Credible Ratings

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ABSTRACT: This paper considers a model of a rating agency with multiple clients, in which each client has a separate market that forms a belief about the quality of the client after the agency issues a rating, and there is no payoff link among the clients in the agency’s utility. When the clients are rated separately (individual rating), the credibility of a good rating in an inflationary equilibrium of the signaling game is limited by the incentive of the agency to exaggerate the quality of the client. In centralized rating, the agency rates all clients together and shares the rating information among all markets. This allows the agency to coordinate the ratings and achieve a higher average level of credibility for its good ratings than in individual rating. In decentralized rating, the ratings are again shared among all markets, but each client is rated by a self-interested rater of the agency with no access to the quality information of other clients. When the underlying qualities of the clients are correlated, decentralized rating leads to a smaller degree of rating inflation and hence a greater level of credibility than in individual rating. Comparing centralized rating with decentralized rating, we find that centralized rating dominates decentralized rating for the agency when the underlying qualities are weakly correlated, but the reverse holds when the qualities are strongly correlated.

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1. Introduction

Consider a rating agency that issues a report on each of its clients. The rating agency is informed of the quality of each client and its report on the client is received as a signal by the market that the client faces. The agency cares about the payoff to each client. Examples of a rating agency with multiple clients include an economics department that ranks its PhD graduates, a stock brokerage firm that deals with multiple stocks, and a consumer electronics magazine that issues ratings on multiple products. We are interested in an environment in which the payoff to each client depends only on the perceived quality of that client, and not on the perceived qualities of other clients, so that there is no direct payoff link among the clients. The only possible link is indirect, and informational: when the markets are given access to all client ratings, the perceived quality of each client can depend on the ratings of other clients, either exogenously through some statistical correlation among client qualities, or endogenously through the reporting strategy of the agency, or both. In the economics department example, the payoff link is likely to be absent if the PhD graduates are in different fields so that their markets are separate, or if the markets are sufficiently thick that each graduate receives a competitive wage, while the informational link will be present if there are strong cohort effects in the graduate program or if the department ranks the students by comparing them. Similarly, for the stock brokerage firm example and the consumer magazine example, there may be little demand substitutability or complementarity in the aggregate so that the price of a rated stock or an electronic product depends only on the valuation of that stock or product, but a positive correlation among the client qualities can still arise, for example, if the future returns of all the stocks are affected by an economy-wide shock or the electronic products share significant common parts or designs. In our model, because the agency cares about the perceived qualities of its clients, credibility of the ratings is at issue. The objective of this paper is to compare the credibility of ratings under three schemes that differ in

\[1\] The literature on asset pricing focuses on the case where the price of a stock depends on the probability distribution of the future cash flow and some “pricing kernel.” In a large market, the cash flow on any single stock does not affect the pricing kernel, and so the payoffs for different stocks are separable. For the electronics example, payoff separability is a more appropriate assumption if the products belong to different categories, or if consumers have strong brand loyalty.
whether the markets have access to all the reports and in whether the raters in the agency share the knowledge about client qualities.

In “individual rating,” the market for each client does not observe the ratings for other clients. This is a natural benchmark due to the absence of any direct payoff linkage. The rating scheme can be analyzed as a simple signaling model with one sender (the rating agency with a single client) and a receiver (the market for the client), with the market only interested in making the right inference about the client’s underlying quality. We make assumptions on the payoff function of the agency regarding its reputational concerns and how these concerns interact with the derived benefits from an improved perception of the client quality. These assumptions imply that the incentive to exaggerate the quality always outweighs the reverse incentive to downplay it regardless of the resulting belief of the market regarding the quality. This “single crossing” property allows us to focus on the “inflationary equilibrium,” which is a semi-pooling equilibrium where the client’s quality is truthfully revealed whenever it is good and sometimes exaggerated when it is bad. The benchmark model of individual rating can be interpreted as a model of credibility, with the equilibrium perception of a good rating as the measure of credibility and a one-to-one correspondence between credibility and the equilibrium ex ante payoff of the agency. The inability of the rating agency to commit to an honest rating policy dilutes the meaning of a good rating without changing the meaning of the bad rating, and therefore reduces the rating agency’s ex ante payoff. We ask the following question in the rest of the paper: can the rating agency obtain a higher ex ante payoff than in the inflationary equilibrium in individual rating by improving credibility of good ratings?

In “centralized rating,” the agency rates all clients together and shares the reports among all markets. Each market can use the ratings of other clients as well as its own client to make inference about the quality of the latter. Sharing the rating information among all markets allows the agency to coordinate the ratings with a correlated randomization between good and bad ratings across clients of bad quality, even when client qualities are statistically independent. It turns out that in an inflationary equilibrium that is not payoff equivalent to the equilibrium under individual rating, there must be a minimal number, larger than zero, of good ratings issued regardless of the actual number of clients of good
quality. Further, ratings are coordinated so that there is more ratings inflation when there are fewer good quality clients; indeed, all clients are given good ratings either when all are indeed good or when all are bad. If such coordination is credible or incentive compatible, it results in a higher payoff to the agency than under individual rating. We show that there exists an equilibrium that weakly dominates the benchmark inflationary equilibrium under individual rating for the agency. This coordinated ratings equilibrium is unique when it strictly dominates the benchmark inflationary equilibrium.

In “decentralized rating,” the ratings are shared among all markets, as in centralized rating, but each client is rated by a self-interested rater of the agency with no access to the quality information of other clients. This means that only independent randomization across clients of bad quality is possible, as in individual rating. However, unlike individual rating, ratings information is shared among all markets. In an inflationary equilibrium the perception of a good rating depends on the total number of good ratings in all markets: the perception improves with more good ratings when the client qualities are positively correlated, and it deteriorates when the qualities are negatively correlated. This endogenous payoff link among the clients makes it more difficult for each rater to fool the market with an exaggerated rating. As a result, the equilibrium probability of an inflationary rating can be lower and the average credibility of a good rating can be higher than in the benchmark inflationary equilibrium under individual rating, leading to a greater equilibrium payoff for the agency than the benchmark.

Comparison between centralized rating and decentralized rating in terms of equilibrium credibility of good ratings and ex ante payoff to the agency depends on the degree of correlation. When the underlying qualities are independently distributed, any inflationary equilibrium under decentralized rating is payoff-equivalent to the benchmark inflationary equilibrium under individual rating, as the ratings of other clients cannot discipline each individual rater and thus there is no gain in credibility. In contrast, under centralized rating the necessary and sufficient condition for an inflationary equilibrium that strictly dominates the benchmark equilibrium is typically satisfied under independence. Thus, centralized rating dominates decentralized rating for the agency under independence. With correlation across the underlying qualities, there is less room to manipulate ratings under both centralized rating and decentralized rating. When the underlying qualities are
almost perfectly correlated, under centralized rating there is no inflationary equilibrium
that strictly dominates the benchmark equilibrium under individual rating, as the strong
correlation across client qualities severely reduces the credibility of coordinated rating. In
contrast, under decentralized rating the discipline on credibility imposed by strong cor-
relation allows the construction of an inflationary equilibrium that is arbitrarily close to
truth-telling. Thus, centralized rating is dominated by decentralized rating for the agency
with strong correlation.

Our comparison results regarding individual rating, centralized rating and decentral-
ized rating have strong implications for how a rating agency can gain credibility of its rat-
ings and improve its welfare. Since there exist inflationary equilibria that weakly dominate
the benchmark under either centralized or decentralized rating schemes, it is always to the
advantage of the agency to share ratings information among all markets it serves. Whether
the agency should share information about client qualities among its raters or commit to a
policy that restricts information access and preserves the raters’ independent concerns for
career reputation, depends on the underlying correlation structure across client qualities.
Our results suggest that the agency should group together clients with weakly correlated
qualities and centralize their rating, but for clients with strongly correlated qualities the
agency should decentralize their rating among the raters.

It is interesting to interpret our comparison results between centralized rating and
decentralized rating in terms of different market structures for rating agencies as opposed
to different information structures for a single rating agency. The centralized rating scheme
naturally corresponds to the monopoly market structure, while the decentralized scheme
can be equivalently viewed as the competitive market structure. Although under the
decentralized scheme there is no direct competition among the agencies because the clients
have separate markets, the agencies indirectly compete for credibility as the ratings are
observed by all markets. Our results then suggest that the comparison between the two
market structures depends on the degree of correlation across the underlying states of
nature. The monopoly structure performs better due to an economy of scale when the
states are weakly correlated. When the states are strongly correlated, the competitive
structure does better because competing ratings constrain the incentive to inflate and
improve the credibility of good ratings.
The paper is organized as follows. Section 2 presents the basic ingredients of our model of rating agencies. We introduce the out-of-equilibrium belief refinement used throughout of the paper, and characterize an inflationary equilibrium under individual rating that serves as the benchmark of comparison. In Section 3 we deal with centralized rating. This turns out to be a signaling model with one-dimensional private information and multi-dimensional signals. We establish the existence of an inflationary equilibrium that weakly dominates the benchmark inflationary equilibrium of individual rating for the agency in terms of expected payoff. We provide a necessary and sufficient condition for the existence of an equilibrium that strictly dominates the benchmark inflationary equilibrium, and show that the equilibrium is unique when it exists. Section 4 presents the model of decentralized rating. We introduce a correlation structure that accommodates possibilities of both positive and negative correlation across client qualities in a multi-dimensional setting. Using the structure we show that there exists an inflationary equilibrium that weakly dominates the benchmark inflationary equilibrium of individual rating for the agency in terms of expected payoff under further assumptions on the payoff functions of the agency. In Section 5 we study how the comparison between centralized rating and decentralized rating depends on the correlation across client qualities. Section 6 provides some remarks on related literature. Proofs of all lemmas can be found in the appendix.

2. A Model of Rating Agencies

A rating agency deals with \( N \) clients. In our model the \( N \) sets of relationship between each client \( i \), \( i = 1, \ldots, N \), and the corresponding market (end-user of the rating for the client) are identical. The underlying quality \( S_i \) of each client \( i \) is either good (\( G \)) or bad (\( B \)); the rating \( s_i \) for the client is either good (\( g \)) or bad (\( b \)). The objective function of the market is to minimize the expectation of the squared difference between a real-valued decision variable \( \delta_i \) and a random variable which is equal to \( \delta_G \) if the the quality of the client is \( G \) and \( \delta_B \) if the quality is \( B \). Let \( q_i \) denote the market’s belief that the quality of the client is good. The solution to the maximization problem is then to set \( \delta_i \) to \( q_i \delta_G + (1 - q_i) \delta_B \), which depends only on the endogenous variable \( q_i \). We write the rating agency’s utility
function from client $i$ as $U(S_i, s_i, q_i)$ for $S_i = G, B$ and $s_i = g, b$. The total utility to the agency is the sum $\sum_{i=1}^{N} U(S_i, s_i, q_i)$.

