Collusion via Resale

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Abstract

The English auction is susceptible to a certain form of tacit collusion when post-auction inter-bidder resale is allowed. We show this by constructing a continuum of equilibria where, with positive probability, one bidder wins the auction without any competition and divides the spoils by optimally reselling the good to the other bidders. Such equilibria support a collusive bidding pattern without requiring the colluders to make any commitment on bidding behavior or post-bidding spoil-division. The equilibria are valid for any number of asymmetric or symmetric bidders, arbitrary reserve prices, and various resale market rules. In symmetric environments, these equilibria interim Pareto dominate (among bidders) the standard value-bidding equilibrium.

Preliminary Version.

1 Introduction

In private-value English auctions without resale it is a dominant strategy for each participant to bid up to her use value. With resale, value-bidding remains an equilibrium outcome, but there is no dominant strategy. Resale opens the possibility that some bidders will optimally drop out at a price below their use values. They prefer to let a competitor win and buy from her in the resale market. The existence of non-value-bidding equilibria is important because the celebrated advantages of the English auction, in particular efficiency, are based on value-bidding, and because resale is possible in most applications.

In this paper we construct a family of non-value-bidding equilibria for the English auction with resale that exist in any independent private value

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environment (symmetric or asymmetric) for any number of bidders. Each equilibrium in this family is identified by the choice of a speculating bidder and a cutoff type, below which all bidders, except the speculating bidder, bid zero. All bidders with types above the cutoff value-bid. In cases where the speculating bidder wins the initial auction and has sufficiently low type, she will offer the item for resale instead of consuming it. Because the bidders in the resale market retain private information about their types the outcome of the resale auction may be inefficient.

Equilibria with a speculating bidder exist for any cutoff between 0 and highest type that is possible for all non-speculating bidders.\(^1\) The speculating bidder is strictly better off than under value-bidding, for any cutoff and for all of her types. Thus any individual bidder would like to play the role of the speculating bidder in one of these equilibria. Moreover, in symmetric environments the equilibria we describe provide bidders with a higher aggregate payoff than under value-bidding. This sets up an opportunity for a form of tacit collusion. By using a publicly observed random device (or sunspot) to select the speculating bidder the bidders can distribute the aggregate payoff in a way that makes everyone better off than under value-bidding, regardless of their use-values. The resale market combined with the sunspot is a collusive device. Hence, resale facilitates noncooperative collusive equilibria that Pareto dominate the standard value-bidding equilibrium.\(^2\)

In the collusive equilibria we describe no binding agreement among the colluding bidders is required. The recommendation made by the sunspot device is not binding. Rather, the sunspot plays the role of a correlating device in a correlated equilibrium (however private recommendations are not required). Once the sunspot identifies the speculating bidder it is in the interest of each bidder to bid accordingly in the initial auction based on the belief that others will follow their assigned roles. After the initial auction, the colluding bidders optimally carry out their final allocation through the resale market, without relying on any form of enforcement.

\(^1\)There also exist equilibria with “extreme cutoffs” where the highest possible type of at least one non-speculating bidder strictly prefers to bid zero. However, there are reasons, having to do with robustness, why intermediate cutoffs are more plausible than extreme cutoffs. These are explained in remark number 5 following Proposition 1.

\(^2\)Readers who are familiar with U.S. litigation history might draw some parallels to the famous phases-of-the-moon bidding ring that was operated by electrical equipment suppliers in the 1950s. While this sounds similar, it is important to note that this scheme earned the phases-of-the-moon designation because it involved an explicit 2-week rotation to determine the low bidder. While it perhaps could have been, bidding was not actually determined by the phase of the moon. See Smith (1961).
Like standard collusive arrangements the proposed equilibria allow the bidders to extract much or all of the surplus from the seller. However, with the proposed equilibria, many of the obstacles to successful collusion inherent in standard mechanisms are avoided. McAfee and McMillan (1992) identify four obstacles to successful collusion. They point out that the most significant factor in the downfall of many cartels is the issue of “dividing the spoils.” Here this is resolved fairly, in a way that is acceptable to all, by the sunspot device.\(^3\) The second obstacle, enforcement, is also avoided here since the participants obtain collusive payoffs through fully rational, noncooperative equilibrium behavior.\(^4\) The fact that formal agreements and enforcement are not required mitigates the destructive impact of new entrants, which is the third obstacle described by McAfee and McMillan. Finally, the equilibria are robust to the introduction of positive reserve prices. Hence, actions taken by the seller to destabilize collusion, the final obstacle raised by McAfee and McMillan, do not eliminate collusive equilibria.

The absence of any sort of preliminary “knock-out” auctions, side payments or enforcement measures in our collusive equilibria suggests that, like other forms of tacit collusion, it is unlikely that participants in the collusive equilibria would be subject to legal prosecution.\(^5\) As such, evidence of such arrangements cannot be found in court records. However, instances of resale among auction participants are not uncommon and these instances have features in common with our construction. For example, Pagnozzi (2007) reports that in the UK 3.4 GHz spectrum auction, one company dropped out of the two auctions at a low price and then later obtained the licenses from the winners (since resale was prohibited this was done via take-overs). Likewise, in procurement auctions horizontal subcontracting to initial auction participants is common and this is a form of inter-bidder resale.\(^6\)

Our equilibrium construction builds on earlier results by Garratt and

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\(^3\)Dividing the spoils is in fact the main issue addressed by McAfee and McMillan (1992), whose solution also involves randomization. They describe an equilibrium for a first-price auction in which bidders submit identical bids and allow the auctioneer to randomly determine the winner.

\(^4\)The simplest collusive mechanisms (eg. Robinson, 1985) require some type of enforcement or punishment to prevent bidders from cheating on the collusive agreement. More elaborate schemes, cf. Graham and Marshall (1987), McAfee and McMillan (1992), and Mailath and Zemsky (1991), are incentive compatible however they require that bidders participate in some type of external auction and adhere to its payment recommendations.

\(^5\)Prosecution under Section 1 of the Sherman Act seems to require some proof of illegal communication, transfer payments, or evidence of enforcement activity.

Tröger (2006). Garratt and Tröger (2006) construct a continuum of equilibria similar to those constructed here, that apply to second-price or English auctions where the aggressive bidder is commonly known to have zero value for the good on sale, i.e., this bidder acts as a pure speculator. Apart from the speculator, an arbitrary number of “regular” bidders with symmetric independent private values participate in the market. The equilibrium constructions provided in Garratt and Tröger (2006) do not utilize dynamic features of the English auction format. In contrast, the current construction utilizes information revealed at each stage of bidding. In particular, the drop-out price of the speculating bidder depends on revealed aspects of her last surviving competitor’s value distribution. This is the key feature of our construction that allows us to consider environments with an arbitrary number of asymmetric bidders.

Our equilibrium construction for the English auction carries over to the second-price auction provided all bidders, except possibly the speculating bidder, have the same value distribution. This is because in this environment the speculating bidder’s equilibrium bid does not change as bidders drop out of the English auction. Hence, in the case of the second-price auction, the current construction is a generalization of Garratt and Tröger (2006) that allows the speculator to have a private value.

Recent work by Lebrun (2007) and Hafalir and Krishna (2007) compare revenue in first- and second-price auctions with resale. Hafalir and Krishna show that in 2-bidder, asymmetric auctions there exists a “general revenue ranking” in favor of first-price auctions, provided bidders play the value-bidding equilibrium in the second price auction. Lebrun shows that this ranking does not necessarily hold when behavior (mixed) strategies are allowed.\(^7\) The existence of the additional equilibria described in this paper does not change the revenue ranking established by Hafalir and Krishna, since the revenue generated by any of our equilibria is no greater than the revenue received by the seller in the value-bidding equilibrium.

Collusive equilibria have been constructed for multi-unit auctions by Milgrom (2000), Brusco and Lopomo (2002) and Engelbrecht-Wiggans and Kahn (2005), however resale does not play a role.\(^8\) In these multi-unit environments bidders signal their preferences in early rounds and then optimally abstain from bidding on other bidders’ preferred items. Interestingly, the

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\(^7\)For the case where two bidders have different value distributions Lebrun (2007) shows that either auction format can have higher expected revenue; it depends on equilibrium selection in second-price auction.

\(^8\)Pagnozzi (2007) analyzes multi-unit auctions with resale in environments with commonly known use values.
open aspect of the ascending English auction is essential in their construction, as it is here in the case of ex-ante asymmetric bidders.

In Section 2 we outline the model. In Section 3 we construct the family of equilibria for the English auction, which are described in Proposition 1. In Section 4 we show how a sunspot device can be used to coordinate equilibrium expectations and allow all bidders, regardless of their type, to obtain a higher expected payoff than in the standard value-bidding equilibrium. These collusive equilibria are outlined in Proposition 2. Some proofs are relegated to the Appendix.

2 Model

We consider environments with \( n \geq 2 \) risk-neutral bidders who are interested in consuming a single indivisible private good. Bidder \( i \in N = \{1, \ldots, n\} \) has the privately known type use value, or type, \( t_i \in T_i = [0, \bar{t}_i] \) \((\bar{t}_i > 0)\) for the good. From the viewpoint of the other bidders, \( t_i \) is distributed according to a probability distribution with cumulative distribution function (c.d.f.) \( F_i \).\(^9\) We assume that \( F_i \) has a density \( f_i \) that is positive and continuous on \( T_i \) and identically 0 elsewhere. We add the standard assumption that the hazard rates are weakly increasing; i.e., for all \( i \in N \), the mapping 

\[ t \mapsto f_i(t)/(1 - F_i(t)) \]

is weakly increasing on \([0, \bar{t}_i)\).

