recall that Ω and the universal type space $U_i(ΔΩ)$ are compact. Let $Q \subseteq U_i(ΔΩ)$ be a countable dense subset of hierarchies. Let $N$ be the set of natural numbers and let $a : N \to N \times Q$ be a bijection. Define $b(0) = 0$ and for $n \geq 1$,

$$b(n) = \sum_{m<n} a_N(m).$$

One can construct a regular hierarchy $u_i^*$ as follows. Take any hierarchy $u_i \in U_i(ΔΩ)$. For any $n$, cut out all the beliefs on levels between $b(n)$ and $b(n+1) - 1$ and replace them by the first $a_N(n)$-order beliefs of hierarchy $a_Q(n)$. In other words, hierarchy $u_i^*$ is constructed by piling up first $a_N(1)$ levels of hierarchy $a_Q(1)$, on top of that first $a_N(2)$ levels of hierarchy $a_Q(2)$, on top of that $a_N(3)$ levels of hierarchy $a_Q(3)$ and so on ... . Hierarchy $u_i^*$ is regular, because, intuitively, any finite hierarchy from a dense subset of all hierarchies is believed with probability 1 at certain level. Thus, there is no closed nontrivial set of hierarchies $W$ such that $u_i^*$ has $p$-common belief in $W$.

More formally, we are going to construct a type space, in which all hierarchies are regular. For any hierarchy $q_i \in Q_i$, let $t_i^{q_i} \in T_i^{q_i}$ be a type in a type space $T_i^{q_i} = (T_i^{q_i}, \mu_i^{q_i})$, such that $u_i(t_i^{q_i}) = q_i$. For any player $j$, any $m \in N$, let $T_j^{q_i,m}$ be an $m$th copy of set $T_j^{q_i}$ of types of
player $j$ where $m = 0, 1, \ldots$ . Let $\eta_{j,m}^{n} : T_j^n \to T_j^{q_j,m}$ be the equivalence mapping. For any player $j$, construct

$$T_j^* = \bigcup_m T_j^{*,m},$$

where, for any $b(n) \leq m < b(n + 1)$

$$T_j^{*,m} = T_j^{q_j,m-b(n)}.$$

Define belief mapping $\mu_j^* : T_j^* \to \Delta (\Omega \times T)$:

- For each $n$, for each $m \leq a_N(n)$ for any $t_i \in T_i^{*,m+b(n)}$, define

$$\mu_i^*(t_i) := \Delta \left[ \frac{\eta_{-i} a_Q(n),m}{\Omega} \right] \mu_{i}^{a_Q(n),m-1} \left( (\eta_{i}^{a_Q(n),m})^{-1}(t_i) \right).$$

In other words, beliefs of type $t_i$ are concentrated on set $T_i^{q_i,m}$. and the joint beliefs over $\Omega$ and types of player $-i$ are equal to the joint beliefs of type $(\eta_{i}^{a_Q(n),m})^{-1}(t_i) \in T_i^n$.

- For each $n$, for each $m < a_N(n)$ for any $t_{-i} \in T_{-i}^{*,m+b(n)}$, define

$$\mu_{-i}^*(t_{-i}) := \Delta \left[ \frac{\eta_{-i} a_Q(n),m+1}{\Omega} \right] \mu_{-i}^{a_Q(n),m-1} \left( (\eta_{-i}^{a_Q(n),m})^{-1}(t_{-i}) \right).$$

In other words, beliefs of type $t_{-i}$ are concentrated on set $T_{i}^{q_i,m}$. and the joint beliefs over $\Omega$ and types of player $i$ are equal to the joint beliefs of type $(\eta_{-i}^{a_Q(n),m})^{-1}(t_{-i}) \in T_{-i}^{q_{-i}}$.

- For each $n$, for any $t_{-i} \in T_{-i}^{*,a_N(n)+b(n)}$, choose $\mu_{-i}^*(t_{-i})$ so that

$$\mu_{-i}(t_{-i}) \left( \Omega \times \left\{ \left( \eta_{i}^{a_Q(n+1),1} \right) \left( t_{i}^{a_Q(n+1)} \right) \right\} \right) = 1.$$

In other words, type $t_{-i}$ believes that the player $i$’s type is equal to $(\eta_{i}^{a_Q(n+1),1}) \left( t_{i}^{a_Q(n+1)} \right)$. For any $n$, define set of hierarchies

$$Q_i^n = \left\{ u_i : \text{proj } u_i = \text{proj } q_i \right\}.$$
Hence, the inductive thesis holds for $m - 1$ and $j = -i$. Similar argument shows that if the thesis holds for $m$ and $j = -i$, then it also holds for $m$ and $j = i$. □

**Lemma 15.** For any $t_i \in T_i^*$, $t_i$ is regular.

*Proof.* Let $t_i \in T_i^{*,m}$ for some $m$. Suppose not, and there is $p > 0$ and closed upper contour proper subset $W \subset U_i (\Delta \Omega)$ such that $u_i (t_i) \in C^p (W)$. Since $Q$ is dense and $W$ is closed proper subset, there is $n$ such that $b (n) > m$ and $Q^n \cap W = \varnothing$. By the previous Lemma, $u_i (t_i) \notin C^p (U_i (\Delta \Omega) \setminus Q^n_i)$. Contradiction. □