For the statistical distribution of client qualities, at this point we assume only that the client qualities are exchangeable random variables: the probability of any realization of the random vector $(S_1, \ldots, S_N)$ depends only on the number of clients of good quality. The joint probability distribution of $(S_1, \ldots, S_N)$ can then be represented by a vector $(\pi_0, \ldots, \pi_N)$, where $\pi_n$ is the probability that there are exactly $n$ clients of good quality. We assume that $\pi_n > 0$ for each $n = 0, 1, \ldots, N$. Define $\pi$ as the probability that any given client is of good quality, which satisfies

$$\pi = \frac{1}{N} \sum_{n=1}^{N} n \pi_n. \tag{1}$$

The assumption of exchangeability introduces symmetry across clients that simplifies our analysis without imposing statistical independence. In the applications of the model that we have in mind, correlated client qualities might be an important feature. For example, student qualities might be correlated through peer effects, stock valuations through some underlying common fundamental, and electronic products through common design features. It turns out that the specific correlation structure does not play any role in our analysis of individual and centralized rating schemes. We will need to make further assumptions on the correlation structure when we analyze decentralized rating.

A few remarks about the setup are in order. First, the specific preference function adopted here for the markets is meant to capture the idea that each client faces competitive bids after the market updates its belief about the quality of the client based on the reports of the agency. This reduces the role of the receiver in our signaling model to forming rational expectations of the client quality, and allows us to focus on the signaling incentives of the agency. Second, the utility of the agency in the relationship with client $i$ is assumed to depend on the market’s belief $q_i$ about client $i$’s quality, which summarizes the payoff to the client. This models the idea that the agency is not an impartial provider of information, in that it cares about the payoff to the client. Third, both the underlying quality $S_i$ and the signal $s_i$ enter the utility function of the agency. This form allows for any two-state, two-signal setup. The general idea is that the utility of the agency is affected both by the payoff
to the client and by its own reputational concerns, and we are using the function $U$ as a reduced-form representation of the agency’s utility. Later we will make further assumptions on how the concern for the client’s payoff and the reputational concerns interact with each other. Finally, the utility of the agency is assumed to be additively separable in the utilities from the $N$ sets of client relationships. This separability assumption is justified if the payoff to each client $i$ only depends on the belief $q_i$ about the client’s quality. As mentioned in the introduction, there are environments in the labor market, the financial market and the goods market in which this assumption is reasonably appropriate. We do not claim that it holds in all relevant situations for rating agencies. Rather, the separability assumption is made to allow us to focus on informational issues of ratings.

We need to make further assumptions on the common utility function $U$. We drop the subscript $i$ for now as there is no risk of confusion. First, we assume that the derivative of $U(S, s, q)$ with respect to $q$, $U_q(S, s, q)$, exists and is strictly positive for each $q \in (0, 1)$.

Assumption 1. $U_q(S, s, q)$ exists and is strictly positive for each $S = G, B$, $s = g, b$ and $q \in (0, 1)$.

Signaling games often have a multiplicity of equilibria. One way to minimize the equilibrium selection issue is to ensure that if the agency weakly prefers $g$ to $b$ when the quality is $B$, then it strictly prefers $g$ to $b$ in state $G$, and conversely, if the agency weakly prefers $b$ to $g$ in state $G$, then it strictly prefers $b$ to $g$ when the quality is $B$. This condition may be referred to as “single-crossing.” It will be used to limit equilibrium signaling to one form of misrepresentation, referred to as “inflationary rating” (issuing a good rating when the quality is bad), and to rule out “deflation” (issuing a bad rating when the quality is good). For the single-crossing result to be effective in eliminating unwanted equilibria, we will need it to hold regardless of how different ratings induce different beliefs:

$$U(G, g, q) - U(G, b, q') > U(B, g, q) - U(B, b, q')$$

for all $q, q' \in [0, 1]$. Condition (2) can be thought of as payoff complementarity between the underlying quality $S$ and the rating $g$, modified to suit the signaling model so that it

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2 This rules out situations where the market’s response to the agency’s rating is discrete, for example, where the only choice of the market is whether or not to acquire the client’s service at some fixed wage.
holds whenever a switch of the underlying quality for the same rating does not affect the belief $q$ while a switch of the rating for the same quality generally will affect $q$.\footnote{Condition (2) is stronger than we need for the purpose of the analysis; single-crossing requires it to hold only when the right-hand-side is non-negative.}

The following assumption on utility functions $U(S, s, q)$, together with Assumption 1, immediately leads to condition (2).\footnote{To see this, note that since $U_q(G, g, q) > U_q(B, g, q)$, we have $U(G, g, q) - U(B, g, q) ≥ U(G, g, 0) - U(B, g, 0)$ for any $q$. Similarly, since $U_q(G, b, q) < U_q(B, b, q)$, we have $U(G, b, q') - U(B, b, q') ≤ U(G, b, 0) - U(B, b, 0)$ for any $q'$. Condition (2) then follows from inequality (2.2) in Assumption 2.}

**Assumption 2.** $U_q(G, g, q) > U_q(B, g, q), U_q(G, b, q) < U_q(B, b, q)$ for any $q ∈ (0, 1)$, and

$$U(G, g, 0) - U(G, b, 0) > U(B, g, 0) - U(B, b, 0).$$

Inequality (3) in the assumption is simply inequality (2) evaluated at $q = q' = 0$. The two conditions on the derivatives of $U$ require that with each rating $s$ the agency benefit more from an improvement in the belief $q$ when the agency is telling the truth about the quality of the client.\footnote{The inequalities are sufficient but not necessary for the single-crossing condition (2). Our analysis of individual rating and decentralized rating goes through so long as (2) holds, but the two inequality conditions on $U_q$ are used for equilibrium construction in the case of centralized rating.} One may interpret the difference $U(G, g, ·) - U(B, g, ·)$ as a measure of the agency’s reputational concern for honesty. Given the same rating $g$ and any belief $q$, $U(B, g, q)$ differs from $U(G, g, q)$ because the agency is concerned that the true quality of the client may be discovered, thus revealing a dishonest rating. Similarly, the difference $U(B, b, ·) - U(G, b, ·)$ is a measure of the agency’s reputational concern for competence: for the same rating $b$ and any $q$, $U(G, b, q)$ differs from $U(B, b, q)$ because when the true quality of the client is discovered, it reveals an inaccurate rating. Assumption 2 requires both differences to be increasing in the client’s perceived quality $q$. This assumption is motivated by the idea that it is more likely (or faster) that the market learns the true quality of the client when the perceived quality is higher. For the consumer magazine example mentioned in the introduction, if an electronic product is new to the market and is of an experience good variety, a higher perceived quality will lead to greater sales and faster consumer learning about its true quality. Similarly, a higher market belief about
the quality of a job candidate is more likely to result in a better and more challenging job placement, which can quickly reveal the true quality of the candidate, and a higher valuation about a rated stock may lead to a greater transaction volume, which motivates more subsequent research.

The next set of assumptions is made to rule out uninteresting equilibria in order to bring out our main results more effectively. It implies that there exist favorable beliefs that will induce the agency to issue an inflationary rating when the quality is \( B \), but there is no incentive to inflate if beliefs cannot be favorably manipulated.

**Assumption 3.** \( U(B, g, 1) > U(B, b, 0) > U(B, g, 0) \).

Assumptions 1 and 3 imply that there is a unique \( q^* \in (0, 1) \) that satisfies

\[
U(B, g, q^*) = U(B, b, 0).
\] (4)

The above equation is the critical indifference condition under quality \( B \) that defines a unique inflationary equilibrium in the benchmark scheme of individual rating. Under individual rating, the market for each client has no access to ratings for other clients. Since the clients are exchangeable, the model reduces to \( N \) identical signaling games involving the agency and the market. In each such game, an inflationary rating strategy is such that the agency gives \( g \) under quality \( G \) and randomizes between \( g \) and \( b \) under quality \( B \).

Suppose that there exists \( p \in (0, 1) \) such that

\[
\frac{\pi}{\pi + (1 - \pi)p} = q^*,
\] (5)

where \( \pi \) is given in equation (1). Then, we have a semi-separating equilibrium in which the agency gives \( b \) under \( B \) with probability \( p \): by equation (4) the agency is indifferent between \( g \) and \( b \) under quality \( B \), which by the single-crossing condition (2) implies that the agency strictly prefers \( g \) to \( b \) under quality \( G \). We refer to this type of inflationary equilibrium as "full support inflationary equilibrium," as the support of the equilibrium strategy is the same as the space of the signals. Since equation (5) can be satisfied by some \( p \in (0, 1) \) only if \( \pi < q^* \), a full support equilibrium does not exist if \( \pi \geq q^* \). Instead, we can construct a "non-full support equilibrium" in which the agency gives \( g \) with probability 1
under $B$. This is accomplished by specifying the out-of-equilibrium belief that the quality of the client is $B$ with probability 1 when $b$ is observed: since the equilibrium belief that the quality is $G$ when $g$ is observed is equal to the prior probability $\pi$, the agency weakly prefers $g$ to $b$ under quality $B$, which implies that it strictly prefers $g$ to $b$ under $G$ by (2). Further, due to the same single-crossing condition (2), the above specification of the out-of-equilibrium belief is the only one consistent with the refinement concept of “Divinity” (Banks and Sobel, 1987).\footnote{More precisely, for any out-of-equilibrium belief $\hat{\varrho}$ that the quality is $G$ after $b$ is observed, $U(G, b, \hat{\varrho}) \geq U(G, g, \pi)$ implies that $U(B, b, \hat{\varrho}) > U(B, g, \pi)$. Thus, $\hat{\varrho} = 0$ under the refinement of Banks and Sobel.} We use this refinement throughout the paper, and we refer to a sequential equilibrium that passes the refinement test simply as equilibrium. It follows that there is a unique inflationary equilibrium under individual rating, which is full support if $q^* > \pi$ and non-full support if $q^* \leq \pi$.\footnote{With additional assumptions, we can show that no other equilibrium exists under individual rating. In particular, if $U(G, g, 1) > U(G, b, 1)$, then we can rule out all “deflationary” equilibria in which the agency gives $b$ with a positive probability under quality $G$. However, since the focus of this paper is on the credibility of good ratings, we are only interested in constructing inflationary equilibria under different rating schemes.}