The type space is denoted\(^10\)

\[ T = T_1 \times \ldots T_n. \]

We consider a 2-period interaction, which begins after each bidder \( i \in N \) has privately observed her use value \( t_i \in T_i \).

In period 1, the good is offered via an English auction. The auctioneer continuously raises the price beginning at price \( p = 0 \) (the impact of a reserve price will be discussed below). Initially, all bidders are “in.” As the price \( p \) rises, each bidder can irrevocably “drop out” at any point. The drop-out events divide the auction into stages. At each stage \( \kappa = 0, 1, \ldots \), the set of bidders who are in is denoted \( S_\kappa \subseteq N \) \((S_0 = N)\) and the current price is denoted \( p \). Each bidder in \( S_\kappa \) has a planned drop-out price in \([p, \infty)\). The

\(^9\)Throughout the paper, any c.d.f. is to be understood as a function defined on \( \mathbb{R} \), even if the support of the underlying distribution is a proper subset of \( \mathbb{R} \).

\(^{10}\)Throughout the paper, we will use boldface letters to denote multidimensional quantities.
auctioneer raises the price to the minimum among the planned drop-out prices, \( p_{\kappa+1} \), and the set \( S_{\kappa+1} \) of bidders that had planned drop-out prices strictly greater than \( p_{\kappa+1} \) stay in. The auction ends at a stage \( \kappa \) where \( S_{\kappa} \) is a singleton or empty, that is, when only one bidder is in or all bidders have dropped out.\(^{11}\) Accordingly, the set of auction histories where bidder \( i \in N \) moves (because she is in and the auction is not over) is denoted\(^{12,13}\)

\[
\mathcal{H}_i = \{(S_0, p_1, S_1, \ldots, p_\kappa, S_\kappa, p) \mid \kappa \geq 0, \ 0 \leq p_1 \leq \ldots \leq p_\kappa \leq p, \ N = S_0 \supset S_1 \supset \ldots \supset S_\kappa \ni i, \ |S_\kappa| \geq 2\},
\]

where \( \kappa = 0 \) yields the initial history \((N, 0)\). The set of terminal histories is denoted

\[
\mathcal{H}_{\text{term}} = \{(S_0, p_1, S_1, \ldots, p_\kappa, S_\kappa, p) \mid \kappa \geq 1, \ 0 \leq p_1 \leq \ldots \leq p_\kappa = p, \ N = S_0 \supset S_1 \supset \ldots \supset S_\kappa, \ |S_\kappa| \leq 1\}.
\]

If the auction ends with one bidder staying in, then this bidder is the auction winner; otherwise all bidders have dropped out, there is a tie, and the auction winner is selected from \( S_{\kappa-1} \) by any deterministic rule.\(^{14}\) For any \( h \in \mathcal{H}_{\text{term}} \), let \( \text{win}(h) \) denote the auction winner. Observe that the price at which the winner was planning to drop out remains her private information, unless there is a tie. The drop-out prices of the losing bidders are called the losing bids. The winner pays the seller the final current price \( p \); this price equals the maximum losing bid \( p_\kappa \) and is called the auction price. The set of all auction histories is denoted

\[
\mathcal{H} = \mathcal{H}_1 \cup \ldots \cup \mathcal{H}_n \cup \mathcal{H}_{\text{term}}.
\]

The auction winner either consumes the good or becomes the period-2 seller, who offers the good in period 2 for resale to the losing bidders, which are called the period-2 buyers. Below we will be more specific about the resale market.

\(^{11}\)Observe that there are at most \( n \) stages, because at each stage at least one bidder drops out.

\(^{12}\)We write \( X \supset Y \) if \( Y \) is a strict subset of \( X \).

\(^{13}\)A simpler notation for histories, recording only which bidder has dropped out at which price, would not be satisfactory. Observe that the “clock stops” when a bidder drops out, so that other bidders may follow by dropping out at the same price.

\(^{14}\)A probabilistic tieing rule, such that, for instance, each bidder in \( S_{\kappa-1} \) is selected with probability \( 1/|S_{\kappa-1}| \), could also be used, but would unnecessarily complicate the definitions (2) and (3), and the payoff formula (4).
2.1 Strategies and beliefs

At each non-terminal history, the bidders who are in consider when to drop out. Accordingly, a bidding profile

\[(\beta_{i,h})_{i \in N, h \in H_i}\]

determines, for each \(t \in T_i, i,\) and \(h,\) the price \(\beta_{i,h}(t)\) at which type \(t\) of bidder \(i\) plans to drop out given the history \(h.\) We assume that the planned drop-out point does not change as long as no drop-out event occurs,

\[\forall i \in N, \ h = (\ldots, p) \in H_i, \ p' \in [p, \beta_{i,h}(t)), \ t \in T_i : \ \beta_{i,h[p']}(t) = \beta_{i,h}(t), \quad (1)\]

where \(h[p']\) denotes the history that is identical to \(h\) except for a possibly higher current price \(p'.\) The justification for condition (1) is that instead of changing her mind along the way, bidder \(i\) can implement her eventual plan right away when the current price is \(p.\) We also assume that \(\beta_{i,h}\) is weakly increasing (i.e., we will focus on equilibria where every type of a given bidder will find it optimal to bid at least as high as all smaller types).

A resale-decision profile is a vector

\[(\gamma_h)_{h \in H_{\text{term}}}, \quad (2)\]

where \(\gamma_h(t) = 1\) if type \(t\) of the winning bidder offers the good for resale at the terminal history \(h,\) and \(\gamma_h(t) = 0\) if the winner consumes the good in period 1. We assume that \(\gamma_h\) is weakly decreasing (i.e., we will focus on equilibria where, if some type \(t\) consumes the good in period 1, all higher types will find it optimal to do the same.)

We restrict attention to equilibria where posterior beliefs remain stochastically independent across all players. Hence we model a belief profile as a vector

\[(G_h)_{h \in H} \quad (3)\]

with the following properties. For all \(h,\) \(G_h\) is the product of stochastically independent probability distributions on \(T_1, \ldots, T_n.\) If \(h\) is a non-terminal history, then \(G_h\) represents the belief about the bidders’ types at history \(h;\) if \(h\) is a terminal history, then \(G_h\) represents the belief about the bidders’ types at the beginning of period 2; that is, if \(h\) is a terminal history, then \(G_h\) incorporates both the information revealed at history \(h\) and the information revealed by the fact that the winner decides to offer the good for resale. The belief that prevails at the terminal history is called the post-auction belief.
Given a commonly known post-auction belief, the period-2 environment features independent private values and no resale.\footnote{An extension that allows for repeated resale will be discussed below.} Hence, the outcome of the period-2 interaction can be derived using standard methods. The period-2 outcome depends on the identity of the period-2 seller, on the post-auction belief, and on the bidders’ types. A period-2 outcome is a vector of functions $$(P_{ij}, Q_{ij})_{i \in N, \ j \in N \setminus \{i\}}$$ where, for every type profile $t = (t_1, \ldots, t_n) = (t_i, t_{-i}) \in T$ and every product measure $J$ on $T$, the number $P_{ij}(t, J)$ denotes the net expected period-2 payment made by bidder $i$, given that bidder $j$ is the period-2 seller and that $J$ represents the post-auction belief about the bidders’ types; $Q_{ij}(t, J)$ denotes the probability that bidder $i$ obtains the good in period 2.

### 2.2 Payoffs

Given a period-2 outcome as denoted above, the probability that period-2 buyer $i \in N$, given her type $t_i \in T_i$, obtains the good from the period-2 seller $j \in N \setminus \{i\}$, given the post-auction beliefs $J$, is denoted

$$q_{ij}(t_i, J) = \int_{T_{-i}} Q_{ij}(t, J) dJ_{-i}(t_{-i}),$$

where $T_{-i} = \prod_{k \neq i} T_k$ and $J_{-i}$ denotes the marginal distribution induced by $J$ on $T_{-i}$. Bidder $i$’s period-2 payoff is denoted

$$l_{ij}(t_i, J) = t_i q_{ij}(t_i, J) - \int_{T_{-i}} P_{ij}(t, J) dJ_{-i}(t_{-i}).$$

The probability that the period-2 seller $j$, given her type $t_j \in T_j$, keeps the good is denoted

$$q_j(t_j, J) = 1 - \sum_{k \in N \setminus \{j\}} \int_{T_{-j}} Q_{kj}(t, J) dJ_{-j}(t_{-j}).$$

The period-2 payoff of the period-2 seller $j$ is denoted

$$w_j(t_j, J) = t_j q_j(t_j, J) + \sum_{k \in N \setminus \{j\}} \int_{T_{-j}} P_{kj}(t, J) dJ_{-j}(t_{-j}).$$

We are now in a position to spell out the expected payoff of bidder $i \in N$ at any history $h \in \mathcal{H}_i$. Suppose that the type profile is given by $t \in T$. 

15 An extension that allows for repeated resale will be discussed below.
Moreover, beginning at \( h \) until the end of the auction, every bidder \( k \in N \setminus \{i\} \) behaves according to \( \beta_k \), and bidder \( i \) behaves according to \( \beta_i \) except that at \( h \) she plans to drop out at price \( b \).ootnote{More precisely, for all \( p' \in [p, b) \), she plans to drop out at price \( b \) at all histories \( h|p' \), where \( h|p' \) is identical to \( h \) except for the current price \( p' \).} Using the shortcut \( x = (i, h, b) \), let

\[
e(x, t) = (\ldots, S(x, t), p(x, t))
\]
denote the resulting terminal history of the auction. Bidder \( i \)'s expected payoff is

\[
u_{i,h}(b, t_i) = \int_{T_{-i}} \left( -p(x, t) + \max\{t_i, \delta w_i(t_i, G_{e(x,t)})\} \right) \mathbf{1}_{\text{win}(e(x,t))\cap i} dt + \delta \sum_{j \neq i} l_{ij}(t_i, G_{e(x,t)}) \mathbf{1}_{\text{win}(e(x,t))\cap j} dt_{j=1} d(G_{h_{-i}}(t_{-i})�(4)\]

where \( \delta \in (0, 1] \) denotes the discount factor, the max term reflects bidder \( i \)'s optimal decision whether to offer the good for resale, and \( (G_{h})_{-i} \) denotes the marginal distribution induced by \( G_{h} \) on \( T_{-i} \).