**Appendix B. Equivalence of topologies**

**Proposition 1.** The strategic topology is equivalent to both topologies $h$ and $C$.

*Proof.* First note that

$$\left\{u_i : h(a_i, G, u_i) > \varepsilon\right\} = \left\{u_i : a_i \notin R(u_i | G, \varepsilon)\right\}$$

and

$$\left\{u_i : h(a_i, G, u_i) < \varepsilon\right\} = \left\{u_i : a_i \in R^c(u_i | G, \varepsilon)\right\}$$

so that the $h$ and $C$ topologies are equivalent. Next,

$$\left\{u_i : R^c(t, \varepsilon) \cap B \neq \emptyset\right\} = \bigcup_{a \in B}\left\{u_i : a \in R^c(u_i | G, \varepsilon)\right\}$$

$$\left\{u_i : R(t, \varepsilon) \subset B\right\} = \bigcap_{a \notin B}\left\{u_i : a \notin R(u_i | G, \varepsilon)\right\}$$

so that the strategic topology is at least as fine as the $C$ topology. Also,

$$\left\{u_i : a \in R^c(u_i | G, \varepsilon)\right\} = \left\{u_i : R^c(t, \varepsilon) \cap \{a\} \neq \emptyset\right\}$$

$$\left\{u_i : a \notin R(u_i | G, \varepsilon)\right\} = \left\{u_i : R(t, \varepsilon) \subset \neg\{a\}\right\}$$

so that the $C$ topology is at least as fine as the strategic topology. □

The next lemma establishes an important continuity property of the mappings $h$. For any two games $G, G'$ with the same set of actions $A_i$ for each player, we denote a distance between games

$$d (G, G') = \max_i \sup_{a \in A_i, \omega \in \Omega} |g_i (a, \omega) - g_i' (a, \omega)|.$$  

**Lemma 16.** If $G$ and $G'$ are two games with the same set of actions $A_i$ for each player, then

$$|h_i (a_i, G, t_i) - h_i (a_i, G', t_i)| \leq 2d (G, G')$$

for all $t_i$ and for all $a_i \in A_i$. 

Proof. Recall the definition of function $\pi_i$ in (2.1) and observe that for all $t_i$ and $a_i$

$$\left| \pi_i^G(a_i, \sigma_{-i}|t_i) - \pi_i^{G'}(a_i, \sigma_{-i}|t_i) \right| \leq d(G, G'). \quad \text{(B.1)}$$

Pick any $\varepsilon > 0$ and consider the $\varepsilon$-rationalizable correspondence $R(\cdot|G, \varepsilon)$ for game $G$. For each $(t_i, a_i) \in R(\cdot|G, \varepsilon)$, there is a behavioral strategy $\sigma_{-i} \in \Sigma_{-i}(R(\cdot|G, \varepsilon))$ of the opponents, such that $a_i$ is an $\varepsilon$-best-response of type $t_i$ in game $G$. By (B.1), $a_i$ is a $[\varepsilon + 2d(G, G')]$-best-response in game $G'$. Since this bound is independent of $t_i$ and $a_i$, it follows that $R(\cdot|G, \varepsilon) \subset R(\cdot|G, \varepsilon + 2d(G, G'))$ which implies

$$h(a_i, G', t_i) \leq h_i(a_i, G, t_i) + 2d(G, G').$$

The lemma follows from switching the roles of $G$ and $G'$ in the preceding argument. \qed

Finally we show that the strategic topology is equivalent to the metric topology introduced by Dekel, Fudenberg, and Morris (2006a). Thus their metric generates the weakest topology consistent with the convergence properties they propose and the weakest topology consistent with continuity of the rationalizable correspondences. In fact, we can dispense with their assumptions of finite $\Omega$ and uniformly bounded payoffs.

**Proposition 2.** The metric topology is equivalent to the strategic topology.

Proof. We show that for any hierarchy $u_i$ and set $U \ni u_i$ open in one topology, there is set $V$ open in the other topology and such that $u_i \in V \subseteq U$.

It is enough to show this relation for all sets $U$ and $V$ in the sub-bases of the respective topologies. For any hierarchy $u_i \in U_i$, any $\varepsilon > 0$, define

$$V^\varepsilon(u_i) = \{u_i' : d(u_i, u_i') < \varepsilon\}.$$  

Then sets $V^\varepsilon(u_i)$ form a sub-basis for the metric topology: for any set $V$ open in metric topology, for any $u_i \in V$, there is $\varepsilon > 0$, such that $u_i \in V^\varepsilon(u_i) \subseteq V$. By Proposition 1, the sets

$$\{u_i : h(a_i, G, u_i) > \varepsilon\} \text{ and } \{u_i : h(a_i, G, u_i) < \varepsilon\}$$

form a sub-basis of the strategic topology.

**Step 1:** Metric topology is finer than strategic topology. Take any game $G$, and let $m$ be the number of actions in $G$. Let $G' \in G_m$ be the game derived from $G$ by normalizing the payoff function. Specifically, let $\bar{g} = \max_{a, \omega} g(a, \omega)$. Then $G'$ is the game with the same action set as $G$ and payoff function $g'$ defined by $g' = \frac{g}{\bar{g}}$. Note that $a_i$ is $\varepsilon$-rationalizable in $G$ iff $a_i$ is $\varepsilon$-rationalizable in $G'$. Write $\bar{\varepsilon} = \frac{\varepsilon}{\bar{g}}$. 
Pick any hierarchy \( u_i \) and \( \varepsilon > 0 \). If

\[
u_i \in \{ u'_i : h(a_i, G, u'_i) < \varepsilon \}
\]

then

\[
u_i \in V^\varepsilon_i (u_i) \subseteq \{ u'_i : h(a_i, G', u'_i) < \varepsilon \} = \{ u'_i : h(a_i, G, u'_i) < \varepsilon \}.
\]

Next, let

\[
u_i \in \{ u'_i : h(a_i, G, u'_i) > \varepsilon \} = \{ u'_i : h(a_i, G', u'_i) > \varepsilon \}
\]

Thus, \( a_i \notin R(u_i|G', \varepsilon) \). By Lemma 5, there is \( \varepsilon' > 0 \), such that \( a_i \notin R(u_i|G', \varepsilon + \varepsilon') \).