The model of individual rating can be interpreted as a model of credibility. The market’s perception of the quality of the client given a good rating is $q^*$ in a full support equilibrium, and is $\pi$ in a non-full support equilibrium. This market belief quantifies equilibrium credibility in our model. From the equilibrium indifference condition (4), we see that the value of $q^*$ depends only on the function $U(B, g, \cdot)$ and the value of $U(B, b, 0)$. When the prior probability of good quality is higher than $q^*$, an increase in the prior translates into an increase in the equilibrium credibility of good ratings by the same amount, which allows the agency to simply pass any client of bad quality as one of good quality. In contrast, when the prior probability is lower than $q^*$, an increase in the prior has no effect on the equilibrium credibility. The increase in the prior probability means that a good rating is too attractive if the agency keeps the probability of reporting $g$ in state $B$ unchanged, and so the probability of inflated good ratings must increase to restore the equilibrium indifference condition. As a result, the equilibrium credibility, and hence the utility to the agency, is pinned down by the indifference condition so long as the
agency reports \( b \) with a positive probability in equilibrium.\(^8\)

The last assumption is a strengthening of the single-crossing condition (2):

**Assumption 4.** For any \( q \in (q^*, 1) \),

\[
\frac{U(G, g, q) - U(G, b, 0)}{U(B, g, q) - U(B, b, 0)} > \frac{U_q(G, g, q)}{U_q(B, g, q)}.
\]

By Assumption 2, both the left-hand-side and the right-hand-side of the above inequality are greater than 1. Assumption 4 strengthens condition (2) for a particular range of market beliefs. Alternatively, the assumption can be viewed as imposing an upper bound on \( U(G, b, 0) \), which is the payoff to the agency from a client of quality \( G \) when it gives the rating \( b \). Assumption 4 thus requires the payoff to be sufficiently low, or the reputational concerns for competence to be sufficiently great. This assumption is used in the construction of inflationary equilibria under centralized rating to regulate the incentives to issue deflationary ratings.\(^9\)

### 3. Centralized Rating: A Model of Multi-dimensional Signals

This section considers centralized rating, in which a single rater of the agency rates all \( N \) clients and shares the rating information among all markets. Although the payoff to each client depends only on the market’s perception of the quality of this client, under centralized rating all the reports are used to make inference about the quality of each client. This means that the agency can potentially coordinate the \( N \) ratings in an attempt to influence market perception.

It may not be intuitive that centralized rating creates opportunities for the agency to increase the credibility of good ratings relative to individual rating, especially if the

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\(^8\) In equilibrium the agency gets its complete information payoff \( U(B, b, 0) \) under quality \( B \), but its equilibrium payoff under quality \( G \) is \( U(G, g, q) \), which is strictly lower than the complete information payoff \( U(G, g, 1) \). Thus, the ex ante payoff to the agency (before the client’s quality is revealed) is lower than what it would obtain if it could commit to truthful revelation of the quality.

\(^9\) With a sufficiently tighter upper bound on the value of \( U(G, b, 0) \), it is possible to rule out all deflationary equilibria. For example, a sufficient condition is that \( U(G, g, 0) - U(G, b, 0) > (N - 1)(U(G, g, 1) - U(G, g, 0)) \), which implies that the smallest loss due to a deflationary rating of a single client exceeds the largest possible gain from all other clients. This assumption would also significantly simplify our analysis for the case of centralized rating. However, it does not hold with an arbitrarily large \( N \).
client qualities are statistically independent. Indeed, it is easy to see that in the case of independent qualities, the equilibrium outcome of individual rating can be supported under centralized rating if the agency independently randomizes between \( g \) and \( b \) for each client of bad quality with the same probability of choosing \( b \) as in individual rating. In this case, the market belief about the quality of any client \( i \) with a good rating remains \( q^* \), regardless of the other ratings, as they provide no information about client \( i \)'s quality under independent qualities and independent randomization. Moreover, this is the only equilibrium outcome under independent randomization. Indeed, a more general result is established below: even when the qualities are correlated and randomizations are coordinated among the clients, any inflationary equilibrium is payoff-equivalent to the benchmark inflationary equilibrium with belief \( q^* \) as long as \( N \) bad ratings are issued with a positive probability in equilibrium.

The key to improved credibility under centralized rating relative to individual rating is to construct an inflationary equilibrium in which the agency never reports \( N \) bad ratings, and we provide a characterization of the structure of any such equilibrium. The main result of this section establishes a necessary and sufficient condition for the existence of an equilibrium with improved credibility. This condition requires the prior probability of having \( N \) bad qualities to be sufficiently low, so that it is credible for the agency never to issue \( N \) bad ratings.

Formally, for the rating agency, the state is now an \( N \)-dimensional vector \((S_1, \ldots, S_N)\) where \( S_i \in \{G, B\} \) for \( i = 1, \ldots, N \). The signal is similarly an \( N \)-dimensional vector \((s_1, \ldots, s_N)\) where \( s_i \in \{g, b\} \) for \( i = 1, \ldots, N \). Given that \( S_1, \ldots, S_N \) are exchangeable, we impose a symmetry requirement that the market belief about any client \( i \)'s quality depend only on the rating \( s_i \) of the client and the total number good ratings issued by the agency. For any \( i = 1, \ldots, N \), let \( q(m) \) be the market belief that \( S_i = G \) when \( s_i = g \) and \( \#\{j : s_j = g\} = m \). Similarly, define \( \hat{q}(m) \) to be the market belief that \( S_i = G \) when \( s_i = b \) and \( \#\{j : s_j = g\} = m \). Given the state, the agency chooses the signal vector \((s_1, \ldots, s_N)\) to maximize the sum of utilities \( \sum_{i=1}^{N} U(S_i, s_i, q_i) \) where \( q_i = q(m) \) if \( s_i = g \) and \( q_i = \hat{q}(m) \) if \( s_i = b \) for all \( m = \#\{j : s_j = g\} \). It directly follows from the single-crossing condition (2) that while the agency may have an incentive to mislead the markets about the total number of clients of good quality, it has no incentive to mislead
the markets about the identity of clients of good quality. That is, for any \( i = 1, \ldots, N \), when \( \{j : S_j = G\} \leq \{j : s_j = g\} \), then \( S_i = G \) implies \( s_i = g \).\(^{10}\) The same is true about the identity of clients of bad quality when the agency deflates the number of clients of good quality. As a result, we can reduce the state space to a one-dimensional variable representing the number of clients of good quality. Denote the signaling strategy of the agency as \( p(m; n) \), the probability of giving \( m \) good ratings when the \( n \) clients are of good quality. Note that the strategy is multi-dimensional because for each number \( n \) we need to specify a vector of probability numbers \( p(m; n) \) for \( m = 0, \ldots, N \). Obviously, we require \( \sum_{m=0}^{N} p(m; n) = 1 \) for all \( n = 0, \ldots, N \).

Let \( W(m; n) \) be the expected payoff to the agency when it chooses \( m \) good ratings when the number of good quality clients is \( n \). For \( m \geq n \), we have

\[
W(m; n) = nU(G,g,q(m)) + (m-n)U(B,g,q(m)) + (N-m)U(B,b,\hat{q}(m)).
\]

For \( m \leq n \), we have

\[
W(m; n) = mU(G,g,q(m)) + (n-m)U(G,b,\hat{q}(m)) + (N-n)U(B,b,\hat{q}(m)).
\]

The follow lemma imposes some restrictions on equilibrium strategies.

**Lemma 1.** (i) For any \( m \leq n < n' \leq m' \), if \( W(m'; n) \geq W(m; n) \) then \( W(m'; n') > W(m; n') \); (ii) for any \( n < n' \leq m, m' \) and \( q(m') > q(m) \), if \( W(m'; n) \geq W(m; n) \), then \( W(m'; n') > W(m; n') \); and (iii) for any \( m, m' \leq n' < n \) and \( \hat{q}(m') > \hat{q}(m) \), if \( W(m'; n) \geq W(m; n) \), then \( W(m'; n') > W(m; n') \).

The first part of the lemma means that the relative incentive to inflate rather than deflate is stronger when the number of clients of good quality is greater. It implies that if in any equilibrium \( p(m'; n) > 0 \) for \( m' > n \), then \( p(m; n') = 0 \) for all \( n' \in \{n+1, \ldots, m'\} \)

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\(^{10}\) To see this, let \( \{j : S_j = G\} = n \) and \( \{j : s_j = g\} = m \). If \( \{j : S_j = G \text{ and } s_j = g\} = n \), the expected utility to the agency is \( nU(G,g,q(m)) + (m-n)U(B,g,q(m)) + (N-m)U(B,b,\hat{q}(m)) \). If instead \( \{j : S_j = G \text{ and } s_j = g\} = n' < n \), the expected utility to the agency is reduced by \( (n-n') \) times \( [U(G,g,q(m)) - U(G,b,\hat{q}(m))] - [U(B,g,q(m)) - U(B,b,\hat{q}(m))] \), which is positive by condition (2).
and \( m \leq n \).\(^{11}\) The second part of the lemma states that the incentive to inflate to a signal with a more favorable belief about good ratings is stronger when there are more clients of good quality, while the third part states that the incentive to deflate to a signal with a more favorable belief about bad ratings is stronger when the agency has more clients of bad quality.

An inflationary strategy satisfies \( p(m; n) = 0 \) for all \( n \) and all \( m < n \). The assumptions made in Section 2, and Lemma 1, are in general insufficient to rule out deflationary equilibrium strategies. Nevertheless, it is natural to focus on inflationary equilibria. Given an inflationary equilibrium let \( T = \{ m : \sum_{n=0}^{N} p(m; n) > 0 \} \) be the set of all signals which are issued with positive probability, and let \( l = \min T \) be the smallest signal (with the lowest number of good ratings). Define \( T_n = \{ m : p(m; n) > 0 \} \) as the set of signals sent with positive probabilities when there are \( n \) clients of good quality. In an inflationary equilibrium, for each \( m \in T \), the market beliefs upon observing \( m \) good ratings are

\[
q(m) = \frac{\sum_{n=0}^{N} \pi_n p(m; n)n}{m \sum_{n=0}^{N} \pi_n p(m; n)},
\]

and \( \hat{q}(m) = 0 \). The following lemma distinguishes two types of inflationary equilibria.

**Lemma 2.** In any inflationary equilibrium, (i) if \( l = 0 \), then \( q(m) = q^* \) for all \( m > 0 \) and \( m \in T \); and (ii) if \( l > 0 \), then either \( q(m) = q^* \) for all \( m \in T \) or \( q(m) > q(m') > q^* \) for \( m, m' \in T \) and \( m < m' \).