### 2.3 Perfect Bayesian equilibrium conditions

The equilibrium condition for the bidding profile is that drop-out prices are chosen optimally:ootnote{We rely on a version of the one-stage-deviation principle.}

\[
\forall i \in N, \ h = (\ldots, p) \in \mathcal{H}_i, \ t \in T_i : \ \beta_{i,h}(t) \in \arg\max_{b \geq p} u_{i,h}(b, t). \quad (5)
\]

The equilibrium condition for the resale-decision profile is

\[
\forall h \in \mathcal{H}_{\text{term}}, \ t \in T_{\text{win}(h)} : \ \gamma_h(t) = \begin{cases} 0 & \text{if } t > \delta w_{\text{win}(h)}(t, G_{h}), \\ 1 & \text{if } t < \delta w_{\text{win}(h)}(t, G_{h}). \end{cases} \quad (6)
\]

In order to state the equilibrium conditions for the belief profile, additional notation is needed. Bidders update their beliefs about other bidders' types based on the information revealed during the auction. A typical piece of information is that a bidder’s type lies in an interval. For all \( i \in N \) and all bounded (open, half-open, or closed) intervals \( I \subseteq \text{supp} F_i \), let \( \hat{F}_{i,I} \) denote
the updated c.d.f. that results from the information that bidder \( i \)'s type belongs to the interval \( I \). Defining \( s = \inf I \) and \( s' = \sup I \),

\[
\hat{F}_{i,I}(t) = \begin{cases} 
F_i(t) - F_i(s) & \text{if } t \in [s, s'), \\
F_i(s') - F_i(s) & \text{if } t \geq s', \\
1 & \text{if } t < s.
\end{cases}
\]

For any \( h = (S_0, p_1, S_1, \ldots, p_\kappa, S_\kappa, p) \in \mathcal{H} \), let the histories preceding \( h \) where drop-out events occurred be denoted by \( h_1, h_2, \ldots, h_\kappa \), and let \( h_0 = (S_0, 0) = (N, 0) \) denote the initial history. If bidder \( i \) has dropped out at stage \( l \), that is, if \( i \in S_l \setminus S_{l+1} \), then it is also known that, for all \( k < l \), at the history \( h_k \), bidder \( i \) was not willing to drop out at any price up to \( p_k \). Hence, the following interval of types is deemed possible for bidder \( i \) at the history \( h \):

\[
U_{i,h} = \begin{cases} 
\bigcap_{k=0}^{\kappa-1} \beta_{i,h_k}^{-1}((p_{k+1}, \infty)) & \text{if } i \in S_l \setminus S_{l+1}, \\
\bigcap_{k=0}^{\kappa-1} \beta_{i,h_k}^{-1}((p_k, \infty)) & \text{if } i \in S_\kappa.
\end{cases}
\] (7)

The equilibrium condition for the beliefs is that Bayes rule is applied whenever possible. At non-terminal histories, the condition is as follows:

\[
\forall i \in N, ~ h \in \mathcal{H} \setminus \mathcal{H}_{term} : \text{ if } U_{i,h} \neq \emptyset \text{ then } (G_h)_i = \hat{F}_{i,U_{i,h}}.
\] (8)

The same condition applies to losing bidders at terminal histories,

\[
\forall h \in \mathcal{H}_{term}, ~ i \in N \setminus \{\text{win}(h)\} : \text{ if } U_{i,h} \neq \emptyset \text{ then } (G_h)_i = \hat{F}_{i,U_{i,h}}.
\] (9)

The post-auction belief about the auction winner takes into account the information that the winner decides to offer the good for resale:

\[
\forall h \in \mathcal{H}_{term}, ~ i = \text{win}(h) : \text{ if } U_{i,h} \cap \gamma_{h}^{-1}(1) \neq \emptyset \text{ then } (G_h)_i = \hat{F}_{i,U_{i,h} \cap \gamma_{h}^{-1}(1)}.
\] (10)

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\(^{18}\)By condition (1), including the other histories does not change the Bayesian inference analysis below.

\(^{19}\)For any distribution \( J \) on \( T \), we denote by \( (J)_i \) the marginal distribution induced on \( T_i \).
Whenever the “if” in (8), (9), or (10) is satisfied, then the respective c.d.f. $G = (G_h)_i$ is either a point distribution or inherits the following properties from $F_i$: (i) the support $\text{supp}(G)$ is a bounded interval, (ii) $G$ has a positive continuous density $g$ throughout its support, and (iii) the virtual valuation function
\[
V_G(t) = t - \frac{1 - G(t)}{g(t)} \quad (t \in \text{supp}(G)).
\]
is strictly increasing. We call any c.d.f. $G$ with the properties (i), (ii), and (iii) regular. If $G$ is a point distribution, define $V_G(t) = t$ for $\{t\} = \text{supp}(G)$.

In the equilibria that we will construct, beliefs will always be regular or point distributions, whether or not the “if” in (8), (9), and (10) is satisfied. Let $\mathcal{G}$ denote the set of product measures $J$ on $T$ such that, for all $i \in N$, the marginal $(J)_i$ is regular or is a point distribution.

The last condition for equilibrium is that the period-2 outcome is induced by a period-2 continuation equilibrium given the post-auction belief. The condition may take various forms depending on the resale market rules.

### 2.4 The resale market

Rather than modelling the period-2 interaction explicitly, we describe general properties that are satisfied by the period-2 continuation equilibria for a range of resale market rules. Several possible resale market rules are described at the end of the section.

Two groups of properties can be derived from incentive compatibility and participation constraints. The third refers to comparative statics with respect to beliefs. The fourth property considers role switching between a period-2 seller and a period-2 buyer. The fifth property refers to the period-2 seller’s share of the gains from trade.

The first group of period-2-continuation-equilibrium properties can be derived from incentive compatibility. The properties relate payoff differences to allocation probabilities:\footnote{For a derivation of properties of the type (11) or (12), see the derivation of the revenue equivalence principle in Myerson (1981). The underlying mathematical principle is the envelope theorem in integral form; see Milgrom and Segal (2002) for a general treatment.}

\[
\text{if } v < t \text{ then } l_{ij}(t, J) - l_{ij}(v, J) = \int_v^t q_{ij}(x, J)dx; \quad (11)
\]
for all $j \in N$, $t, v \in T_j$, and $J \in \mathcal{G}$,

\[
\text{if } v < t \text{ then } w_j(t, J) - w_j(v, J) = \int_v^t q_j(x, J)dx. \quad (12)
\]
The second group of properties can be derived from voluntary participation in the resale market. The period-2 seller would not obtain less than what she gets by consuming the good, and cannot obtain more than the available surplus: for all \( j \in N \), \( t \in T_j \), and \( \mathbf{J} \in \mathcal{G} \),

\[
t \leq w_j(t, \mathbf{J}) \leq \int_{T_{\neg j}} \max\{t, \max_{k \in N \setminus \{j\}} t_k\} d\mathbf{J}_{\neg j}(t_{\neg j});
\]

\( (13) \)

any period-2 buyer cannot lose in expectation, and cannot obtain more than the surplus that remains if the period-2 seller keeps the good: for all \( i \in N \), \( j \in N \setminus \{i\} \), \( t \in T_i \), and \( \mathbf{J} \in \mathcal{G} \),

\[
0 \leq l_{ij}(t, \mathbf{J}) \leq \int_{T_{\neg i}} \max\{t, \max_{k \in N \setminus \{i\}} t_k\} d\mathbf{J}_{\neg i}(t_{\neg i}) - \int_{T_j} t d\mathbf{J}_j(t_j).
\]

\( (14) \)

The third group of properties is more novel; it refers to shifts in beliefs. For any \( i \in N \) and \( \mathbf{J}, \mathbf{H} \in \mathcal{G} \), we say that the belief about bidder \( i \) shifts upwards \((\mathbf{H} \succeq_i \mathbf{J})\), if \((\mathbf{H})_i\) dominates \((\mathbf{J})_i\) in terms of the hazard rate\(^{21}\) and \((\mathbf{H})_k = (\mathbf{J})_k\) for all \( k \in N \setminus \{i\} \). If the belief about a period-2 buyer shifts upwards, then her period-2 payoff is reduced: for all \( i \in N \), \( j \in N \setminus \{i\} \), \( t \in T_i \), and \( \mathbf{J}, \mathbf{H} \in \mathcal{G} \),

\[
\text{if } \mathbf{H} \succeq_i \mathbf{J} \text{ then } l_{ij}(t, \mathbf{H}) \leq l_{ij}(t, \mathbf{J}).
\]

\( (15) \)

Similarly, if the belief about the period-2 seller shifts upwards, then her period-2 payoff is increased: for any \( j \in N \), \( t \in T_j \), and \( \mathbf{J}, \mathbf{H} \in \mathcal{G} \),

\[
\text{if } \mathbf{H} \succeq_j \mathbf{J} \text{ then } w_j(t, \mathbf{H}) \geq w_j(t, \mathbf{J}).
\]

\( (16) \)

The intuition behind properties (15) and (16) is that the period-2 seller will aim at a higher price if any bidder is more likely to have a high use value.