Therefore,

\[
u_i \in \{ u'_i : h(a_i, G', u'_i) > \varepsilon + \varepsilon' \}.
\]

Notice that for any \( u'_i \in V^\varepsilon_i (u_i) \),

\[
h(a_i, G', u'_i) \geq h(a_i, G', u_i) - \varepsilon' > \varepsilon.
\]

This implies that

\[
u_i \in V^\varepsilon_i (u_i) \subseteq \{ u'_i : h(a_i, G', u'_i) > \varepsilon \}.
\]

**Step 2**: Strategic topology is finer than metric topology. For any \( \varepsilon \), let \( m(\varepsilon) \) satisfy

\[
\frac{\gamma \varepsilon m(\varepsilon)}{1 - \beta} \leq \frac{\varepsilon}{4}.
\]

We can identify the space \( G_m \) with the space of functions

\[
G_m = \{ g : \{1, \ldots, m\}^2 \times \Omega \to [-1, 1]^2 \}.
\]

This space is compact in the sup topology. Hence, for any \( \varepsilon < 0 \), there is a finite subset \( F_m^\varepsilon \subset G_m \) with the property that for any \( G \in G_m \), there is \( G' \in F_m^\varepsilon \) such that

\[
d(G, G') \leq \frac{1}{m(\varepsilon)} \varepsilon.
\]

We can bound the distance between two hierarchies as follows

\[
d(u_i, u'_i) \leq \frac{\varepsilon}{4} + \sum_{m=1}^{m(\varepsilon)} \beta^m \sup_{G \in G_m} \sup_{a_i \in A_i} |h(a_i, G, u_i) - h(a_i, G, u'_i)|
\]

\[
\leq \frac{\varepsilon}{4} + \sum_{m=1}^{m(\varepsilon)} \beta^m \sup_{G' \in F_m} \sup_{a_i \in A_i} |h(a_i, G', u_i) - h(a_i, G', u'_i)| + 2 \frac{1}{m(\varepsilon)} \varepsilon \sum_{m=1}^{m(\varepsilon)} \beta^m
\]

\[
\leq \frac{\varepsilon}{2} + \sum_{m=1}^{m(\varepsilon)} \sup_{G' \in F_m} \sup_{a_i \in A_i} |h(a_i, G', u_i) - h(a_i, G', u'_i)|.
\]

Now pick any \( u_i \) and \( \varepsilon > 0 \), and consider the open set in the metric topology \( V^\varepsilon_i (u_i) \). The previous inequality implies

\[
\bigcap_{m \leq m(\varepsilon)} \bigcap_{G' \in F_m} \bigcap_{a_i \in A_i} \left\{ u'_i : h(a_i, G, u'_i) - \frac{\varepsilon}{m(\varepsilon)} < h(a_i, G, u'_i) < h(a_i, G, u_i) + \frac{\varepsilon}{m(\varepsilon)} \right\} \subseteq V^\varepsilon_i (u_i).
\]
and the set on the left-hand side is a finite intersection of $u_i$-neighborhoods in the strategic topology. □

APPENDIX C. PROOF OF LEMMA 5

Define assessment
$$R^u_i (\cdot | G, \varepsilon) := \bigcap_{\varepsilon' > \varepsilon} R_i (\cdot | G, \varepsilon).$$

We check that this assessment is closed with respect to best response property, i.e.
$$R^u_i = R_i. \quad (C.1)$$

By the second part of Theorem 1, there exists a type space $(L_i, \mu^L_i)$ over $\Omega$ and such that
- $\varepsilon$-rationalizable correspondence $R_i (\cdot | G, \varepsilon)$ is upper hemi continuous on $L_i$. In particular, define set $R_i (G, \varepsilon) \subseteq L_i \times A_i$ as
$$R_i (G, \varepsilon') = \{(l_i, a_i) : a_i \in R_i (l_i | G, \varepsilon')\}.$$

Then, such a set is closed.
- for any type space and any type $t_i \in T_i$, there is a corresponding type $l_i \in L_i$, such that for any game $G$ any $\varepsilon \geq 0$
$$R_i (t_i | G, \varepsilon) = R_i (l_i | G, \varepsilon). \quad (C.2)$$

We show that (C.1) holds on type space $(L_i, \mu^L_i)$. Because of (C.2), this is sufficient to establish the Lemma.

Suppose that action $a_i$ is $\varepsilon'$-rationalizable for $l_i$ and any $\varepsilon' > \varepsilon$. Then, there is a sequence of strategies $\sigma^\varepsilon_{-i} : L_{-i} \rightarrow \Delta A_{-i}$, such that $a_i \in B (\varepsilon' | l_i, \sigma^\varepsilon_{-i})$ and $\sigma^\varepsilon_{-i}$ is $\varepsilon'$-rationalizable, i.e.
$$\sigma^\varepsilon_{-i} (l_{-i}) (R_{-i} (l_{-i} | G, \varepsilon')) = 1.$$

The sequence of strategies generates corresponding conjectures $\psi^\varepsilon_{-i} \in \Delta (\Omega \times L_{-i} \times A_{-i})$, such that
$$\operatorname{marg}_{\Omega \times L_{-i}} \psi^\varepsilon_{-i} (l_i) = \mu^L_i (l_i)$$

and for any continuous function $f : L_{-i} \times A_{-i} \rightarrow R$,
$$\int_{L_{-i} \times A_{-i}} f (l_i, a_i) d\psi^\varepsilon_{-i} (l_i, a_i) = \int_{L_{-i}} \left[ \int_{A_{-i}} f (l_i, a_i) d\sigma^\varepsilon_{-i} (l_{-i}) (a_i) \right] d\psi^\varepsilon_{-i} (l_i).$$

Moreover, each of the conjectures assigns probability 1 to the fact that the opponent strategy is $\varepsilon'$-rationalizable:
$$\psi^\varepsilon_{-i} \left( \Omega \times R^u_{-i} (G, \varepsilon') \right) = 1.$$
By compactness, we can find a convergent subsequence with a limit \( \psi_{\varepsilon_{i}} \). Then, by continuity, \( a_{i} \) is \( \varepsilon \)-best response against conjecture \( \psi_{\varepsilon_{i}} \) and

\[
\psi_{\varepsilon_{i}}(\Omega \times R_{-i}^{u}(G, \varepsilon)) \geq \lim_{\varepsilon' \to \varepsilon} \psi_{\varepsilon_{i}}(\Omega \times R_{-i}(G, \varepsilon')) = 1,
\]

which follows from the definition of the correspondence \( R^{u} \) and the fact that it is u.h.c.