An inflationary equilibrium with \( l = 0 \) does not have full support if \( T \neq \{0, 1, \ldots, N\} \). However, part (i) of Lemma 2 establishes that any inflationary equilibrium with \( l = 0 \) is payoff-equivalent to the full support inflationary equilibrium under individual rating.

Although each market can use the ratings of other clients as well as its own client to make inference about the quality of the latter, the rating agency gains no credibility relative to

\(^{11}\) Conversely, if in any equilibrium \( p(m; n') > 0 \) for \( m < n' \), then \( p(m'; n) = 0 \) for all \( n \in \{m, \ldots, n'-1\} \) and \( m' \geq n \). Although we restrict to inflationary equilibria in the following analysis, this and the other two results in Lemma 1 are needed for restricting out-of-equilibrium beliefs. Note that part (i) of Lemma 1 does not imply that if \( W(m'; n') \geq W(m; n) \) for some \( m' > m \) then \( W(m'; n') > W(m; n') \) for all \( n' > n \). In other words, the incentive to increase the number of good ratings is not necessarily single-crossing in the number of good quality clients. Indeed, what satisfies single-crossing is the incentive to inflate as opposed to deflate. Similarly (ii) and (iii) of Lemma 1 are not single-crossing conditions either, because they require restrictions on the endogenous variables \( q \).
individual rating. In any such equilibrium, when all clients have bad quality, the agency is indifferent between issuing zero good rating and issuing any number of good ratings in $T$. These indifference conditions reduce centralized rating to individual rating in terms of payoff to the agency.\footnote{The proof of this result (in the appendix) is more complicated than indicated by this reasoning, because we have to allow for non-full support strategies. This requires the use of the refinement. Later, we will show that all inflationary equilibria have the threshold property that $T = \{l, ..., N\}$. However, if we restrict to strategies that satisfy this property, then Lemma 2 and part (i) through part (iii) of Lemma 3 below can be established using the equilibrium conditions, without resorting to the refinement.} Part (ii) of the above lemma establishes that in an equilibrium with $l > 0$, either the same indifference conditions are again at work and the market belief corresponding to a good rating is the same regardless of the number of good ratings issued and equal to $q^*$, or the market beliefs are all strictly greater than $q^*$. In the second case, the beliefs decrease in the number of good ratings issued, for otherwise the agency would inflate as much as possible. The second type of inflationary equilibria are more interesting, because the agency’s ex ante payoff is higher than in the benchmark full support individual rating case. From now on, we distinguish equilibria according to whether they are payoff-equivalent to the full support equilibrium under individual rating: equilibria with $l > 0$ and $q(l) > q^*$ are referred to as non-full support equilibria, and those with $q(m) = q^*$ for all $m \in T$ are referred to as full support equilibria regardless of whether $l = 0$ or $l > 0$. The next lemma provides a partial characterization of the structure of the equilibrium signaling strategy in a non-full support equilibrium.

**Lemma 3.** In any non-full support equilibrium, (i) $T_l \ni l$; (ii) $T_m = \{m\}$ if $m \in T$ and $m > l$; (iii) $\min T_m \geq \max T_{m+1}$ for all $m < l$; (iv) $T = \{l, ..., N\}$; and (v) $q(m) = 1$ and $\hat{q}(m) = 0$ for all $m < l$.

The structure of the equilibrium strategy described by Lemma 3 is illustrated in Figure 1. In the figure, an arrow from node $n$ to $m$ indicates that $p(m; n) > 0$ in a non-full support equilibrium. When the number of clients of good quality is greater than the minimum number $l$ of good ratings issued, the agency issues a truthful report with probability 1. When the number of clients of good quality is less than $l$, the agency exaggerates the number of good quality clients; indeed it issues more good ratings when there are fewer
clients of good quality.\textsuperscript{13} This characterization follows from the result in Lemma 2 that the credibility of a good rating decreases with the total number of good ratings, and the result in Lemma 1 that the agency has a stronger incentive to inflate to a more credible signal when there are more clients of good quality. Part (iv) of the above lemma establishes that in any non-full support equilibrium the aggregate support of the equilibrium strategy, $T$, satisfies the threshold property that all signals $m \geq l$ are sent with positive probability. Finally, part (v) of the lemma specifies a unique set of out-of-equilibriums beliefs $q(m)$ and $\hat{q}(m)$ for $m \notin T$ that satisfy the refinement. It is established by showing that if the agency finds it weakly optimal to send an out-of-equilibrium signal $m < l$ when there are $n \neq m$ good quality clients, then the signal is strictly optimal when there are exactly $m$ good quality clients.

The main result of this section is Proposition 1 below. Since the proof in the appendix is rather involved, it is useful here to describe the main steps. We start by showing that the restrictions imposed by Lemma 3 on the structure of $T_n$ in each state $n > 0$, together with necessary equilibrium conditions that there are no profitable deviations, result in certain iterative constraints on the equilibrium reporting strategy $p(m; n)$ given the reporting strategies in states $n + 1, \ldots, N$ (Definition A.1 and Lemma A.1 in the appendix). Next we show that, given the reporting strategies in state $n = 1, \ldots, N$, all the equilibrium conditions can be satisfied by choosing the value of $p(m; 0)$ appropriately.

\textsuperscript{13} When the number of clients of good quality is equal to $l$, the agency may tell the truth in equilibrium, or it may randomize between issuing $l$ or issuing more than $l$ good ratings (as depicted in Figure 1).
for each $m = 0, \ldots, N$, and an equilibrium obtains when such values satisfy $\sum p(m; 0) = 1$ (Definition A.2 and Lemma A.2). Then, we show that the set of reporting strategies for state $1, \ldots, N$ satisfying the necessary conditions of an equilibrium is closed and connected and, under an appropriately defined binary relation, is strictly ordered (Definition A.3, Lemma A.3 and Lemma A.4). In the final step of the proof we show that the set of values $p(m; 0)$ that completes the equilibrium conditions has the property that $\sum p(m; 0)$ strictly decreases continuously over the collection of reporting strategies in state $1, \ldots, N$ satisfying the necessary equilibrium conditions (Lemma A.5). We use this property to conclude the proof by showing that exactly one equilibrium exists and it has $l > 0$ if and only if $q^* < 1 - \pi_0$.

**Proposition 1.** An inflationary equilibrium exists under centralized rating. Further, a unique non-full support equilibrium exists if and only if $q^* < 1 - \pi_0$.

The necessary and sufficient condition for the existence of a unique non-full support equilibrium has a simple interpretation. It requires the the probability $\pi_0$ that no client is of good quality to be too small to distribute to each positive message $m > 0$ to achieve the indifference in the state when there are no good quality clients between issuing $m$ good ratings and all bad ratings. If this condition is violated, a randomization strategy $p(m; n)$ with non-full support and $p(0; 0) = 0$ is not credible. On the other hand, when this condition is satisfied, more probability mass from low states (small numbers of good quality clients) can be distributed to higher states through the randomization described in the proof of Proposition 1, implying that $l > 0$. Recall that a non-full support equilibrium exists under individual rating if and only if $q^* < 1 - \pi$, which implies $q^* < 1 - \pi_0$. Thus, correlated randomization under centralized rating can allow the agency to credibly apply a non-full support signaling strategy and achieve a higher average level of credibility, when

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14 The proof of Proposition 1 considers just one particular kind of full support equilibrium, with $l = 0$ and $p(n; n) = 1$ for all $n > 1$. However, there are typically multiple equilibria with $l = 0$ and $q(m) = q^*$, and they can also coexist with a non-full support equilibrium. For example, consider the model with $N = 2$, $\pi_0 = \pi_2 = \rho/2$ and $\pi_1 = 1 - \rho$ for some $\rho \in (0, 1)$, and $q^* > 1/2$. We can show that for $\rho$ in the interval between $2(1 - q^*)(2q^* - 1)/q^*$ and $2(1 - q^*)$, there are three equilibria: one full support equilibrium with $p(1; 1) = 1$, another full support equilibrium with $p(2; 1) > 0$, and one non-full support equilibrium with $l = 1$. 

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independent randomization under individual rating implies that the credibility of good ratings is fixed at $q^*$. The existence result of Proposition 1 allows us to consider what happens when the number of clients $N$ becomes arbitrarily large. The simplest case is when the qualities are independently distributed; by the law of large numbers, the fraction of good quality clients converges to $\pi$ when $N$ converges to infinity. For each $N$ let the quality distribution be given by $\{\pi^N_n\}$. Denote the equilibrium threshold as $l^N$, the equilibrium strategy as $p^N(m;n)$, and the equilibrium belief as $q^N(n)$. We claim that if for all $\mu \in (\pi, 1]$

$$\pi U(G, g, 1) + (\mu - \pi) U(B, b, 0) > \pi U(G, g, \mu) + (\mu - \pi) U(B, g, \mu),$$

or in words if the agency strictly prefers truthful rating to issuing any higher fraction $\mu$ of good ratings when it is common knowledge that with probability 1 a fraction of $\pi$ is good quality, then the equilibrium outcome is arbitrarily close to truthful reporting in the limit: for all $\epsilon > 0$,

$$\lim_{N \to \infty} \sum_{n \in [N(\pi - \epsilon), N(\pi + \epsilon)]} \sum_{m > N(\pi + \epsilon)} \pi^N_n p^N(m;n) = 0.$$ 

To establish this claim, we first argue that $l^N/N$ cannot be smaller than $\pi$ in the limit: $\liminf_{N \to \infty} l^N/N \geq \pi$. If this were not true, then for all small enough $\epsilon > 0$ there exists a subsequence such that $l^N/N < \pi - \epsilon$ for all $N$ in the subsequence. By Lemma 3, for each $N$ in the subsequence $p^N(n;n) = 1$ for all $n \in [N(\pi - \epsilon/2), N(\pi + \epsilon/2)]$. Since the sum of $\pi^N_n$ over $n \in [N(\pi - \epsilon/2), N(\pi + \epsilon/2)]$ converges to 1, for $\epsilon$ sufficiently small and for $N$ sufficiently large there exists $n \in [N(\pi - \epsilon/2), N(\pi + \epsilon/2)]$ such that $q^N(n)$ is arbitrarily close to 1, implying $W(l^N; l^N) < W(n; l^N)$, a contradiction. Next, we argue that the limit of $q^N(l^N)$ exists and is 1; otherwise, since $q^N(l^N - 1) = 1$ by Lemma 3, for $N$ sufficiently large $W(l^N - 1; n) > W(l^N; n)$ for all $n$, a contradiction. Since in the limit $l^N/N$ is greater than or equal to $\pi$, it follows from Lemma 3 that $q^N(\pi N) = 1$ in the limit, and thus the payoff in any state $n$ close to $\pi N$ from truthful reporting is close to $n U(G, g, 1) + (N - n) U(B, b, 0)$. By assumption, this payoff is strictly larger than

\[15\] In fact, our analysis of the limit outcome implies that the limit of $l^N/N$ exists and is equal to $\pi$. 