The fourth property states that, for each type of each bidder, the probability that the bidder ends up with the good is higher when she is the period-2 seller than when she is a period-2 buyer: for all \( i \in N \), \( j \in N \setminus \{i\} \), \( \mathbf{J} \in \mathcal{G} \), and \( t \in \text{supp}(\mathbf{J})_i \),

\[
q_i(t, \mathbf{J}) \geq q_{ij}(t, \mathbf{J}).
\]

\( (17) \)

\(^{21}\)Hazard rate dominance is defined by the following conditions: \( \min \text{supp}(\mathbf{H})_i \geq \min \text{supp}(\mathbf{J})_i \), \( \max \text{supp}(\mathbf{H})_i \geq \max \text{supp}(\mathbf{J})_i \), and \( V_{(\mathbf{H})_i}(t) \leq V_{(\mathbf{J})_i}(t) \) for all \( t \in \text{supp}(\mathbf{J})_i \cap \text{supp}(\mathbf{H})_i \). Hazard rate dominance implies first-order stochastic dominance. See, e.g., Krishna (2002, Appendix B) for details.
The fifth period-2-continuation-equilibrium property states that, if there are expected gains from trade with every period-2 buyer, then the period-2 seller captures a non-zero share of the gains: for all \( j \in N \setminus \{j\} \), and \( t \in T_i \),

if \( \max \text{supp} G_i > t \) for all \( i \in N \setminus \{j\} \), then \( w_j(t, J) > t \). \hfill (18)

Properties (11) to (18) are satisfied if the period-2 seller uses an optimal mechanism (i.e., period-2 payoff maximizing mechanism), as derived in Myerson (1981).\(^{22}\) Myerson shows that the seller allocates the good to the bidder with the highest virtual valuation, unless her own use value is higher than every bidder’s virtual valuation, in which case she keeps the good. Using the shortcuts \( J_k = (J)_k \) and \( V_k = V_{J_k} \) for all \( k \in N \), we obtain the allocation probabilities: for all \( i \in N \), \( j \in N \setminus \{i\} \), \( t \in T_i \), and \( J \in G \),\(^{23}\)

\[
q_{ij}^{\text{Myerson}}(t, J) \overset{\text{def}}{=} J_j(V_i(t)) \prod_{k \in N \setminus \{i,j\}} J_k((V_k)^{-1}(V_i(t))). \tag{19}
\]

To verify (15), observe from (19) that

\[
q_{ij}^{\text{Myerson}}(t, H) \leq q_{ij}^{\text{Myerson}}(t, J)
\]

and use (11) together with the fact that in an optimal mechanism the believed lowest type of bidder \( i \) obtains a period-2 payoff of 0. Property (16) holds with equality. Property (17) holds because \( q_{ij}^{\text{Myerson}}(t, J) \) is defined as:

\[
q_{ij}^{\text{Myerson}}(t, J) \overset{\text{def}}{=} \prod_{k \in N \setminus \{i,j\}} J_k((V_k)^{-1}(t)) \geq q_{ij}^{\text{Myerson}}(t, J), \tag{19}
\]

\(^{22}\)In contrast to Myerson’s assumptions, we allow for the possibility that the period-2 seller is privately informed about her type. As shown in Milovanov and Tröger (2006), the resulting informed-principal game has an equilibrium such that the period-2 seller offers the same resale mechanism as when her type is publicly known, as assumed by Myerson.

\(^{23}\)Concerning the virtual valuation function \( V \) for any c.d.f. \( G \), we use the convention

\[
V(t) = \begin{cases} 
\max \text{supp} G & \text{if } t > \max \text{supp} G, \\
-\infty & \text{if } t < \min \text{supp} G.
\end{cases}
\]

Concerning the inverse virtual valuation function \( V^{-1} \), we use the convention

\[
V^{-1}(t) = \begin{cases} 
\infty & \text{if } t > \max \text{supp} G, \\
\min \text{supp} G & \text{if } t < \min (V(\text{supp} G)).
\end{cases}
\]

Recall that \( G(\infty) = 1 \) and \( G(-\infty) = 0 \).
where the inequality follows from the fact that any type’s virtual valuation is at most as large as her use value \( V_j^{-1}(t) \geq t \geq V_i(t) \).

Properties (11) to (18) are also satisfied if the period-2 seller is restricted to use an English auction with no reserve price, and with the option to reject all bids, provided that \( n \geq 3 \). As observed by Haile (2000), such a mechanism is common in practice and may serve as a model of a resale market where the bargaining power is shifted to the period-2 buyers. Because further resale after period-2 is not possible, we may assume that the period-2 seller rejects all bids unless the highest bid exceeds her use value; every period-2 buyer plans to stay in until the current price reaches her use value and, once she is the only bidder left in the auction, has a final chance to increase her bid (the purpose of the final bid is to reduce the probability of rejection by the period-2 seller). Property (15) holds with equality because the beliefs about any period-2 buyer have no impact on behavior. Property (16) holds because the final bid of the period-2 buyer with the highest use value weakly increases if her belief about the period-2 seller shifts upwards.

To verify (17), observe that a period-2 buyer ends up with the good only if she has the highest use value in the market, and in this event she will keep the good if she is the period-2 seller. Hence,

\[
q^\text{English reject}_i(t, J) \geq \prod_{k \in N \setminus \{i\}} J_k(t) \geq q^\text{English reject}_{ij}(t, J).
\]

Property (18) is satisfied because \( n \geq 3 \).

A yet different resale market structure is obtained if we assume that bargaining power is bidder-specific, as, e.g., in Calzolari and Pavan (2006). Suppose that \( n = 2 \), and bidder 1 always has the power to make a posted-price offer (that is, if bidder 1 is the period-2 seller, then she sells if bidder 2 is willing to buy at the posted price, and if she is a period-2 buyer, then she obtains the good if bidder 2 is willing to sell at the posted price). One readily checks that properties (15) and (16) are satisfied. To verify (17), observe that as a period-2 buyer, bidder 1 will ask a price below her use value, while she will ask a price above her use value if she is the period-2 seller, implying \( q_{12}(t, J) \leq J_2(t) \leq q_1(t, J) \) and \( q_{21}(t, J) \leq J_1(t) \leq q_2(t, J) \). Property (18) does in general not hold for \( j = 2 \). However, if we change the resale market rule such that each bidder makes a posted-price offer with a strictly positive probability, then all properties are satisfied.

Generally, for any finite collection of resale market rules satisfying the properties (11) to (18), any combination where each rule is applied with a fixed probability also satisfies the properties.\[^{24}\]

\[^{24}\text{The resale markets considered above are assumed to be single-period. We suspect...} \]
3 Equilibria for English auctions with resale

In this section, we construct a family of equilibria for the English auction with active resale markets. The equilibrium outcomes differ from the well-known value-bidding equilibrium outcome. The basic structure of the equilibria is similar to the structure of the equilibria constructed in Garratt and Tröger (2006) for English and second-price auctions with resale where all bidders up to one are ex-ante symmetric, and where the bidder who becomes the period-2 seller has no private information, but is known to have the use value 0. Here, bidders are asymmetric, every bidder has some private information, and we consider a bigger range of resale market structures, so that computing payoffs is more complicated. Hence, while the basic structure of the equilibria is similar to Garratt and Tröger (2006), it is more difficult now to establish that the proposed strategy profiles are in fact equilibria. Moreover, as will be explained below, while the equilibrium construction always works for the English auction, it works for the second-price auction only under additional assumptions.

Some notation is needed to describe the equilibria. Each equilibrium is parameterized by some number \( t^* \in (0, \bar{t}) \), where

\[
\bar{t} = \min_{i \geq 2} t_i
\]

denotes the largest type that is possible for all bidders except bidder 1. For any type \( t \in T_1 \), the belief resulting from the information that bidder 1’s type is at most \( t \) and the other bidders’ types are below \( t^* \) is denoted

\[
J_{t, t^*} = \hat{F}_{1,[0,t]} \times \prod_{i=2}^{n} \hat{F}_{i,[0,t^*]}.
\]

Let \( \tau(t^*) \in T_1 \) denote the smallest \(^26\) type of bidder 1 that, if she is the auction winner and the post-auction beliefs are \( J_{\tau(t^*), t^*} \), weakly prefers (i) consuming the good to (ii) offering the good for resale. By construction, every type below \( \tau(t^*) \) prefers (ii). The following lemma shows that moreover

\[^25\text{See Haile (1999) for the proof that the value-bidding equilibrium remains valid when a resale opportunity exists.}\]

\[^26\text{More precisely, smallest here means “infimum”, where the infimum of the empty set is defined as } \bar{t}_1.\]

that the properties (11) to (18) hold for a range of rules that allow for multiple periods and repeated resale. In any case, the equilibrium outcomes constructed in Proposition 1 survive repeated resale under certain conditions; see remark number 2 below Proposition 1.
every type above $\tau(t^*)$ prefers (i). Also, there are always some types which prefer (ii), and with vanishing discounting almost all types below $t^*$ prefer (ii). The proof can be found in the Appendix.

**Lemma 1** Let $t^* \in (0, \bar{t})$. For all $t \in (\tau(t^*), \bar{t}_1]$, we have $t \geq \delta w_1(t, J_{\tau(t^*), t^*})$. Also, $0 < \tau(t^*) \leq t^*$, and if $\delta \to 1$ then $\tau(t^*) \to t^*$.