Define a strategy \( \sigma_{\varepsilon_{i}} : L_{-i} \to \Delta A_{-i} \) as

\[
\sigma_{\varepsilon_{i}}(l_{-i}) := \text{marg}_{A_{-i}} \psi_{\varepsilon_{i}}(\cdot | l_{-i}).
\]

Then, \( \sigma_{\varepsilon_{i}}(l_{-i}) \in R_{\varepsilon_{i}}^{u}(l_{-i}|G, \varepsilon) \) (one may need to modify \( \sigma_{\varepsilon_{i}} \) on the set of types \( l_{-i} \) with a total \( \mu^{L}_{i}(l_{i}) \)-mass equal to 0), and \( a_{i} \in B(\varepsilon | l_{i}, \sigma_{\varepsilon_{i}}) \). This ends the proof.

**Appendix D. Proofs of Section 4**

**D.1. Proof of Lemma 6.** Note that \( [B^{p}]^{k}(W) \) is a product event and

\[
B_{j}^{p}[B^{p}]^{k}(W) = B_{j}^{p}[B^{p}]^{k-1}(W) \cap B_{j}^{p}B_{-j}^{p}[B^{p}]^{k-1}(W).
\]

By definition

\[
C^{p}(W) = \bigcap_{k \geq 0} \left[ B_{j}^{p}[B^{p}]^{k}(W) \cap B_{j}^{p}[B^{p}]^{k}(W) \right]
\]

and so \( C^{p}(W) \) is a product set and

\[
C_{j}^{p}(W) = \bigcap_{k \geq 0} B_{j}^{p}[B^{p}]^{k}(W)
\]

\[
= B_{j}^{p}(W) \cap \bigcap_{k \geq 1} B_{j}^{p}[B^{p}]^{k}(W)
\]

\[
= B_{j}^{p}(W) \cap \bigcap_{k \geq 1} B_{j}^{p}B_{-j}^{p}[B^{p}]^{k-1}(W)
\]

\[
= W_{j} \cap B_{j}^{p}(W_{-j}) \cap \bigcap_{k \geq 1} B_{j}^{p}B_{-j}^{p}[B^{p}]^{k-1}(W)
\]

\[
= W_{j} \cap B_{j}^{p} \left[ W_{-j} \cap \bigcap_{k \geq 1} B_{-j}^{p}[B^{p}]^{k-1}(W) \right]
\]

\[
= W_{j} \cap B_{j}^{p}C_{-j}^{p}(W).
\]

The last equality follows from the first (reversing the roles of players.)
D.2. **Proof of Lemma 7.**

**Proof.** We fix an arbitrary \( l \) and prove the lemma by induction on \( k > l \). Let \( z^k \in Y^k_l \), so that \( z^k(\Delta \Omega \times \{ y^l \}) = 1 \). We define \( \varphi^k_l(z^k) \in U^{k+1} \) as follows.

1. \( \operatorname{marg}_{\Delta \Omega} \varphi^k_l(z^k) = \operatorname{marg}_{\Delta \Omega} z^k \)
2. \( \varphi^k_l(z^k)[\Delta \Omega \times \{ y^k \}] = 1 \)

The second part is equivalent to part 2 of the lemma. To verify that part 1 of the lemma is satisfied, recall

\[
\operatorname{proj}_{k+1}^k \varphi^k_l(z^k) = \Delta \left[ \operatorname{id}_{\Delta \Omega} \times \operatorname{proj}_{k-1}^k \varphi^k_l(z^k) \right]
\]

(see Section 2.2.)

Since \( k = l + 1 \), the coherency of \( y \) implies \( \operatorname{proj}_{k-1}^k y^k = y^l \), thus the right-hand side is the unique measure \( \nu \) satisfying

1. \( \operatorname{marg}_{\Delta \Omega} \nu = \operatorname{marg}_{\Delta \Omega} \varphi^k_l(z^k) = \operatorname{marg}_{\Delta \Omega} z^k \)
2. \( \nu(\Delta \Omega \times \{ y^l \}) = 1 \).

which is \( z^k \).

For the inductive step, assume that for the mapping \( \varphi^{k-1}_l \) has been defined and satisfies the two conditions. We define

\[
\varphi^k_l = \Delta \left[ \operatorname{id}_{\Delta \Omega} \times \varphi^{k-1}_l \right] : Y^k_l \to U^{k+1}.
\]

By the induction hypothesis,

\[
\operatorname{id}_{\Delta \Omega} \times \operatorname{proj}_{k-1}^k \circ \operatorname{id}_{\Delta \Omega} \times \varphi^{k-1}_l = \operatorname{id}_{\Delta \Omega} \times \operatorname{id}_{Y^{k-1}_l}
\]

hence

\[
\operatorname{proj}_{k+1}^k \circ \varphi^k_l = \Delta \left[ \operatorname{id}_{\Delta \Omega} \times \operatorname{proj}_{k-1}^k \right] \circ \Delta \left[ \operatorname{id}_{\Delta \Omega} \times \varphi^{k-1}_l \right]
\]

\[
= \Delta \left[ \operatorname{id}_{\Delta \Omega} \times \operatorname{id}_{Y^{k-1}_l} \right]
\]

\[
= \Delta \left[ \operatorname{id}_{\Delta \Omega \times Y^{k-1}_l} \right]
\]

\[
= \operatorname{id}_{\Delta(\Delta \Omega \times Y^{k-1}_l)}
\]

\[
= \operatorname{id}_{Y^{k}_l}
\]

establishing the first part of the lemma. For the second part, let \( z^k \in Y^k_l \).