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the payoff from issuing \( m \) good ratings, if \( m \) is bounded away from \( N\pi \) and if \( q^N(m) \) is sufficiently close to \( n/m \). In the appendix (Lemma A.6), we formally establish that if it is not true that the equilibrium outcome is arbitrarily close to truthful reporting in the limit, then in equilibrium there must exist some state \( n \) close to \( \pi N \) such that with positive probability \( m \) good ratings are issued with \( m \) bounded away from \( N\pi \) and \( q^N(m) \) close to \( n/m \). This contradicts the assumption that the agency strictly prefers truthful rating when it is common knowledge that with probability 1 a fraction of \( \pi \) is good quality, and establishes that in the limit when \( N \) goes to infinity the equilibrium outcome converges to truthful reporting. When the assumption does not hold, there will be ratings inflation even when \( N \) is arbitrarily large.

4. Decentralized Rating: A Model of Competing Signals

In decentralized rating, rating information is shared among all markets, as in centralized rating, but each client is rated by a self-interested rater of the agency with no access to the quality information of other clients. The implicit assumption is that it is possible for the agency to limit the information about client quality available to each rater to the single client that the rater is assigned to.\(^{16}\) In terms of strategy space, decentralized rating is the same as individual rating, as only independent randomization across clients is feasible. If the underlying client qualities are independently distributed, decentralized rating produces identical equilibrium outcome as in individual rating. However, since ratings information is shared among all markets, when the underlying qualities are correlated, each market can use the other ratings to make inference about the quality of its own client.

In this section we construct an inflationary equilibrium under decentralized rating. Unlike the case of centralized rating, the analysis of decentralized rating requires a model of quality correlation across the clients. In Definition 1 below, we give precise formulations for positive and negative correlations among client qualities. These formulations allow us to give sharp characterizations of inflationary equilibria: under positive (negative) correlation

\(^{16}\) How to structure the incentives within the agency to motivate the raters and to restrict their information access is beyond the scope of this paper. We are instead interested in an analysis of credibility from a signaling perspective assuming that the agency has full control over information sharing within the organization.
each rater expects a greater number of good ratings conditional on \( G \) than conditional on \( B \), in the sense of first order stochastic dominance, and credibility of a good rating is increasing (decreasing) in the total number ratings issued. The main result of this section establishes the existence of a symmetric inflationary equilibrium under decentralized rating, and the necessary and sufficient condition for a full support equilibrium. It turns out that this condition is identical to the condition under individual rating. We postpone to the next section a discussion of how in a decentralized scheme the rating agency can gain in credibility under correlated qualities and therefore become better off relative to individual rating.

Define a random variable \( X_i, i = 1, \ldots, N \), such that \( X_i = 1 \) if \( S_i = G \) and \( X_i = 0 \) if \( S_i = B \). Let \( f(X_1, \ldots, X_N) \) represent the joint probability mass function of the random vector \( X = (X_1, \ldots, X_N) \).

**Definition 1.** We say that \( X \) is multivariate totally positive of order 2 (MTP\(_2\)) if, for all \( x, y \in \{0, 1\}^N \),

\[
f(x \vee y)f(x \wedge y) \geq f(x)f(y),
\]

where \( x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_N, y_N\}) \); \( x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_N, y_N\}) \).

We say that \( X \) is multivariate reverse rule of order 2 (MRR\(_2\)) if the above inequality is reversed.

The definition of MTP\(_2\) is the same as log-supermodularity, also referred to as affiliation. It is a commonly used concept of positive dependence among random variables in the statistics literature (see, for example, Joe, 1997) and in the auction literature (see, for example, Milgrom and Weber, 1982). Similarly, MRR\(_2\) can be used to capture the idea of negative dependence among random variables. These dependence concepts are stronger than the notion of “regression dependence” used by Lehmann (1966).

We focus on symmetric inflationary equilibria in which for each \( i = 1, \ldots, N \), the common signaling strategy satisfies \( \Pr[s_i = g | S_i = G] = 1 \) and \( \Pr[s_i = g | S_i = B] = p \) for some \( p \in [0, 1] \).\(^{17}\) Define the random variable \( Y_i, i = 1, \ldots, N \), such that \( Y_i = 1 \) if

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\(^{17}\) In the proof of Proposition 2 below, we use Assumption 2 to establish that the indifference condition between \( g \) and \( b \) under \( B \) is sufficient to imply truth telling under \( G \). But this result presumes the signaling structure of inflationary equilibria. The single crossing condition of Assumption 2 is generally insufficient to rule out deflationary strategies.
$s_i = g$ and $Y_i = 0$ if $s_i = b$. Also let $Z_i = \sum_{j \neq i} X_j$ and $\tilde{Z}_i = \sum_{j \neq i} Y_j$. The following result is straightforward.

**Lemma 4.** If $X$ is exchangeable and MTP$_2$, then, for any $i$, $(X_i, Y_i, Z_i, \tilde{Z}_i)$ is MTP$_2$. If $X$ is exchangeable and MRR$_2$, then, for any $i$, $(X_i, Y_i, N - Z_i, N - \tilde{Z}_i)$ is MTP$_2$.

Fix some $i = 1, \ldots, N$. For each $m = 1, \ldots, N$, let $r^G(m)$ be the probability of a total number $m$ of good ratings conditional on $S_i = G$ and $s_i = g$:

$$r^G(m) = \Pr[\tilde{Z}_i = m - 1 | X_i = 1, Y_i = 1].$$

Similarly, let

$$r^B(m) = \Pr[\tilde{Z}_i = m - 1 | X_i = 0, Y_i = 1].$$

Note that $r^G(0) = r^B(0) = 0$. Intuitively, for any fixed $p$, under MTP$_2$ each individual rater expects to find more good ratings when the quality of his own client is good than when it is bad, while the reverse is true under MRR$_2$. This idea is formalized in the following lemma.

**Lemma 5.** In any inflationary equilibrium, $\{r^G(m)\}$ first order stochastic dominates $\{r^B(m)\}$ under MTP$_2$; the reverse is true under MRR$_2$.

Given any inflationary equilibrium, the beliefs $q(m)$, $m = 1, \ldots, N$, are given by

$$q(m) = \Pr[X_i = 1 | Y_i = 1, \tilde{Z}_i = m - 1].$$

Let $\beta(t, k, p)$ represent the probability of $k$ successes out of $t$ Bernoulli trials with independent probability of success $p$; that is,

$$\beta(t, k, p) = \binom{t}{k} p^k (1 - p)^{t-k}. $$

Then, $q(m)$ can be written more explicitly as

$$q(m) = \frac{1}{m} \sum_{n=0}^{m} \pi_n \beta(N - n, m - n, p)_n. \quad (7)$$

The above formula is valid so long as the denominator is strictly positive, which happens if $p < 1$. We refer to an inflationary equilibrium with $p < 1$ as a full support equilibrium.
Lemma 6. In any full support inflationary equilibrium, \( q(m) \) is increasing in \( m \) under MTP\(_2\) and is decreasing in \( m \) under MRR\(_2\).

The above result is quite intuitive. In an inflationary equilibrium the perception of a good rating depends on the total number of good ratings in all markets: the perception improves with more good ratings when the client qualities are positively correlated, and it deteriorates when the qualities are negatively correlated. We are now ready to use Lemma 5 and Lemma 6 to establish existence of an inflationary equilibrium. Note that in any inflationary equilibrium, \( \hat{q}(m) = 0 \) for all \( m = 0, \ldots, N - 1 \).

Proposition 2. There exists an inflationary equilibrium under decentralized rating. Further, if \( \pi < q^* \), there is a full support inflationary equilibrium.

Proof. A necessary and sufficient condition for the existence of a full support inflationary equilibrium is that there exists \( p \in (0, 1) \) such that (i) \( s_i = g \) is weakly preferred to \( s_i = b \) if \( S_i = G \),

\[
\sum_{m=1}^{N} r_G(m)U(G, g, q(m)) \geq U(G, b, 0);
\]

and (ii) \( s_i = g \) and \( s_i = b \) yield the same expected payoff if \( S_i = B \),

\[
\sum_{m=1}^{N} r_B(m)U(B, g, q(m)) = U(B, b, 0). \tag{8}
\]

Under MTP\(_2\), Lemma 5 states that \( \{r^G(m)\} \) first order stochastic dominates \( \{r^B(m)\} \), while Lemma 6 states that \( q(m) \) is increasing \( m \). Therefore,

\[
\sum_{m=1}^{N} r^G(m)U(G, g, q(m)) \geq \sum_{m=1}^{N} r^B(m)U(G, g, q(m)).
\]

It follows from Assumption 2 that condition (ii) implies condition (i). Under MRR\(_2\), \( \{r^B(m)\} \) first order stochastic dominates \( \{r^G(m)\} \) while \( q(m) \) is decreasing \( m \), so again condition (ii) implies condition (i) by Assumption 2. Now, consider the indifference condition ii). If \( p = 0 \), we have \( q(m) = 1 \) for all \( m = 1, \ldots, N \). By Assumption 3,

\[
\sum_{m=1}^{N} r^B(m)U(B, g, q(m)) > U(B, b, 0)
\]
when \( p = 0 \). When \( p = 1 \), we have \( q(N) = \sum_n \pi_n n/N = \pi \) and the left-hand-side of condition ii) becomes \( U(B, g, \pi) \). Under Assumption 2, the refinement implies that the out-of-equilibrium belief \( \hat{q}(N - 1) \) is equal to 0. Thus, if \( U(B, g, \pi) < U(B, b, 0) \), or equivalently \( \pi < q^* \), then by the intermediate value theorem there exists \( p \in (0, 1) \) such that the equilibrium condition ii) is satisfied, and hence there is a full support inflationary equilibrium. If instead \( \pi \geq q^* \), with the out-of-equilibrium belief \( \hat{q}(N - 1) \) set to 0, \( g \) is weakly preferred to \( b \) under quality \( B \), implying that \( g \) is strictly preferred to \( b \) under \( G \). We thus have a non-full support equilibrium with \( p = 1 \).