For all bidders $i \in N \setminus \{1\}$, let $b_i(t^*)$ denote the price that makes type $t^*$ of bidder $i$ indifferent between (i) winning the auction at price $b_i(t^*)$ and consuming the good, and (ii) participating in a resale market where bidder 1 is the period-2 seller and the post-auction beliefs are $J_{\tau(t^*), t^*}$:

$$b_i(t^*) = t^* - \delta l_{i1}(t^*, J_{\tau(t^*), t^*}).$$

(20)

The following lemma provides bounds for $b_i(t^*)$ and shows that types above $t^*$ prefer (i) while types below $t^*$ prefer (ii).

**Lemma 2** Let $t^* \in (0, \bar{t})$. For all $i \in N \setminus \{1\}$, we have $0 < b_i(t^*) \leq t^*$. Moreover, for all $t \in T_i$,

$$ if \quad t < t^* \quad then \quad t - b_i(t^*) \leq \delta l_{i1}(t, J_{\tau(t^*), t^*}).$$

$$ > \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qua

Proposition 1 describes our equilibria. Bidder 1 takes the role of the “speculating bidder.” In (21), we describe bidder 1’s drop-out points at any history with current price 0 where bidder 1 moves and where only one other bidder $i$ is in. If bidder 1’s use value is below $\tau(t^*)$, then she plans to drop out at $b_i(t^*)$; if her use value is between $\tau(t^*)$ and $t^*$, then she plans to drop out at price $t^*$; if her use value is above $t^*$, then she plans to drop out if the price equals her use value. In (22) we describe the drop-out points of any bidder $i \geq 2$ at the initial history. She drops out at the beginning of the auction if her use value is below $t^*$, and otherwise drops out if the price equals her use value. In (23) we describe the post-auction beliefs if all bidders except bidder 1 drop out at the beginning of the auction; bidder 1 then offers the good for resale if and only if her type is below $\tau(t^*)$. In (24) we describe off-equilibrium-path beliefs. The belief about any bidder $i \geq 2$ who does not drop out at the beginning of the auction, but drops out at a price below $b_i(t^*)$ and loses to bidder 1, is shifted upwards relative to the belief that would prevail had she dropped out at the beginning.
Proposition 1 Let \( t^* \in (0, T) \). Then there exists an equilibrium with the following properties. For all \( i \geq 2 \) and \( h = (\ldots, \{1, i\}, 0) \),

\[
\forall t \in T_1 : \quad \beta_{i,h}(t) = \begin{cases} 
  b_i(t^*) & \text{if } t < \tau(t^*), \\
  t^* & \text{if } t \in (\tau(t^*), t^*), \\
  t & \text{if } t > t^*.
\end{cases}
\] (21)

For all \( i \geq 2 \) and \( h = (N, 0) \),

\[
\forall t \in T_i : \quad \beta_{i,h}(t) = \begin{cases} 
  0 & \text{if } t < t^*, \\
  t & \text{if } t > t^*.
\end{cases}
\] (22)

For \( h = (N, 0, \{1\}, 0) \),

\[
G_h = J_{\tau(t^*), t^*}, \quad \forall t \in T_1 : \quad \gamma_h(t) = \begin{cases} 
  1 & \text{if } t < \tau(t^*), \\
  0 & \text{if } t > \tau(t^*).
\end{cases}
\] (23)

For all \( i \geq 2 \) and \( p \in [0, b_i(t^*)) \), if \( h = (\ldots, \{1, i\}, p, \{1\}, p) \) (and \( p > 0 \) if \( n = 2 \)), then

\[
G_h \geq_i J_{\tau(t^*), t^*}.
\] (24)

The properties of these \( t^* \)-equilibria generalize those of Garratt and Tröger (2006, Proposition 2, and online supplement). The equilibria feature an active resale market. If bidder 1 has a use value below \( \tau(t^*) \) and all other bidders have use values below \( t^* \), then bidder 1 wins the auction at price 0 and, with positive probability, resells the good to a losing bidder with a higher use value. Bidder 1 is strictly better off than at the value-bidding equilibrium, for all of her types (she sometimes wins at a lower price and can obtain additional profit from resale). For typical resale market rules, the final allocation is inefficient with positive probability even if there is no discounting; the inefficiency arises because bidders retain some private information when they enter the resale market.

A number of steps towards the proof of Proposition 1 are straightforward. Bidder 1 has no incentive to deviate. If her use value is below \( t^* \), then she expects to win at price 0 whenever she wins; if her use value is above \( t^* \), then dropping out when the price reaches her use value is optimal for reasons analogous to the reasons in an English auction without resale.

The bidders \( i \geq 2 \) have no incentive to deviate. Any bidder \( i \) who decides to not drop out at the beginning can hope to gain only if she stays in beyond the price \( b_i(t^*) \) (otherwise she loses the auction by (21), and the only possible
effect of her deviation is that the belief about her type shifts upwards by (24), which reduces her period-2 payoff by (15)). By Lemma 2, bidder $i$ if her use value is below $t^*$ prefers (i) waiting for the resale market to (ii) staying in beyond the price $b_i(t^*)$ and consuming the good upon winning; vice versa if bidder $i$’s use value is above $t^*$. This is in accordance with (22).

The crucial remaining step is to show that any bidder $i \geq 2$ with a use value below $t^*$ prefers possibility (i) to possibility (iii) staying in beyond the price $b_i(t^*)$ and choosing optimally between consuming the good and offering the good for resale upon winning. To understand why the deviation (iii) is not profitable, assume for simplicity that there is no discounting, so that it is always optimal to offer the good for resale upon winning. The deviation switches bidder $i$’s role from a period-2 buyer to a period-2 seller, leaving the the post-auction beliefs $J_{\tau(t^*),t^*}$ unchanged. By (13), $w_i(t^*,J_{\tau(t^*),t^*}) = t^*$. Hence, (20) implies that type $t^*$ is indifferent between the two roles:

$$w_i(t^*,J_{\tau(t^*),t^*}) - b_i(t^*) = l_{i1}(t^*,J_{\tau(t^*),t^*}) \quad \text{if } \delta = 1. \quad (25)$$

From (11), (12), and (17), the payoff difference between type $t^*$ and any type $t < t^*$ is larger in the seller role than in the buyer role:

$$w_i(t^*,J_{\tau(t^*),t^*}) - w_i(t,J_{\tau(t^*),t^*}) \geq l_{i1}(t^*,J_{\tau(t^*),t^*}) - l_{i1}(t,J_{\tau(t^*),t^*}).$$

This together with (25) shows that all types below $t^*$ prefer to be a period-2 buyer rather than a period-2 seller, showing that the proposed deviation is not profitable. With discounting, the argument is slightly more complicated; details can be found in the Appendix. This completes the proof of Proposition 1.

Some remarks.

1. Suppose that the initial seller sets a reserve price $r > 0$. Equilibria with an active resale market and an inefficient final allocation still exist if $r < \bar{t}$, that is, if the reserve price does not exceed any bidder’s highest possible type. Let $t^* \in (r,\bar{t})$. Let $\hat{t}$ be such that, if bidder 1 with type $\hat{t}$ wins the good at price $r$ and offers the good for resale, given the belief that the types in $[0,t^*)$ of other bidders participate in the resale market, then her expected payoff equals 0. If $\delta$ is sufficiently close to 1, then $\hat{t} < r$, and an

\footnote{Formally, the game is extended as follows. The auction starts with a current price that corresponds to “no sale.” Like at any other current price, bidders have the chance to drop out simultaneously and sequentially at no-sale; i.e., when somebody drops out at no-sale, then the clock stops, and others have the chance to drop out. Once no more bidders drop out at no-sale, the clock jumps to the current price $r$.}
equilibrium with the following properties exists. Initially, bidder 1 drops out at no-sale if and only if her type is below \( \hat{t} \). Once bidder 1 has dropped out at no-sale, other bidders act as in the absence of a resale opportunity: any bidder drops out at no-sale if her use value is below \( r \), and otherwise drops out when the price equals her use value. If bidder 1 does not drop out at no-sale, then the bidders’ subsequent actions are analogous to the equilibria described in Proposition 1, where bidding 0 is replaced by dropping out at no sale. In particular, on the equilibrium path resale occurs given the belief that (i) bidder 1’s type is distributed on \([\hat{t}, \tau]\) for some \( \tau \in (r, t^*) \), and (ii) the other bidders’ types are distributed on \([0, t^*]\).

2. If repeated resale is allowed, the \( t^* \)-equilibrium outcomes remain valid if \( n = 2 \) (because the period-2 seller has no incentive to re-buy), and remain valid under certain conditions if \( n \geq 3 \). Suppose that the period-2 seller selects a mechanism that is optimal given that she cannot prevent the next owner from selecting another mechanism in order to resell again, and so on, and the game ends if and only if no sale occurs. For such a game, Zheng (2002) constructs a perfect Bayesian equilibrium that induces the Myerson optimal-auction allocation if bidders’ type distributions satisfy certain conditions. Embedding Zheng’s equilibrium into our model translates into corresponding conditions on the type distribution profile for the period-2 buyers on the equilibrium path, \( \prod_{i=2}^{n} \hat{F}_{i, [0, t^*]} \).\(^{28}\)\(^{29}\) For all \( n \geq 4 \), these conditions are satisfied if and only if the distribution \( \hat{F}_{3, [0, t^*]} = \ldots = \hat{F}_{n, [0, t^*]} \) has a weakly decreasing density and dominates \( \hat{F}_{2, [0, t^*]} \) in terms of the hazard rate.\(^{30}\)

3. The \( t^* \)-equilibrium construction makes essential use of the sequential structure of the English auction format, because bidder 1’s bidding behavior depends on the other bidders’ observed drop-out points. In effect, bidder 1 if her value is below \( \tau(t^*) \) bids \( b_1(t^*) \) against bidder \( i \geq 2 \). For this, and only this, reason the equilibrium construction does not generally extend to second-price auctions. The construction does extend if \( b_2(t^*) = \ldots = b_n(t^*) \), which is the case if bidders 2 to \( n \) are ex-ante symmetric.