\[
\operatorname{marg}_{U^k} \varphi^k_l(z^k) = \operatorname{marg}_{U^k} \left[ \operatorname{id}_{\Delta \Omega} \times \varphi^{k-1}_l \right](z^k)
\]

\[
= \Delta \left[ \varphi^{k-1}_l \right] \operatorname{marg}_{U^{k-1}} z^k
\]
and by the induction hypothesis, \( \varphi_l^{k-1}(Y_l^{k-1}) \subseteq Y_{l+1}^k \), hence
\[
\Delta \varphi_l^{k-1}(\Delta Y_l^{k-1}) \subseteq \{ \nu \in \Delta U^k : \nu(Y_l^k) = 1 \}
\]
therefore, in particular since \( z \in Y_l^k \implies z^k(\Delta \Omega \times Y_l^{k-1}) = 1 \),
\[
\Delta [\varphi_l^{k-1}] \text{ marg } z^k(Y_l^k) = 1
\]
implying \( \text{marg}_{U^k} \varphi_l^k(z^k)(Y_l^k) = 1 \), i.e. \( \varphi_l^k(z^k) \in Y_{l+1}^{k+1} \).

D.3. Technical result. Here, we prove a useful technical result.

**Lemma 17.** Suppose that \( E \) is separable and metrizable, \( A \) is a finite set, and let \( \{V_a\}_{a \in A} \) be an open covering of \( E \). Let \( \mu \in \Delta E \) be a measure over \( E \). Consider a mapping \( \sigma : E \to \Delta A \) such that \( \sigma \) is adapted to the covering \( \{V_a\}_{a \in A} \), i.e.
\[
\sigma(e)(a) > 0 \implies e \in V_a.
\]

There is a sequence of continuous mappings \( \sigma^m : E \to \Delta A \), each adapted to \( \{V_a\}_{a \in A} \) such that
\[
\sigma^m \to \sigma, \ \mu\text{-almost surely}.
\]

**Proof.** By standard topological arguments, for each \( V_a \), there exists a sequence of continuous functions \( \alpha_a^m : E \to [0,1] \) such that \( \alpha_a^m(e) > 0 \) if and only if \( e \in V_a \) and \( \alpha_a^m \) converges pointwise to the indicator function for \( V_a \). Also there is a sequence of continuous mappings \( \tau^m : E \to \Delta A \), such that \( \tau^m \to \sigma, \mu\text{-almost surely} \).

Note that for any \( e \in E \), \( \sum_{a \in A} \alpha_a^m(e) > 0 \). Construct the sequence of mappings \( \sigma^m : E \to \Delta A \) as follows. For any \( e \in E \), for any \( a \in A \), let
\[
\sigma^m(e)(a) := \frac{\alpha_a^m(e) \left[ \tau^m(e)(a) + \frac{1}{m} \right]}{\sum_{a' \in A} \alpha_{a'}^m(e) \left[ \tau^m(e)(a') + \frac{1}{m} \right]}.
\]

By construction, \( \sigma \) is continuous. Moreover, for each \( m \), \( \sigma^m(e)(a) > 0 \) if and only if \( e \in V_a \). For \( \mu \)-almost all \( e \in V_a \),
\[
\lim_{m \to \infty} \alpha_a^m(e) \tau^m(e)(a) \to \sigma(e)(a)
\]
Thus \( \sigma^m \to \sigma, \ \mu\text{-almost surely} \).
D.4. \textit{m-rationalizable correspondence.} In order to prove Lemma 8, we need some results about \textit{m}-rationalizable actions. We start with a definition. In Ely and Peski (2006), we showed that the set of rationalizable actions can be obtained through the iterative elimination of not best responses (similar fact for correlated rationalizability is used in Dekel, Fudenberg, and Morris (2006b)). This can be easily extended to \( \varepsilon \)-rationalizability. Let us denote by \( R_i^m(\cdot|G, \varepsilon) \) the correspondence of actions obtained in the \( m \)th round of elimination for player \( i \) in game \( G \). Then, the quoted results state that

\[
R_i(\cdot|G, \varepsilon) = \bigcap_{m} R_i^m(\cdot|G, \varepsilon) .
\]

We refer to \( R_i^m(\cdot|G, \varepsilon) \) as the correspondence of \textit{m}-rationalizable actions.

It is easy to show that the set of type \( t_j \)'s \textit{m}-rationalizable correspondence depends only on type \( t_j \)'s hierarchy of beliefs: for any player \( j \), any types \( t_j, t'_j \in T_j \), any game \( G \) and any \( \varepsilon > 0 \), if \( u_j^T(t_j) = u_j^T(t'_j) \), then \( R_j^m(t_j|G, \varepsilon) = R_j^m(t'_j|G, \varepsilon) \). \footnote{This can be shown using the product game construction in the proof of Lemma 19.} Thus we can view \( R_j^m(\cdot|G, \varepsilon) \) to be correspondence whose domain is \( U_j(\Delta \Omega) \). By standard arguments, this correspondence is upper hemi-continuous. For any \( m \), player \( j \), game \( G \), action \( a_j \) and hierarchy \( u_j \), define

\[
h_j^m(a_j, G, u_j) = \inf \{ \varepsilon : a_j \in R_j^m(u_j|G, \varepsilon) \}.
\]

\textbf{Lemma 18.} \( h_j^m(a_j, G, u_j) \) is continuous in \( u_j \).