The condition for the existence of a full support inflationary equilibrium is identical to the condition for the existence of the unique full support inflationary equilibrium under individual rating. When \( S_1, \ldots, S_N \) are independently distributed, we have \( \pi_n = \beta(N, n, \pi) \) for each \( n = 0, \ldots, N \). Then, for each \( m = 1, \ldots, N \), direct calculations reveal that

\[
q(m) = \frac{1}{m} \sum_{n=1}^{m} \frac{(\pi/(1-\pi)p)^n(1/((m-n)!(n-1)!))}{\sum_{n=0}^{m} \frac{\pi/(1-\pi)p^n(1/((m-n)!n!)}} = \frac{\pi}{\pi + (1-\pi)p}.
\]

Thus, under independence, decentralized rating reduces to individual rating.

Unlike in the case of centralized rating, we cannot rely on a model of independent quality distribution for an equilibrium analysis when the number of clients is arbitrarily large. Consider then the following specialized model with common shocks and idiosyncratic shocks. Suppose that there are two aggregate states, \( \bar{G} \) and \( \bar{B} \), with prior probability \( \theta \) and \( 1 - \theta \) respectively. Conditional on \( \bar{B} \), each client’s quality \( S_i \) is \( B \) with probability \( \pi \) and \( 1 - \pi \) respectively. Note that as \( \theta \) varies from \( \pi \) to 1, this model goes from perfect correlation to independence, so \( \theta \) is

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\[18\] In general, multiple full support inflationary equilibria may occur. For example, in the model of \( N = 2 \), \( \pi_0 = \pi_2 = \rho/2 \) and \( \pi_1 = 1 - \rho \) for some \( \rho \in (0, 1) \), and \( q^* > 1/2 \), the indifference condition is given by \( r^B(2)U(B, g, q(2)) + (1 - r^B(2))U(B, g, q(1)) = U(B, b, 0) \). Both \( q(2) \) and \( q(1) \) are decreasing functions of \( p \), while \( r^B(2) \) increases with \( p \). With positive correlation \( (\rho > 1/2) \), we have \( q(2) > q(1) \), and thus the left-hand-side of the indifference condition is not necessarily monotone in \( p \), leading to possibly multiple full support inflationary equilibria. We note that the comparison result in the next section are not affected by the possibility of multiple equilibria.
a measure of quality independence across clients. It can be shown that this specification satisfies the assumption of MTP\(_2\).\(^{19}\) Consider any full support inflationary equilibrium with probability of ratings inflation \(p \in [0,1]\). For each \(m = 0, \ldots, N\),

\[
q(m) = \frac{\theta \beta(N, m, \pi_g (1 - \pi_g)p)}{\theta \beta(N, m, \pi_g (1 - \pi_g)p) + (1 - \theta) \beta(N, m, p) \pi_g (1 - \pi_g)p},
\]

where the first fraction is the probability that the aggregate state is \(G\) conditional on observing \(m\) good ratings, and the second fraction is the probability that any client with a good rating is of good quality conditional on \(G\). From the viewpoint of the rater for client \(i\), conditional on \(S_i = B\), the rater believes that the aggregate state is \(G\) with probability

\[
\frac{\theta(1 - \pi_g)}{\theta(1 - \pi_g) + 1 - \theta} = \frac{\theta - \pi}{1 - \pi}.
\]

Thus, for any \(m = 1, \ldots, N\),

\[
r^B(m) = \frac{\theta - \pi}{1 - \pi} \beta(N - 1, m - 1, \pi_g + (1 - \pi_g)p) + \frac{1 - \theta}{1 - \pi} \beta(N - 1, m - 1, p).
\]

Given the above expressions for \(q(m)\) and \(r^B(m)\), the equilibrium \(p\) as determined by equation (8) in the proof of Proposition 2 exists if \(\pi < q^*\). When \(N\) becomes arbitrarily large, in any such equilibrium the aggregate state can be inferred perfectly from the fraction of good ratings. As a result, in the limit the equilibrium probability of ratings inflation \(p\), if positive, satisfies

\[
\frac{\theta - \pi}{1 - \pi} U(B, g, \pi_g (1 - \pi_g)p) + \frac{1 - \theta}{1 - \pi} U(B, g, 0) = U(B, b, 0).
\]

Thus, if

\[
\frac{\theta - \pi}{1 - \pi} U(B, g, 1) + \frac{1 - \theta}{1 - \pi} U(B, g, 0) < U(B, b, 0)
\]

then any sequence of full support equilibria converges to honest rating with \(p = 0\) when \(N\) converges to infinity.\(^{20}\) If the above condition is reversed, then there will still be some ratings inflation even when the number of clients \(N\) is arbitrarily large.

\(^{19}\) Indeed, the condition in Definition 1 holds with strict inequality when \(x \wedge y = (0, \ldots, 0)\), and otherwise holds with equality.

\(^{20}\) It can be shown that when \(U(B, g, \cdot)\) is a linear function, \(\sum_{m=1}^N r^B(m) U(B, g, q(m))\) decreases with \(N\) for any \(p\), so that the equilibrium probability of inflation \(p\) decreases with \(N\). The proof is available upon request.
5. Comparing Rating Schemes: Credibility and Welfare

Comparison between centralized rating and decentralized rating in terms of equilibrium credibility of good ratings and ex ante payoffs to the agency generally depends on the underlying correlation structure. In Proposition 1, we have established that there always exists an inflationary equilibrium under centralized rating that does at least as well as the full support inflationary equilibrium under individual rating. Moreover, when \( q^* < 1 - \pi_0 \), there is a unique non-full support equilibrium that does strictly better. This condition is rather weak, and is easily satisfied when the qualities are independently distributed, as long as \( N \) is not too small. In contrast, with independently distributed qualities, the unique inflationary equilibrium under decentralized rating is payoff-equivalent to the full support inflationary equilibrium under individual rating. Thus, we expect centralized rating to dominate decentralized rating for the agency when there is weak correlation among the qualities.

The next set of results shows that both equilibrium credibility of good ratings and ex ante payoff to the agency under decentralized rating improve relative to the benchmark of individual rating when the qualities are correlated. Unlike the above discussion about centralized rating, comparison of credibility between decentralized rating and individual rating requires a precise definition of equilibrium credibility of good ratings. For any \( p \) (the probability of inflating), consider the following expression:

\[
\sum_{n=0}^{N} \frac{n \pi_n}{N \pi} \sum_{m=1}^{N} \beta(N-n,m-n,p)q(m).
\]

(9)

The above may be thought of as an average measure of credibility of good ratings under decentralized rating, as the credibility of a single given good rating depends on the total number of good ratings. It is an average across states, with each state weighted both by the prior probability of the state and by the number of good quality clients in the state. Under individual rating, the same expression (9) applies, but \( q(m) \) is constant and equal to \( q \) because the markets are separate. Since \( \sum_{m=1}^{N} \beta(N-n,m-n,p) = 1 \), the above definition of credibility is consistent with the definition given under individual rating. Further, if we replace \( \beta(N-n,m-n,p) \) with \( p(m;n) \) in (9), we have a measure of
credibility under centralized rating. It follows that for any non-full support equilibrium, the equilibrium credibility is greater than $q^*$, the equilibrium credibility in the benchmark full support inflationary equilibrium under individual rating. To make comparison of equilibrium credibility between decentralized rating and individual rating, we first note that

$$r^G(m) = \sum_{n=1}^{N} \Pr[\tilde{Z}_i = m - 1 \mid X_i = 1, Y_i = 1, Z_i = n - 1] \Pr[Z_i = n - 1 \mid X_i = 1, Y_i = 1]$$

$$= \sum_{n=1}^{m} \beta(N - n, m - n, p) \frac{n \pi_n}{N \pi}.$$

Hence the credibility measure is simply

$$\sum_{n=0}^{N} \frac{n \pi_n}{N \pi} \sum_{m=1}^{N} \beta(N - n, m - n, p) q(m) = \sum_{m=1}^{N} r^G(m) q(m).$$

Thus, the average measure of credibility of good ratings under decentralized rating is equal to the market belief expected by a rater with a good quality client. Next, we present a preliminary result.

**Lemma 7.** Under decentralized rating, for any $p < 1$,

$$\sum_{m=1}^{N} r^B(m) q(m) \leq \frac{\pi}{\pi + (1 - \pi)p}.$$

Under individual rating, the market’s belief upon observing $g$ is given by $\pi/(\pi + (1 - \pi)p)$ if $p$ is the probability that rating $g$ is issued under quality $B$. Thus, under decentralized rating the weighted average of the market belief conditional on a bad quality client is lower than the market belief under individual rating for the same probability of rating inflation. That is, a rater that issues an inflated rating on a bad quality client expects on average a less favorable market belief under either positive or negative correlation than when qualities are

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21 It turns out that the definition (9) of credibility corresponds one-to-one with the average expected loss of the $N$ markets. With the quadratic loss function given in Section 2, the average expected loss under individual, centralized, decentralized rating schemes is $(\delta_G - \delta_B)^2 \pi (1 - Q)$, where $Q$ is given by (9) for the corresponding scheme.
statistically independent. The intuition is that under either positive correlation ($MTP_2$) or negative correlation ($MRR_2$), for the rater with a bad quality client, the weights are smaller for higher market beliefs $q$, so that the weighted average is lower than the average when the qualities are independently distributed for independent randomizations with the same probability of inflation $p$. For example, under positive correlation, a higher market belief $q$ is associated with a greater number of good ratings, but since a rater with a bad quality client expects statistically fewer good quality clients and thus fewer good ratings, a higher market belief receives a smaller weight. Conversely, under negative correlation, a higher market belief $q$ is associated with a smaller number of good ratings, and again receives a smaller weight in the belief of the rater with a bad quality client.

Lemma 7 illustrates the idea that in our model of credibility correlation across client qualities imposes a discipline on incentives to inflate by making it harder for each individual rater to fool its own market. In fact, if $U(B, g, q)$ is weakly concave in $q$, then at any full support inflationary equilibrium in decentralized rating, the equilibrium probability of inflation is lower than the full support equilibrium probability of inflation under individual rating. Furthermore, under the same condition on $U(B, g, \cdot)$, the equilibrium credibility is higher under decentralized rating than under individual rating.

**Proposition 3.** Suppose $U(B, g, q)$ is concave in $q$. In any full support inflationary equilibrium under decentralized rating, the probability of inflation is lower and the credibility is higher than in the full support inflationary equilibrium under individual rating.