4. Equilibria with small \( t^* \) are more “robust” than equilibria with large

\(^{28}\) No conditions are needed concerning resale off the equilibrium path, because, by the revelation principle, repeated resale can only reduce the deviator’s payoff relative to an environment without repeated resale.

\(^{29}\) Observe that \( n \) bidders in our model correspond to \( n - 1 \) bidders in Zheng (2002), because only bidders 2 to \( n \) have an incentive to participate in repeated resale.

\(^{30}\) See Mylovanov and Troger (2006, Corollary 1); this paper also characterize the conditions that are relevant in case \( n = 3 \), which corresponds to the 2-bidder case in Zheng (2002).
In a $t^*$-equilibrium, common knowledge of prior probability distributions is required only on the interval $[0, t^*]$.

5. The $t^*$-equilibria are not the only equilibria that differ from the value-bidding equilibrium. There exist “extreme equilibria” where bidder 1’s planned drop-out price is so high that, for at least one bidder $i \geq 2$, even the highest type $\tilde{t}_i$ finds it optimal to drop out at the beginning of the auction. We find such extreme equilibria less plausible than the $t^*$-equilibria. Bidder $i$ can throw an extreme equilibrium off its path before she loses in the English auction, while she cannot do that to a $t^*$-equilibrium. At an extreme equilibrium, once the English auction starts, bidder $i$ should have dropped out. Thus, if bidder $i$ continues to bid in the English auction, it becomes a commonly known off-path event. In contrast, if at a $t^*$-equilibrium bidder $i$ remains in the auction, she is regarded to be following her equilibrium strategy with a type above the threshold $t^*$, so the off-path action is not known by others. Consequently, the response to (observed) off-path actions needed for the extreme equilibrium is more delicate than that for a $t^*$-equilibrium. If we require that an equilibrium be robust against a small involuntary probability with which bidder 1 drops out prematurely in response to a (commonly known) off-path action, one can show that the extreme equilibrium is ruined, while the $t^*$-equilibria are unaffected.

This distinction between the extreme equilibrium and the $t^*$-equilibria is closely related to the dynamic nature of an English auction. If the English auction is replaced by a second-price auction, a simultaneous-move game, there is no chance for bidder $i$ to produce a publicly known off-path event in period 1. Hence, extreme equilibria may be considered somewhat more plausible in a second-price auction.\textsuperscript{31}

4 Collusion

In this section, we consider the possibility that bidders can condition behavior on a public randomization device, or sunspot. In symmetric environments with no discounting, a sunspot variable which chooses each bidder to play the role of the speculating bidder with equal probability, allows all bidders of all types to obtain a higher payoff than in the value-bidding equilibrium. Hence, moving from the value-bidding equilibrium to the new equilibrium is self-enforcing at the interim stage. Our construction may be dubbed a collusive equilibrium, because the assignment of roles based on the

\textsuperscript{31}Zheng (2000, Section 5.2) constructs an “extreme equilibrium” in a second-price-auction-type mechanism.
sunspot variable is a form of tacit collusion. Observe that, in contrast to the collusion devices that have been discussed in the literature, coordination on our collusive equilibrium does not require a repeated-auction setup, nor does it require any (possibly illegal) post-auction enforcement between bidders.

Consider a sunspot that selects each bidder with probability $1/n$, and suppose a $t^*$-equilibrium with the selected bidder as the speculating bidder is then played. We call the resulting overall equilibrium a $t^*$-collusive equilibrium.

Proposition 2 Suppose that bidders are symmetric and there is no discounting,

$$F_1 = \ldots = F_n, \quad \delta = 1.$$  \hspace{1cm} (26)

If $t^* > 0$ is sufficiently close to 0, then in a $t^*$-collusive equilibrium all types of bidders are strictly better off than in an equilibrium where everybody bids their value and no resale occurs.

The proof proceeds via a series of lemmas. Let $\bar{t} = \bar{t}_i$. For all $t \in [0, \bar{t}]$, let $U_{\text{val}}(t)$ denote the payoff of type $t$ in an equilibrium where everybody bids their value and no resale occurs. For all $t, t^* \in [0, \bar{t}]$, let $U_{\text{col}}(t)$ denote the payoff of type $t$ in a $t^*$-collusive equilibrium.\(^{32}\)

The first lemma states that, if type $t^*$ is better off in the $t^*$-collusive equilibrium, then all types above $t^*$ are also better off. The reason is simple. Conditional on the event that all competing bidders’ types are below $t^*$, type $t^*$ consumes the good, whether she is assigned the role of the speculating bidder or not. Hence, the relevant question for her is what price she pays in the auction and in the resale market: if the expected price is smaller in a $t^*$-collusive equilibrium compared to a value-bidding equilibrium, then she is better off in a $t^*$-collusive equilibrium. The same expected price difference is relevant to all types above $t^*$, so that they face the same payoff comparison as type $t^*$.

Lemma 3 Suppose that condition (26) holds.

For all $t^*$, if $U_{\text{col}}(t^*) > U_{\text{val}}(t^*)$, then $U_{\text{col}}(t) > U_{\text{val}}(t)$ for all $t > t^*$.

Proof. straightforward.

\(^{32}\)Observe that $U_{\text{col}}(t)$ is also a function of $t^*$, but we suppress this dependence in order to save on notation. We do the same for some other functions defined below.
By Lemma 3, it is sufficient to show that, if $t^*$ is sufficiently small, then all types up to $t^*$ are strictly better off in a $t^*$-collusive equilibrium compared to a value-bidding equilibrium.

We use the shortcut $F = F_i$.

Consider any type $t \leq t^*$. The probability that she obtains the good in the value-bidding equilibrium is $F(t)^{n-1}$. Hence, by the envelope theorem, type $t$'s payoff in the value-bidding equilibrium is

$$U^{val}(t) = \int_0^t F(x)^{n-1}dx.$$  \hspace{1cm} (27)$$

The probability that type $x \leq t^*$ of bidder 1 obtains the good in a $t^*$-equilibrium is

$$q_1(x) = F((V^*)^{-1}(x))^{n-1},$$  \hspace{1cm} (28)$$

where $(V^*)^{-1}$ denotes the inverse of the (post-auction) virtual valuation function $V^*$ defined by

$$V^*(v) = v - \frac{1 - F(v)}{F(t^*)} = v - \frac{F(t^*) - F(v)}{f(v)} \quad (v \in [0, t^*]).$$  \hspace{1cm} (29)$$

To understand (28), observe that bidder 1 wins the auction if and only if all other bidders’ types are below $t^*$ and, given that she wins, finally keeps the good if and only if all other bidders’ virtual valuations are below $x$, that is, if and only if all other bidders’ types are below $(V^*)^{-1}(x)$.

The payoff of type $t^*$ of bidder 1 in a $t^*$-equilibrium equals $t^*F(t^*)^{n-1}$, because she obtains the good at price 0 and consumes it whenever she wins the auction. Together with (28), the envelope theorem implies that the payoff of type $t \leq t^*$ of bidder 1 in a $t^*$-equilibrium is

$$U_1(t) = t^*F(t^*)^{n-1} - \int_t^{t^*} q_1(x)dx$$

$$= t^*F(t^*)^{n-1} - \int_t^{t^*} F((V^*)^{-1}(x))^{n-1}dx.$$  \hspace{1cm} (30)$$

The probability that type $x \leq t^*$ of bidder 2 obtains the good in a $t^*$-equilibrium is

$$q_2(x) = F((V^*)(x))F(x)^{n-2},$$  \hspace{1cm} (31)$$

because she obtains the good if and only if, given the information that her type is below $t^*$, her virtual valuation $(V^*)(x)$ exceeds bidder 1’s type, and
all remaining bidders’ types are below $x$. Together with (31), the envelope theorem implies that the payoff of type $t \leq t^*$ of bidder 2 in a $t^*$-equilibrium is

$$U_2(t) = \int_0^t q_2(x)dx = \int_0^t F((V^*)(x))F(x)^{n-2}dx. \quad (32)$$

Putting (30) and (32) together, we find the payoff of type $t \leq t^*$ in a $t^*$-collusive equilibrium as

$$U^{col}(t) = \frac{1}{n}U_1(t) + \frac{n-1}{n}U_2(t) \quad (33)$$

In the following, we will consider $t^*$-collusive equilibria with $t^*$ close to 0. To this end, the linearization of $F$ at 0 will be very useful,

$$F(x) = f(0) x + h_1(x), \quad (x \geq 0), \quad (34)$$

where $|h_1(x)|/x \to 0$ as $x \to 0$.

Also, for any $k \geq 0$, we will use the shortcut $o((t^*)^k)$ for any function $h(x,t^*)$ such that $\sup_{x \in [0,t^*]} |h(x,t^*)|/(t^*)^k \to 0$ as $t^* \to 0$.

The next lemma approximates the payoffs in a value-bidding equilibrium.

**Lemma 4** For all $t^*$ and all $t \leq t^*$,

$$U^{val}(t) = f(0)^{n-1} \frac{1}{n} t^n + o((t^*)^n).$$

**Proof.** From (27) and (34),

$$U^{val}(t) = \int_0^t F(x)^{n-1}dx$$

$$= \int_0^t (f(0)x + h_1(x))^{n-1}dx$$

$$= \int_0^t (f(0)^{n-1}(x)^{n-1} + h_2(x)) \, dx$$

$$= f(0)^{n-1} \frac{1}{n} t^n + \int_0^t h_2(x) \, dx,$$

where $|h_2(x)|/(x)^{n-1} \to 0$ as $x \to 0$.