\textit{Proof.} The upper semi-continuity of \( h_j^m(a_j, G, \cdot) \) follows from the upper hemi-continuity of the correspondence \( R(\cdot|G, \varepsilon) \). We prove lower semi-continuity by induction on \( m \). When \( m = 0 \), by definition \( h_j^0(a_j, G, u_j) \equiv 0 \). Now assume that the Lemma holds for \( m - 1 \). Fix a player \( j \), a game \( G \), an action \( a_j^* \) and a hierarchy \( u_j^* \). Let \( h^* = h_j^m(a_j^*, G, u_j^*) \). We are going to show that for any \( \varepsilon > 0 \), there is a neighborhood \( V \ni u_j^* \), such that \( h_j^m(a_j^*, G, u_j) < h^* + \varepsilon \) for any \( u_j \in V \).

Let \( L = (L_j, \mu_j) \) be the type space from the second part of Theorem 1. Find a type \( t_j^* \in L_j \), such that \( u_j^L(t_j) = u_j^* \). Let \( \sigma_{-j} \) be a strategy of player \(-j\) that makes \( a_j^* \) an \( h^* \)-interim best response for type \( t_j^* \). For any action \( a_{-j} \in A_{-j} \), define

\[
V_{a_{-j}} = \{ a_{-j} : h_j^m(a_{-j}, G, u_{-j}) < h^* + \varepsilon/2 \}.
\]

By the induction hypothesis, the collection \( \{ V_{a_{-j}} \}_{a_{-j} \in A_{-j}} \) is an open covering of \( L_{-j} \). Also, if \( \sigma_{-j} (t_{-j}) (a_{-j}) > 0 \) for some type \( t_{-j} \), then \( t_{-j} \in V_{a_{-j}} \). By Lemma 17, there is a sequence of continuous strategies \( \sigma_{-j}^n \) converging \( \sigma_{-j} (t_{-j}) (a_{-j}) > 0 \), then \( t_{-j} \in V_{a_{-j}} \).

Take any sequence of hierarchies \( u_j^k \rightarrow u_j^* \) and find a sequence of types \( t_j^k \in L_j \), such that \( u_j^k(t_j^k) = u_j^* \) and \( \lim_{k \rightarrow \infty} \mu_j(t_j^k) = \mu_j(t_j) \) in the weak* topology (such a sequence exists by
the last part of Theorem 1). Then, for any action \(a_j \in A_j\)
\[
\lim_{n \to \infty} \lim_{k \to \infty} \mu_j(t^k_j) \left[ g_j(a^*_j, \sigma'^-j(t_{-j}), \omega) - g_j(a_j, \sigma^n_j(t_{-j}), \omega) \right] \\
= \lim_{n \to \infty} \mu_j(t^k'_j) \left[ g_j(a^*_j, \sigma'^-j(t_{-j}), \omega) - g_j(a_j, \sigma^n_j(t_{-j}), \omega) \right] \leq h^*.
\]
This implies that there is a neighborhood \(V \ni u^*_j\) such that for all types \(u_j \in V\), \(a^*_j\) is
\((h^* + \varepsilon/2)\)-interim rationalizable, i.e.
\[
h^*_j(a^*_j, G) < h^* + \varepsilon.
\]
\[\square\]

**Lemma 19.** Suppose that \(v_i \notin u^*_i\). There are game \(\bar{G} = (\bar{A}_j, g_j)\), action \(a_i \in \bar{A}_i\), \(\varepsilon > 0\) and \(m\) such that
\[
a_i \in R^m_i(v_i|\bar{G}, 0) \quad \text{and} \quad a_i \notin R^m_i(u^*_i|\bar{G}, \varepsilon).
\]

**Proof.** By the definition of order, there is game \(\bar{G}\) and action \(a_i\), such that \(a_i \in R_i(v_i|\bar{G}, 0)\)
and \(a_i \notin R_i(u^*_i|\bar{G}, 0)\). By Lemma 5, there is \(\varepsilon > 0\), such that \(a_i \notin R_i(u^*_i|\bar{G}, \varepsilon)\). By (D.2),
there is \(m\), such that the thesis of the Lemma holds. \(\square\)

**Lemma 20.** Suppose that \(v_i \notin u^*_i\). There are neighborhood \(u_i \in U_i\), game \(\bar{G} = (\bar{A}_j, g_j)\),
action \(a_i \in \bar{A}_i\), \(\varepsilon > 0\) and \(m\) such that
\[
a_i \in R^m_i(u_i|\bar{G}, 0) \quad \text{for all} \quad u_i \in U_i \quad \text{and} \quad a_i \notin R^m_i(u^*_i|\bar{G}, \varepsilon).
\]

**Proof.** Let \(\bar{G}', a'_i\), be the game from Lemma 19. Because of Lemma 18, there is a neighborhood
\(U_i \ni v_i\) open in the product topology and \(\varepsilon > \varepsilon' > 0\), such that
\[
a'_i \in R^m_i(u_i|\bar{G}', \varepsilon') \quad \text{for all} \quad u_i \in U_i \quad \text{and} \quad a'_i \notin R^m_i(u^*_i|\bar{G}', \varepsilon).
\]
The Lemma follows from the argument similar to the proof of Lemma 5. \(\square\)

**D.5. Proof of Lemma 8.** We first prove a preliminary result.

**Lemma 21.** Suppose that \(v_i \notin u^*_i\). There is \(\varepsilon > 0\), a neighborhood \(U_i \ni v_i\), a game
\(G = (A_j, g_j)\), such that \(A_i = A^0_i \times A^1_i\) and

(1) For any \(a_{-i} \in A_{-i}\), any \(a^1_i \in A^1_i\), any \(a^0_i, a^0_{-i} \in A^0_i\), any \(\omega\),
\[
g_{-i}(a_{-i}, (a^0_i, a^1_i), \omega) = g_{-i}(a_{-i}, (a^0_{-i}, a^1_i), \omega).
\]