**Proof.** In a full support inflationary equilibrium, for each $i = 1, \ldots, N$, we must have the indifference condition between $s_i = g$ and $s_i = b$. This condition gives

$$\sum_{m=1}^{N} r^B(m)U(B, g, q(m)) = U(B, g, q^*).$$

This comparison between equilibrium probabilities of inflation under decentralized and individual rating holds so long as the function is not too convex. An even stronger result can be obtained if one imposes more structure on the form of quality correlation than $MTP_2$ or $MRR_2$. Consider for example the model of $N = 2$, $\pi_0 = \pi_2 = \rho/2$ and $\pi_1 = 1 - \rho$ for some $\rho \in (0, 1)$, and $q^* > 1/2$. We can show that if $U(B, g, q)$ is concave in $q$, then an increase in $\rho$ for $\rho > 1/2$ or a decrease in $\rho$ for $\rho < 1/2$ reduces the equilibrium probability of ratings inflation.
Since $U(B, g, q)$ is concave in $q$, we have
\begin{equation}
U(B, g, \sum_{m=1}^{N} r^B(m)q(m)) \geq U(B, g, q^*). \tag{11}
\end{equation}

It then follows from Lemma 7 that
\begin{equation}
U(B, g, \pi/(\pi + (1 - \pi)p)) \geq U(B, g, q^*),
\end{equation}
where $p$ is the equilibrium probability of inflation. Comparing the above inequality to equation (5) in a full support inflationary equilibrium under individual rating, we immediately have that the equilibrium probability of inflation is lower under decentralized rating than under individual rating.

From Lemma 5 and Lemma 6, we have that \( \{r^G(m)\} \) first order stochastically dominates \( \{r^B(m)\} \) and $q(m)$ is increasing under MTP\(_2\), while \( \{r^B(m)\} \) stochastically dominates \( \{r^B(m)\} \) and $q(m)$ is decreasing under MRR\(_2\). In either case, we have
\begin{equation}
\sum_{m=1}^{N} r^G(m)q(m) \geq \sum_{m=1}^{N} r^B(m)q(m).
\end{equation}

From inequality (11) we then have
\begin{equation}
\sum_{m=1}^{N} r^G(m)q(m) \geq q^*,
\end{equation}
implying that the equilibrium credibility is higher under decentralized rating. \textit{Q.E.D.}

For welfare comparison between decentralized rating and individual rating under either MTP\(_2\) or MRR\(_2\), we say that $U(B, g, \cdot)$ is “more concave” than $U(G, g, \cdot)$ if there is a weakly concave function $H$ such that $U(B, g, q) = H(U(G, g, q))$. We have the following result.

**Proposition 4.** Suppose $U(B, g, q)$ is more concave in $q$ than $U(G, g, q)$. Then, the agency’s payoff in a full support inflationary equilibrium under decentralized rating is higher than the full support inflationary equilibrium under individual rating.

**Proof.** If $U(B, g, q)$ is more concave in $q$ than $U(G, g, q)$, the indifference condition (10) implies
\begin{equation}
\sum_{m=1}^{N} r^B(m)U(G, g, q(m)) \geq U(G, g, q^*).\end{equation}
Under MTP$_2$, $\{r^G(m)\}$ first order stochastically dominates $\{r^B(m)\}$ and $q(m)$ is increasing, and so

$$\sum_{m=1}^{N} r^G(m)U(G, g, q(m)) \geq U(G, g, q^*).$$

Under MRR$_2$, $q(m)$ is decreasing but $\{r^B(m)\}$ stochastically dominates $\{r^B(m)\}$, so again the inequality is true. Q.E.D.

Compared to individual rating, in decentralized rating each client $i$ is exposed to a greater risk when $S_i = G$ because of the uncertainty regarding the ratings of other clients. However, the beliefs are more favorable under $G$ than under $B$ in the sense of first order stochastic dominance. Thus, welfare improves so long as the agency is not too much more risk-averse when $S_i = G$ than when $S_i = B$.

Since the strategy space in decentralized rating is the same as in individual rating, the above results show that the gains in credibility and welfare in decentralized rating come from sharing ratings information among the markets. We expect that the gains are larger when the correlation is stronger. Indeed, the next proposition establishes that when the correlation across client qualities is almost perfect, there is a limit inflationary equilibrium with “truth-telling,” i.e., the equilibrium probability of inflation converges to 0. Let $\{\pi_k^0, \ldots, \pi_k^N\}$ be a sequence of probability distributions that satisfy MTP$_2$, such that (i) $\lim_{k \to \infty} \sum_{n=1}^{N-1} \pi_k^n = 0$; and (ii) $\lim_{k \to \infty} \pi_k^N / (\pi_k^N + \pi_k^0) < q^*$. The first condition means that the states become almost perfectly positively correlated as $k$ becomes arbitrarily large. The second condition guarantees that there exists no pooling equilibrium with $p(N; n) = 1$ for all $n$ when $k$ is large.

PROPOSITION 5. Under decentralized rating, truth-telling is a limit inflationary equilibrium when $k$ goes to infinity.

PROOF. Equation (8) is necessary and sufficient for an inflationary equilibrium under decentralized rating. As in the proof of Proposition 2, for $p = 0$, the left-hand-side of (8) is strictly larger than the right-hand-side for any $k$. Next, for all $p > 0$, the limit of left-hand-side as $k$ goes to infinity is strictly less than

$$p^{N-1}U(B, g, 1) + (1 - p^{N-1})U(B, g, 0).$$
This is because in the limit when the correlation is perfect, from equation (7) we have $q(m) = 0$ for all $m < N$, while $q(N) < 1$. Let $\hat{p}$ be the value of $p$ that solves

$$p^{N-1}U(B, g, 1) + (1 - p^{N-1})U(B, g, 0) = U(B, b, 0).$$

Then, for all $0 < p < \hat{p}$, the limit of the left-hand-side of (8) as $k$ goes to infinity is strictly smaller than the right-hand-side. Hence, for each $p$ there exists $k(p)$ such that for all $k > k(p)$ there is an inflationary equilibrium with the probability of inflation strictly between 0 and $p$. Since this construction of $\hat{p}$ and $k(p)$ holds for all $p$, by taking $p$ arbitrarily close to 0, we can establish truth-telling (i.e., $p = 0$) as a limit point of a sequence of inflationary equilibria for $k$ going to infinity. \[Q.E.D.\]

While strong correlation enhances credibility and improves welfare in decentralized rating, the opposite is true in centralized rating. To see this, note that the conditions made on the convergence of the sequence of the distributions $\pi^k$ imply that in the limit of $k \to \infty$, there is no non-full support equilibrium by Proposition 1. Thus, centralized rating cannot improve upon individual rating when correlation is almost perfect.\footnote{In Proposition 5 we have considered only the limit case of perfect positive correlation. For the model of $N = 2$, $\pi_0 = \pi_2 = \rho/2$ and $\pi_1 = 1 - \rho$ for some $\rho \in (0, 1)$, and $q^* > 1/2$, perfect negative correlation is well-defined. Under decentralized rating, there is unique inflationary equilibrium with negative correlation ($\rho < 1/2$). As $\rho$ converges to 0, the equilibrium converges to truth-telling, with $p = 0$. Under centralized rating, when $\rho$ is sufficiently small (precisely, when $\rho < 2(1 - q^*)/(2q^* - 1)$), the unique non-full support equilibrium has $l = 1$, $p(1; 1) = 1$, and $p(1; 0), p(2; 0) > 0$. As $\rho$ decreases, $p(1; 0)$ increases and $p(2; 0)$ decreases. In the limit as $\rho$ converges to 0, both $p(1; 0)$ and $p(2; 0)$ are strictly positive, so the limit equilibrium strategy is not truth-telling. However, the ex ante payoff of the agency in the limit equilibrium is the same as in truth-telling.}

Correlation of the underlying qualities reduces the manipulation room both under decentralized rating and under centralized rating. Under decentralized rating the constraint imposed by correlation makes it harder for a rater to fool the market with a good rating, and forces the individual raters to tone down the exaggeration. This then results in a greater ex ante payoff relative to individual rating. In contrast, strong correlation makes correlated randomization under centralized rating less effective.

For analysis involving non-extreme values of correlation, the notion of the “degree of dependence” is ambiguous, and a more specific description of the multivariate probability
process is required. We illustrate how the degree of dependence affects the welfare properties of centralized rating versus decentralized rating using the probability distribution given in Joe (1997):

$$\pi_n = \binom{N}{n} \frac{\Pi_{i=0}^{n-1} (\pi + i\gamma)\Pi_{i=0}^{N-n-1} (1 - \pi + i\gamma)}{\Pi_{i=0}^{N-1} (1 + i\gamma)},$$

where $\gamma \geq -(N-1)^{-1} \min\{\pi, 1-\pi\}$. This probability mass function is fairly general as it encompasses four classes of probability distributions: (i) beta-binomial; (ii) binomial; (iii) hypergeometric; and (iv) Polya-Eggenberger distribution. It satisfies MTP$_2$ when $\gamma > 0$ and satisfies MRR$_2$ when $\gamma < 0$. For this distribution, Joe (1997) shows that $E[Y_i] = \pi$ and $\text{cor}(Y_i, Y_j) = (1 + \gamma^{-1})^{-1}$. Therefore, the higher the absolute value of $\gamma$, the greater is the degree of (positive or negative) dependence among the qualities of different clients. We use this distribution, with $N = 10$ and $\pi = 0.5$, to calculate the equilibria under centralized rating and the inflationary equilibria under decentralized rating for different degrees of dependence.\textsuperscript{24} Figure 2 shows the ex ante welfare for the rating agency under the two

\textsuperscript{24} When $N > 2$, the maximum degree of negative correlation among a group of exchangeable binary random variables is bounded away from $-1$. In our probability distribution with $N = 10$ and $\pi = 0.5$, the maximum degree of negative correlation is approximately $-0.059$. We cannot compare centralized rating with decentralized rating beyond this degree of negative correlation.
scenarios for different values of $\gamma$, for payoff functions $U(G, g, q) = 1 + q$, $U(B, b, q) = 0.8(1 + q)$, $U(G, b, q) = 0.6(1 + q)$ and $U(B, g, q) = 0.5(1 + q)$. For centralized rating, equilibrium threshold decreases weakly with $\gamma$. When the correlation is about zero, there is a non-full support equilibrium with threshold $l = 4$. When $\gamma > 0.61$, the only equilibrium is the full support equilibrium ($l = 0$) with payoff equal to that under individual rating. For decentralized rating, equilibrium probability of inflation is at a maximum when $\gamma = 0$, and decreases monotonically as $\gamma$ becomes either more positive or more negative. In the figure, centralized rating dominates decentralized rating for all $\gamma < 0.56$.