Let $\epsilon > 0$. If $t^*$ is sufficiently small, then $|h_2(x)| \leq \epsilon (x)^{n-1}$ for all $x \leq t^*$. Therefore

$$\left| \int_0^t h_2(x)dx \right| \leq \int_0^t |h_2(x)| \, dx \leq \epsilon \int_0^t x^{n-1}dx \leq \epsilon ((t^*)^n).$$

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which completes the proof. \( QED \)

Define
\[
\kappa(t^*) = \frac{f(0)}{\min_{x \in [0,t^*]} f(x)} \quad (t^* \in [0,t]).
\] (35)

Observe that \( \kappa(t^*) \to 1 \) as \( t^* \to 0 \).

The next lemma provides an approximate lower bound for the virtual valuation function if \( t^* \) is small.

**Lemma 5** For all \( t^* \in [0,t] \) and all \( t \in [0,t^*] \),
\[
V^*(t) \geq t - \kappa(t^*)(t^* - t) + o(t^*).
\] (36)

**Proof.** Using (29) and (34),
\[
V^*(t) = t - \frac{f(0)t^* + h_1(t^*) - f(0)t - h_1(t)}{f(t)}
\]
\[
= t - \frac{f(0)}{f(t)}(t^* - t) + h(t,t^*)
\]
\[
\geq t - \kappa(t^*)(t^* - t) + h(t,t^*),
\] (36)

where
\[
h(t,t^*) := \frac{h_1(t^*) - h_1(t)}{f(t)}.
\] (37)

Observe that (34) implies
\[
\sup_{x \in [0,t^*]} \frac{|h_1(x)|}{t^*} \leq \sup_{x \in [0,t^*]} \frac{|h_1(x)|}{x} \to 0 \quad \text{as} \quad t^* \to 0.
\]

Hence, defining \( f(t^*) = \min_{x \in [0,t^*]} f(x) \), we have
\[
\sup_{x \in [0,t^*]} \frac{|h(x,t^*)|}{t^*} \leq \frac{1}{f(t^*)} \left( \frac{|h_1(t^*)|}{t^*} + \sup_{x \in [0,t^*]} \frac{|h_1(x)|}{t^*} \right) \to 0
\]
as \( t^* \to 0 \). \( QED \)

Next we establish a result concerning the inverse virtual valuation function that is analogous to Lemma 5.
Lemma 6. For all $t^* \in [0, T]$ and all $t \in [0, t^*],$

\[(V^*)^{-1}(t) \leq \frac{t + \kappa(t^*)t^*}{1 + \kappa(t^*)} + o(t^*).\]

Proof. Let $h$ be defined as in (37). Rewriting (36) in inverse form,

\[t \geq (V^*)^{-1}(t) - (V^*)^{-1}(t^*) + h((V^*)^{-1}(t^*)),\]

\[= (V^*)^{-1}(t)(1 + \kappa(t^*)) - \kappa(t^*)t^* - y(t, t^*) + o(t^*),\]

(38)

where

\[y(t, t^*) := -\frac{h((V^*)^{-1}(t), t^*)}{1 + \kappa(t^*)},\]

(39)

Solving (38) for $(V^*)^{-1}(t)$ yields

\[(V^*)^{-1}(t) \leq \frac{t + \kappa(t^*)t^*}{1 + \kappa(t^*)} + y(t, t^*).\]

Using (39) and the fact that $(V^*)^{-1}(t) \in [0, t^*],$

\[
\sup_{x \in [0, t^*]} \frac{|y(x, t^*)|}{t^*} \leq \sup_{x \in [0, t^*]} \frac{|h(x, t^*)|}{(1 + \kappa(t^*))t^*} \to 0 \quad \text{as} \quad t^* \to 0,
\]

because $h(x, t^*) = o(t^*).$ \hfill QED

For all $y \in [0, 1]$ and $k > 0,$ let

\[
\phi_1(y, k) = 1 - \frac{1 + k}{n} + \frac{(y + k)^n}{n(1 + k)^{n-1}}.
\]

Lemma 7. For all $t^* \in [0, T]$ and all $t \in [0, t^*],$

\[U_1(t) \geq f(0)^{n-1}(t^*)^n \phi_1\left(t, \frac{t^*}{t^*}, \kappa(t^*)\right) + o((t^*)^n).\]

Proof. Using (34) and Lemma 6,

\[\int_t^{t^*} F((V^*)^{-1}(x))^{n-1} dx = \left(f(0)(V^*)^{-1}(x) + h_1((V^*)^{-1}(x))\right)^{n-1} dx\]

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\[
\leq \int_{t}^{t^*} \left( f(0) \frac{x + \kappa(t^*)t^*}{1 + \kappa(t^*)} + o(t^*) \right)^{n-1} dx \\
= f(0)^{n-1} \int_{t}^{t^*} \left( \frac{x + \kappa(t^*)t^*}{1 + \kappa(t^*)} \right)^{n-1} dx + o((t^*)^n) \\
= f(0)^{n-1} \left( \frac{1 + \kappa(t^*)}{n} (t^*)^n - \frac{(t + \kappa(t^*)t^*)^n}{n(1 + \kappa(t^*))^{n-1}} \right) + o((t^*)^n) \\
= f(0)^{n-1} (t^*)^n \left( \frac{1 + \kappa(t^*)}{n} - \frac{ \left( \frac{t}{n} + \kappa(t^*) \right)^n}{n(1 + \kappa(t^*))^{n-1}} \right) + o((t^*)^n).
\]

Hence, the result follows from (30) and the fact that \( t^* F(t^*)^{n-1} = f(0)^{n-1}(t^*)^n + o((t^*)^n) \).

\[QED\]

For all \( y \in [0, 1] \) and \( k > 0 \), let

\[
\phi_2(y, k) = \frac{1 + k}{n} y^n - \frac{k}{n-1} y^{n-1} + \frac{k^n}{n(n-1)(1+k)^{n-1}} \text{ if } y > \frac{k}{1+k},
\]

and otherwise \( \phi_2(y, k) = 0 \). It is straightforward to check that \( \phi_2 \) is continuous.

**Lemma 8** For all \( t^* \in [0, T] \) and all \( t \in [0, t^*] \),

\[
U_2(t) \geq f(0)^{n-1}(t^*)^n \phi_2(\frac{t}{t^*}, \kappa(t^*)) + o((t^*)^n).
\]

**Proof.** Using (34),

\[
F(V^*(x)) = f(0)V^*(x) + h_1(V^*(x)) \text{ if } V^*(x) > 0, \\
\]

and \( F(V^*(x)) = 0 \) otherwise. Hence, using that \( V^*(x) \in [0, t^*] \) if \( V^*(x) > 0 \),

\[
F(V^*(x)) = f(0) \max\{0, V^*(x)\} + o(t^*).
\]

Therefore, using Lemma 5,

\[
F(V^*(x)) \geq f(0) \max\{0, x - \kappa(t^*)(t^* - x)\} + o(t^*).
\]

Hence,

\[
\int_{0}^{t} F((V^*)'(x))F(x)^{n-2}dx \\
\geq f(0)^{n-1} \int_{0}^{t} \max\{0, x - \kappa(t^*)(t^* - x)\} x^{n-2}dx + o((t^*)^n).
\]
Observe that the following equivalence holds:

\[ x - \kappa(t^*)(t^* - x) > 0 \iff x > \frac{\kappa(t^*)}{1 + \kappa(t^*)} t^*. \]

Suppose first that \( t/t^* > \kappa(t^*)/(1 + \kappa(t^*)) \). Using (32) and (41),

\[
U_2(t) \geq f(0)^{n-1} \int_{\frac{t}{1 + \kappa(t^*)}}^{t^*} (x - \kappa(t^*)(t^* - x)) (x)^{n-2} dx + o((t^*)^n)
\]

\[
= f(0)^{n-1}(t^*)^{n} \left( \frac{1 + \kappa(t^*)}{n} \left( \frac{t}{t^*} \right)^n + \frac{\kappa(t^*)^n}{n(n-1)(1 + \kappa(t^*))^{n-1}} \right) - \frac{\kappa(t^*)}{n-1} \left( \frac{t}{t^*} \right)^{n-1} + o((t^*)^n)
\]

\[
= f(0)^{n-1}(t^*)^{n} \phi_2(\frac{t}{t^*}, \kappa(t^*)) + o((t^*)^n).
\]

If \( t/t^* < \kappa(t^*)/(1 + \kappa(t^*)) \), then (32) and (41) yield

\[
U_2(t) \geq o((t^*)^n)
\]

\[
= f(0)^{n-1}(t^*)^{n} \phi_2(\frac{t}{t^*}, \kappa(t^*)) + o((t^*)^n),
\]

because \( \phi_2(t/t^*, \kappa(t^*)) = 0 \).