(2) There are correspondences \(A^0 : U_i(\Delta\Omega) \Rightarrow A^0_i, A^1 : U_i(\Delta\Omega) \Rightarrow A^1_i\), such that for all
\(u_i\),
\[
R_i(u_i|G, 0) = A^0(u_i) \times A^1(u_i).
\]
There is an action \( a_0^* \subseteq A_0 \), such that
\[
\{a_0^*\} \times A_1^*(u_i) \subseteq R_i(u_i|G, 0) \quad \text{for all } u_i \in U_i,
\]
\[
\{a_0^*\} \times A_1^*(u_i^*) \cap R_i(u_i^*|G, \varepsilon) = \emptyset.
\]

**Proof.** Let \( u_i \in U_i \), game \( \vec{G} = (\vec{A}_j, \vec{g}_j) \), action \( a_i \in \vec{A}_i \), \( \varepsilon > 0 \) and \( m \) be as in Lemma 20. Define game \( G = (A_j, g_j) \) : for any player \( j \), let \( A_j = (\vec{A}_j)^m \) and
\[
g_j((a_j^1, \ldots, a_j^m), (a_{-j}^1, \ldots, a_{-j}^m), \omega) = \sum_{k=1}^{m-1} g_j(a_j^k, a_{-j}^{k+1}, \omega).
\]
Notice that for any \( \varepsilon' \geq 0 \),
\[
R_i(\cdot|G, \varepsilon') = R_i^m(\cdot|G, \varepsilon') \times \ldots \times R_i^0(\cdot|G, \varepsilon').
\]
Let \( A_0^i = \vec{A}_i \) and \( A_1^i = (\vec{A}_i)^{m-1} \). The thesis of the Lemma follows. □

**Proof of Lemma 8.** For any \( v_i \in W, v_i \not\in u_i^* \) and we can apply Lemma 21 to find \( \varepsilon(v_i) > 0 \), neighborhoods \( U_i(v_i) \ni v_i \), games \( G(v_i) \). Let \( v_1^i, \ldots, v^K_i \in W \) be a finite sequence of hierarchies, such that \( W \subseteq \bigcup_k U_i(v_k^i) \). To shorten the notation, let \( G^k = (A^k_j, g_j) := G(v_k^i) \). Define game \( G \) as the product game \( G = G^1 \times \ldots G^K : \)
\[
A_0^i := \prod_{k=1}^{K} A_i^{0,k}, A_1^i := \prod_{k=1}^{K} A_i^{1,k} \quad \text{and}
\]
\[
A_0^* = \left\{ a_i^0 \in A_0^i : (a_i^0)_k = a_i^{0,k} \text{ for some } k = 1, \ldots, K \right\},
\]
and for any \( (a_j^k) \in A_j, (a_{-j}^k) \in A_{-j} \), let
\[
g_j((a_j^k), (a_{-j}^k), \omega) = \sum_{k=1}^{K} g_j^k(a_j^k, a_{-j}^k, \omega).
\]
Notice that for any \( \varepsilon \geq 0 \),
\[
R(\cdot|G, \varepsilon) = R(\cdot|G^1, \varepsilon) \times \ldots \times R(\cdot|G^K, \varepsilon)
\]
The thesis of the Lemma follows from the construction. □

**D.6. Proof of Lemma 9.** We assume here that
\[
\max_{j,a,\omega} |g_j(a, \omega)| \leq 1. \quad (D.3)
\]
This is w.l.o.g., as one can always scale the payoffs without affecting the set of rationalizable actions.
We prove the Lemma by induction on $k$ and player $j$. In order to shorten the notation, define

$$E_j^k := U_j (\Delta \Omega) \setminus B_j^p \left( [B^p]^k (W_i \times U_{-i} (\Delta \Omega)) \right).$$

Note that, by construction, sets $E_j^k$ are open. When $k = 0$ and $j = i$, then

$$E_j^0 = E_i^0 = U_i^A.$$

Then, the Lemma is a consequence of the fact that $U_i^A$ is open and for any hierarchy in $U_i^A$, any $\varepsilon \leq \varepsilon^A$, the set of $\varepsilon$-rationalizable actions is equal to $A_i^\varepsilon$.

Assume that the thesis of (Lemma 9) holds for $k \geq 0$ and $j = i$. Take any $u_j^0 \in E_j^k$ and suppose that $a_{-i}^*$ is a rationalizable action at $u_{-i}^0$. Let $L = (L_j, \mu_j)$ be the type space from the second part of Theorem 1. Let $l_{-i}^0 \in L_{-i}$ be a type with a hierarchy $u_{L_{-i}}^0 (l_{-i}^0) = u_{-i}^0$. Let $l_{-i}^n \in L_{-i}$ be a sequence of types of player $-i$ with hierarchies $u_{L_{-i}}^n (l_{-i}^n) = u_{-i}^n$. By construction of type space $L_i$ (see Ely and Peski (2006)),

$$\mu_{L_i}^0 (l_{-i}^0) \to \mu_{L_i}^0 (l_{-i}^0) \quad \text{(D.4)}$$

in the weak sense.

Say that behavioral strategy $\sigma_j : L_j \to \Delta A_j$ is $\delta$-rationalizable, if for any type $l_j \in L_j$, actions played by $u_j$ are $\delta$-rationalizable, $\sigma_j^0 (u_j) (R (l_j | G, \delta)) = 1$. Let $\sigma_i^0$ be a 0-rationalizable strategy of player $i$, against which action $a_{-i}^*$ is a best response of type $l_{-i}^0$. We need an intermediate result.

**Lemma 22.** There is a sequence of 6$p$-rationalizable strategies $\sigma_i^m : L_i \to \Delta A_i$, such that $\sigma_i^m$ is continuous on $(u_i^L)^{-1} \left( E_i^k \right)$ and

$$\sigma_i^m \to \sigma_i^0, \mu_{L_i}^0 (l_{-i}^0) \text{-almost surely.} \quad \text{(D.5)}$$

**Proof.** By the inductive hypothesis, there are sets of hierarchies

$$V_i (a_i) \subseteq E_i^k,$$

such that for each $a_i$ set $V_i (a_i)$ is open in the product topology and

$$\{ u_i \in E_i^k : a_i \in R (u_i | G, 0) \} \subseteq V_i (a_i) \subseteq \{ u_i \in E_i^k : a_i \in R (u_i | G, 6p) \}.$$

In particular, $\{ V_i (a_i) \}_{a_i \in A_i}$ is an open cover of $E_i^k$. By Lemma 17, there is a sequence of functions $\sigma_i^{m*} : E_i^k \to \Delta A_i$, such that for any $u_i \in E_i^k$,

$$\sigma_i^{m*} (u_i) (a_i) > 0 \text{ iff } u_i \in V_i (a_i) \text{ and}$$

and $\sigma_i^{m*} \to \sigma_i^0$ almost surely with respect to measure $\mu_{L_i}^0 (l_{-i}^0) \left( \cdot \right) A_i^k$. 

Define $\sigma^m_i : L_i \to \Delta A_i$ as

$$\sigma^m_i (l_i) = \begin{cases} \sigma^m_i (u^L_i (l_i)) , & l_i \in E^k_i , \\
\sigma^0_i (l_i) , & \text{otherwise}. \end{cases}$$

Since $u^L_i$ is continuous (see Theorem $\sigma^m_i$).

Let $\alpha^m : L_i \to [0,1]$ be a sequence of continuous functions, such that $\alpha^m_i (l_i) = 0$ for any type $l_i \notin (u^L_i)^{-1} E^k_i$ and $\lim_{m \to \infty} \alpha^m_i (l_i) = 1$ for any $l_i \in (u^L_i)^{-1} E^k_i$. Since $E^k_i$ is open and $u^L_i$ is continuous, $(u^L_i)^{-1} (E^k_i)$ is open and such a sequence trivially exists. For each $a_{-i} \in A_{-i}$,

$$\begin{align*}
|\mu^L_i (l_{-i}) [g_{-i} (a_{-i}, \sigma^m_i, \omega)] - \mu^L_i (l_0_{-i}) [g_{-i} (a_{-i}, \sigma^0_i, \omega)]| \\
\leq |\mu^L_i (l_{-i}) [g_{-i} (a_{-i}, \sigma^m_i, \omega)] - \mu^L_i (l_0_{-i}) [g_{-i} (a_{-i}, \sigma^m_i, \omega)]| \\
+ |\mu^L_i (l_0_{-i}) [g_{-i} (a_{-i}, \sigma^m_i, \omega)] - \mu^L_i (l_0_{-i}) [g_{-i} (a_{-i}, \sigma^0_i, \omega)]| \\
\leq |\mu^L_i (l_{-i}) [g_{-i} (a_{-i}, \sigma^m_i, \omega)] - \mu^L_i (l_0_{-i}) [g_{-i} (a_{-i}, \sigma^m_i, \omega)]| \\
+ |\mu^L_i (l_0_{-i}) [(1 - \alpha^m_i)]| + |\mu^L_i (l_0_{-i}) [(1 - \alpha^m_i)]| \\
+ |\mu^L_i (l_0_{-i}) [(g_{-i} (a_{-i}, \sigma^m_i, \omega) - g_{-i} (a_{-i}, \sigma^0_i, \omega))]\alpha^m_i| \\
+ |\mu^L_i (l_0_{-i}) [(g_{-i} (a_{-i}, \sigma^m_i, \omega) - g_{-i} (a_{-i}, \sigma^0_i, \omega))] (1 - \alpha^m_i)|.
\end{align*}$$

In the second inequality, we used (D.3). Because of the convergence (D.4), and the fact that $g_{-i} (a_{-i}, \sigma^m_i (l_i), \omega) \alpha^m_i (l_i)$ is a continuous function of $l_i$,

$$\lim_{n \to \infty} \sup \left| \mu^L_{-i} (l_{-i}) [g_{-i} (a_{-i}, \sigma^m_i, \omega) \alpha^m_i] - \mu^L_{-i} (l_0_{-i}) [g_{-i} (a_{-i}, \sigma^m_i, \omega) \alpha^m_i] \right| = 0.$$

Because of (D.5),

$$\lim_{m \to \infty} \sup \left| \mu^L_{-i} (l_0_{-i}) [(g_{-i} (a_{-i}, \sigma^m_i, \omega) - g_{-i} (a_{-i}, \sigma^0_i, \omega))] \alpha^m_i \right| = 0,$$

Since

$$\lim_{m \to \infty} \mu^L_{-i} (l_0_{-i}) [1 - \alpha^m_i] = 1 - \mu^L_{-i} (l_0_{-i}) (E^k_i) \leq p,$$

we have

$$\lim_{m \to \infty} \sup_{n \to \infty} \left[ \left| \mu^L_{-i} (l_{-i}) [1 - \alpha^m_i] \right| + \left| \mu^L_{-i} (l_0_{-i}) [1 - \alpha^m_i] \right| \right]$$

$$\leq \lim_{m \to \infty} \sup_{n \to \infty} \left( 4 \mu^L_{-i} (l_0_{-i}) [1 - \alpha^m_i] \right) \leq 4p.$$
This shows that for high $n$ and $k$, $a^*_{-i}$ is a $6p$-best response of $l^m_{-i}$ against $\sigma^m_i$, and it shows th inductive thesis for $k$ and $j = -i$.

Assume that the thesis of (Lemma 9) holds for $k$ and $j = -i$. A very similar argument to the one above demonstrates that the inductive thesis holds for $k + 1$ and $j = i$. This ends the proof of the Lemma.

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