QED

**Lemma 9** If \( k \) is sufficiently close to 1, then

\[
\min_{y \in [0,1]} \frac{1}{n} \phi_1(y, k) + \frac{n-1}{n} \phi_2(y, k) - \frac{1}{n} y^n > 0.
\]

**Proof.** By continuity, it is sufficient to consider \( k = 1 \); i.e., to show for all \( y \in [0,1] \),

\[
\frac{1}{n} \phi_1(y, 1) + \frac{n-1}{n} \phi_2(y, 1) - \frac{1}{n} y^n > 0.
\]

(42)

Defining

\[
\psi(y) = \phi_1(y, 1) - y^n = 1 - \frac{2}{n} + \left( \frac{y + 1}{n} \right)^{n-1} - y^n,
\]

we have

\[
\psi'(y) = \frac{(y + 1)^{n-1}}{2^{n-1}} - ny^{n-1} = y^{n-1} \left( \frac{1}{2^{n-1}} (1 + \frac{1}{y})^{n-1} - n \right) \text{ strictly decreasing in } y.
\]

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Hence, there exists $y^*$ such that $\psi'(y) > 0$ if $y < y^*$ and $\psi'(y) < 0$ if $y > y^*$. Therefore, $\psi$ takes its minimum on $[0, 1]$ at 0 or at 1. Because $\psi(0) > 1 - 2/n \geq 0$ and $\psi(1) = 0$,

$$\frac{1}{n}\phi_1(y, 1) - \frac{1}{n}y^n = \frac{1}{n}\psi(y) > 0 \quad \text{if} \quad y < 1. \quad (43)$$

From (40),

$$\frac{\partial}{\partial y}\phi_2(y, 1) = 2y^{n-1} - y^{n-2} = y^{n-2}(2y - 1) > 0 \quad \text{if} \quad y > \frac{1}{2}. \quad (44)$$

Hence, if $y > 1/2$, then $\phi_2(y, 1) > 0$, because $\phi_2(\frac{1}{2}, 1) = 0$. Combining this with (43) and the fact that $\phi_2(y, 1) = 0$ if $y < 1/2$, we obtain (42). $QED$

For all $t^* \in [0, \bar{T}]$ and all $t \in [0, t^*]$, using (33), Lemma 4, Lemma 7, and Lemma 8,

$$\frac{U^{\text{col}}(t) - U^{\text{val}}(t)}{f(0)^{n-1}(t^*)^n} \geq \frac{1}{n}\phi_1\left(\frac{t}{t^*}, \kappa(t^*)\right) + \frac{n-1}{n}\phi_2\left(\frac{t}{t^*}, \kappa(t^*)\right) - \frac{1}{n}(\frac{t}{t^*})^n + o(1).$$

If $t^*$ is sufficiently close to 0, then $\kappa(t^*)$ is arbitrarily close to 1, and thus (44) together with Lemma 9 implies that

$$\min_{t \in [0, t^*]} (U^{\text{col}}(t) - U^{\text{val}}(t)) > 0.$$

This together with Lemma 3 proves Proposition 2.

Some remarks.

1. The collusion result extends to environments with a discount factor sufficiently close to 1; this follows from Proposition 1 and Lemma 1.

2. The collusion result extends to environments with asymmetric bidders that are approximately symmetric in an appropriate sense; this follows from continuity and Proposition 1. By using a non-uniformly distributed sunspot variable it may be possible to extend the collusion result to a larger class of asymmetric environments. In some very asymmetric environments, however, collusion is not possible using a small $t^*$. Consider environments with at least two private-value bidders and a pure speculator as in Garratt and Tröger (2006): if $t^*$ is small, then the initial seller’s revenue in a $t^*$-equilibrium is larger than in the value-bidding equilibrium, which means that the bidders’
aggregate payoff is smaller (Garratt and Tröger, 2006, online supplement, Proposition 5).

3. The collusion result extends to $t^*$-equilibria with arbitrary $t^*$ if the type distribution $F$ is uniform.$^{33}$ This is intuitive because the essential idea behind the idea of Proposition 2 was to use the linearization (34).

4. The gains to playing a collusive equilibrium can be quite large. Table 1 shows the gains to a bidder with type $t^*$ in environments with $F$ uniform on $[0,1]$, for various numbers of participants. The gains to type $t^*$ are the minimum gains over all types.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$U^v(.9)$</th>
<th>$U^s(.9)$</th>
<th>% increase</th>
</tr>
</thead>
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<tr>
<td>2</td>
<td>0.405</td>
<td>0.50625</td>
<td>25</td>
</tr>
<tr>
<td>5</td>
<td>0.1181</td>
<td>0.19043</td>
<td>61.24</td>
</tr>
<tr>
<td>10</td>
<td>0.03487</td>
<td>0.06277</td>
<td>80.01</td>
</tr>
</tbody>
</table>

Table 1: $F(t) = t$, $t^* = .9$

5 Appendix

Proof of Lemma 1. Let $t > \tau(t^*)$. By definition of $\tau(t^*)$, there exists $v \in (\tau(t^*), t)$ such that

$$v \geq \delta w_1(v, J_{\tau(t^*)}, t^*).$$

(45)

\[\text{From (12) and } q_1(s, J_{\tau(t^*)}, t^*) \leq 1, \]
\[w_1(t, J_{\tau(t^*)}, t^*) - w_1(v, J_{\tau(t^*)}, t^*) \leq t - v. \quad (46)\]

Putting (45) and (46) together we obtain $t \geq \delta w_1(t, J_{\tau(t^*)}, t^*)$.

\[\text{From (13), } \tau(t^*) \leq t^*. \text{ To prove that } \tau(t^*) > 0, \text{ observe that } w_1(0, J_{0,t^*}) > 0 \text{ by (18). Hence, } \hat{t} \overset{\text{def}}{=} \delta w_1(0, J_{0,t^*}) > 0. \text{ Hence, for all } t < \hat{t}, \]
\[t < \delta w_1(0, J_{0,t^*}) \overset{(12)}{\leq} \delta w_1(t, J_{0,t^*}) \overset{(16)}{\leq} \delta w_1(t, J_{t,t^*}), \]

implying $\tau(t^*) \geq \hat{t}$.

To prove the limit result, let $\epsilon > 0$ and define

$$\delta = \sup_{t \leq t^* - \epsilon} \frac{t}{w_1(t, J_{0,t^*})}.$$
By (12), \( w_1(t, \mathbf{J}_{0,t^*}) \) is continuous in \( t \). Hence, (18) implies that \( \bar{\delta} < 1 \). For all \( \delta > \bar{\delta} \) and \( t \leq t^* - \epsilon \),

\[
\delta w_1(t,J_{t,t^*}) \geq \delta w_1(t,J_{0,t^*}) > \delta w_1(t,J_{0,t^*}) \geq t,
\]

implying that \( \tau(t^*) \geq t^* - \epsilon \). \( \square \)

**Proof of Lemma 2.** By Lemma 1, \( \tau(t^*) > 0 \). Hence, condition (14) implies \( l_{i1}(t^*, J_{\tau(t^*),t^*}) < t^* \). Hence, (20) implies \( 0 < b_i(t^*) \). The inequality \( b_i(t^*) \leq t^* \) follows from the lower bound 0 in (14).

To prove the “Moreover”-part, let \( t < t^* \). From (11),

\[
l_{i1}(t^*, J_{\tau(t^*),t^*}) - l_{i1}(t, J_{\tau(t^*),t^*}) \leq t^* - t.
\]

Hence,

\[
t - b_i(t^*) = t^* - b_i(t^*) - (t^* - t) \leq l_{i1}(t, J_{\tau(t^*),t^*}).
\]

The cases \( t > t^* \) are treated analogously. \( \square \)

**Proof of Proposition 1.** The remaining step is to show that, if \( b' \in (b_i(t^*), t^*) \), \( \forall i \geq 2, \ t < t^* : u_{i,(N,0)}(b', t) \leq u_{i,(N,0)}(0, t) \). (48)

Consider the event \( \mathcal{E} \) that bidder 1’s type is below \( \tau(t^*) \) and all other bidders’ types are below \( t^* \). Let \( e = F_1(\tau(t^*))F_2(t^*) \cdots F_n(t^*) \) denote the probability of event \( \mathcal{E} \). Let \( h = (N,0,\{1\},0) \) denote the equilibrium terminal history in event \( \mathcal{E} \). Let \( h' = (\ldots, \{1,i\}, b_i(t^*), \{i\}, b_i(t^*)) \) denote the terminal history in event \( \mathcal{E} \) if bidder \( i \) deviates by planning to drop out at \( b' \).

Observe that \( \mathbf{G}_h = J_{\tau(t^*),t^*} = \mathbf{G}_{h'} \). Hence,

\[
u_{i,(N,0)}(0, t) = \delta l_{i1}(t, J_{\tau(t^*),t^*}) e,
\]

\[
u_{i,(N,0)}(b', t) = (\max\{t, \delta w_i(t, J_{\tau(t^*),t^*})\} - b_i(t^*)) e.
\]

Hence,\(^{34}\)

\[
\frac{\partial u_{i,(N,0)}}{\partial t}(0, t) = \delta q_{i1}(t, J_{\tau(t^*),t^*}) e \leq \delta q_{i1}(t, J_{\tau(t^*),t^*}) e \leq \frac{\partial u_{i,(N,0)}}{\partial t}(b', t).
\]

Together with

\[
\delta l_{i1}(t^*, J_{\tau(t^*),t^*}) = t^* - b_i(t^*) = \max\{t^*, \delta w_i(t^*, J_{\tau(t^*),t^*})\} - b_i(t^*),
\]

we obtain (48). \( \square \)

\(^{34}\)Observe that the mapping \( t \mapsto u_{i,(N,0)}(b', t) \) is absolutely continuous, and is therefore differentiable almost everywhere and is the integral over its derivative.

If \( g(t) \) and \( h(t) \) are absolutely continuous functions, then \( k(t) = \max\{g(t), h(t)\} \) is absolutely continuous, and \( dk/dt \geq \min\{dg/dt, dh/dt\} \).
6 References


PAGNOZZI, M. (2007): ...