6. Concluding Remarks

Providers of information often care about the way their information is used. The desire to create favorable beliefs about its clients may cause the rating agency to inflate its assessment of the quality of its clients. The exuberant stock recommendations made during the internet boom, and the failure of auditors to raise alerts in a number of recent corporate scandals have heightened the public’s concern about the potential conflict of interests inherent in situations where raters are advocates for the rated. Moore et al. (2005) study this kind of problems and their possible solutions from a variety of perspectives. Gentzkow and Shapiro (2006) study how competition and the concern for reputation may constrain biased reporting by the mass media. Chan, Li and Suen (forthcoming) use a signaling model to understand why grades in academia tend to be exaggerated. None of these papers, however, examines how the credibility of ratings can be improved by coordinating or decentralizing the rating decisions, which is the main focus of our paper.

In the literature on reputational cheap talk, a bad sender type may provide useful information to the receiver to establish the credibility as a good sender type so as to extract future surplus (Sobel 1985; Benabou and Laroque 1992; Morris, 2001; Morgan and Stockton, 2003). This effect arises in a cheap talk game where the sender has private information on both the relevant state-of-world and his personal bias. As a costly signaling model of credibility, our model of individual rating has a single source of private information. The equilibrium credibility of a good rating is quantifiable in our model and corresponds one-to-one with the welfare of the rating agency. These features make our model of credibility a natural benchmark for comparisons with centralized and decentralized rating schemes.
This paper is related to the small literature on multi-dimensional signaling (Quinzii and Rochet, 1985; Engers, 1987). This literature focuses on the conditions under which separation of types occurs. Technically, the models in the existing literature are concerned with multi-dimensional private information for the sender and one-dimensional signals. Our signaling model of centralized rating assumes exchangeability of the components of the state vector, so that the private information is the number of good clients, which is one-dimensional. However, the signal space is multi-dimensional, as a strategy specifies a number of good ratings for each number of good clients. As a result, the single crossing condition in the benchmark case of individual rating is not completely effective in either centralized rating or decentralized rating. This feature complicates the analysis but enriches the comparison analysis for the different schemes of rating. Chakraborty and Harbaugh (forthcoming) show that in a cheap talk game where a send and a receiver interact on several unrelated issues, the sender can credibly communicate to the receiver the ranking of the private signals even if the conflicts between them are too great to permit credible communication of the signal on any single issue.\(^{25}\) Their result has the interpretation that bundling independent reports may help information transmission, which is related to our result for centralized rating. However, their result follows from the observation that the sender has no incentive to deceive the receiver about the ranking of the signals, while our analysis is based on coordination of the reports in a costly signaling model.

In the literature on signaling games, there are a few models that involve multiple senders (Bagwell and Ramsey, 1991; Hertzendorf and Overgaard, 2001). In these models, the senders know each other’s types and interact with each other directly through their signals. In contrast, the raters in our model of decentralized rating have private information about their own types and have no direct interaction except that their signals are jointly used by the receivers to make inference about the types of the senders. Our model of decentralized rating is therefore a model of competing signals, rather than a model of competing senders.

\(^{25}\) The idea that linking decisions can be payoff-improving also appears in the literature on bundling in monopoly pricing (Adam and Yellen, 1976; McAfee, McMillan, Whinston, 1979) and incentive design (Maskin and Tirole, 1990; Jackson and Sonnenschein, forthcoming).
Appendix

A.1. Proof of Lemma 1

(i) The difference of differences \([W(m'; n') - W(m; n')] - [W(m'; n) - W(m; n)]\) is equal to \((n' - n)\) times
\[
[U(G, g, q(m')) - U(G, b, \hat{q}(m))] - [U(B, g, q(m')) - U(B, b, \hat{q}(m))],
\]
which is positive by equation (2).

(ii) The difference between \(W(m'; n') - W(m; n')\) and \(W(m'; n) - W(m; n)\) is \((n' - n)\) times
\[
[U(G, g, q(m')) - U(G, g, q(m))] - [U(B, g, q(m')) - U(B, g, q(m))],
\]
which is positive by Assumption 2 since \(q(m') > q(m)\).

(iii) The difference between \(W(m'; n') - W(m; n')\) and \(W(m'; n) - W(m; n)\) is \((n - n')\) times
\[
[U(B, b, \hat{q}(m')) - U(B, b, \hat{q}(m))] - [U(G, b, \hat{q}(m')) - U(G, b, \hat{q}(m))],
\]
which is positive by Assumption 2 since \(q(m') > q(m)\).

A.2. Proof of Lemma 2

Let \(\overline{m}_1 = N\), and iteratively define \(\overline{m}_k\) as the smallest integer such that \(\{\overline{m}_k, \ldots, \overline{m}_k\} \subseteq T\) and \(\overline{m}_{k+1}\) as the largest integer smaller than \(\overline{m}_k\) such that \(\overline{m}_{k+1} \in T\). We have the following claims regarding equilibrium and out-of-equilibrium beliefs.

(1) In any inflationary equilibrium, if \(q(m) < q^*\) for some \(m \in T\) and \(m - 1 \in T\), then \(q(m - 1) < q(m) < q^*\). Otherwise, \(W(m - 1; n) > W(m; n)\) for all \(n \leq m - 1\), and either \(m \not\in T\) or \(q(m) = 1\), a contradiction in either case.

(2) If \(q(m) < q(m')\) for all \(m, m' \in \{\overline{m}_k, \ldots, \overline{m}_k\}\) and \(m < m'\), then \(p(n; n) < 1\) for all \(n \in \{\overline{m}_k, \ldots, \overline{m}_k - 1\}\). Otherwise, \(W(n; n) \geq W(n + 1; n)\) implies \(W(n; n') > W(n + 1; n')\) for all \(n' < n\) by part (ii) of Lemma 1, implying that either \(n + 1 \not\in T\) or \(q(n + 1) = 1\), a contradiction in either case.
(3) For any \( x \not\in T \), if for each \( k \) such that \( m_k > x \) we have \( q(m) < q(m') < q^* \) for all \( m, m' \in \{m_k, \ldots, m_k\} \) and \( m < m' \), then the out-of-equilibrium belief \( \hat{q}(x) = 0 \). Suppose instead \( \hat{q}(x) > 0 \). We will show that \( W(x; n) \geq W(t_n; n) \) for any \( n > x \) and \( t_n \in T_n \) implies that \( W(x; n - 1) > W(t_{n-1}; n - 1) \) for any \( t_{n-1} \in T_{n-1} \). An iteration of this result then leads to \( \hat{q}(x) = 0 \) by the refinement, a contradiction that establishes the claim. For any \( n > x \), there are two cases. In the first case, either there is no \( n = \overline{m}_k + 1 \), which by claim (2) above implies that there exists a \( t_{n-1} \in T_{n-1} \) such that \( t_{n-1} \geq n \), or \( n = \overline{m}_k + 1 \) for some \( k \) but \( p(n - 1; n - 1) < 1 \), which implies again that there exists a \( t_{n-1} \in T_{n-1} \) such that \( t_{n-1} \geq n \). Then, since \( W(x; n) \geq W(t_n; n) \), we have \( W(x; n) \geq W(t_{n-1}; n) \) by optimality, which implies \( W(x; n - 1) > W(t_{n-1}; n - 1) \) by part (i) of Lemma 1. In the second case, \( n = \overline{m}_k + 1 \) for some \( k \) and \( p(n-1; n-1) = 1 \). Since \( \hat{q}(x) > 0 \) and \( \hat{q}(n-1) = 0 \), by part (iii) of Lemma 1, \( W(x; n) \geq W(n-1; n) \) implies that \( W(x; n - 1) > W(n-1; n - 1) \).

(4) If for some \( \overline{m}_k > 0 \) we have \( q(m) < q^* \) for all \( m > \overline{m}_k \), then \( q(\overline{m}_k) < q^* \). To see this, suppose that \( q(\overline{m}_k) \geq q^* \). By construction \( \overline{m}_k + 1 \not\in T \) and by claim (3) \( \hat{q}(\overline{m}_k + 1) = 0 \). Next, for any \( n \leq \overline{m}_k \) and any \( t_n \in T_n \), \( W(\overline{m}_k + 1; n) \geq W(t_n; n) \) implies \( W(\overline{m}_k + 1; n) \geq W(\overline{m}_k; n) \). Since \( \hat{q}(\overline{m}_k + 1) = 0 \), and \( q(\overline{m}_k) \geq q^* \) by assumption, \( W(\overline{m}_k + 1; n) \geq W(\overline{m}_k; n) \) implies \( q(\overline{m}_k + 1) \geq q^* \). It follows that \( W(\overline{m}_k + 1; \overline{m}_k + 1) > W(t_{\overline{m}_k+1}; \overline{m}_k + 1) \) for any \( t_{\overline{m}_k+1} \in T_{\overline{m}_k+1} \). The refinement then implies \( q(\overline{m}_k + 1) = 1 \), a contradiction.

Using the above four claims, we now establish that in any equilibrium \( q(m) \geq q^* \) for all \( m \in T \). Suppose instead \( q(m) < q^* \) for some \( m \in T \). Then, \( q(\mathcal{N}) < q^* \); otherwise, \( W(N; n) > W(m; n) \) for all \( n \leq m \), contradicting the assumption that \( m \in T \). Claims (1) and (4) above then imply that \( q(m) < q^* \) for all \( m \in T \) and \( m > 0 \). If \( l > 0 \), we have \( W(0; 0) > W(m; 0) \) for all \( m \in T \) regardless of \( \hat{q}(0) \), a contradiction. If \( l = 0 \) and \( 1 \in T \), we then have \( p(0; 0) = 1 \) and \( q(1) = 1 \), again a contradiction. Finally, if \( l = 0 \) and \( 1 \not\in T \), since \( \hat{q}(1) = 0 \) by claim (3) above and \( q(t_1) < q^* \) for any \( t_1 \in T_1 \), we have that \( W(1; 1) > W(t_1; 1) \) whenever \( W(1; 0) \geq W(0; 0) \), which then implies \( q(1) = 1 \) by the refinement, again a contradiction.

(i) For the first part of the lemma, note that if \( q(m) > q^* \) for some \( m > 0 \) and \( m \in T \), then \( W(m; 0) > W(0; 0) \). This contradicts the assumption that \( l = 0 \). Thus, \( q(m) = q^* \) for all \( m > 0 \) such that \( m \in T \).
(ii) For the second part, note that if \( q^* \leq q(m') < q(m) \) or if \( q^* < q(m') = q(m) \) for some \( m, m' \in T \), and \( m > m' \), we have \( W(m; n) > W(m'; n) \) for all \( n \leq m' \), contradicting the assumption that \( m' \in T \). Thus, it remains to prove that if \( q(m) = q^* \) for some \( m \in T \), then \( q(m') = q^* \) for all \( m \in T \) and \( 0 < m' < m \). To establish this last claim, suppose \( q(m) = q^* \) for some \( m \in T \). There are two cases. First suppose \( m - 1 \in T \). If \( q(m - 1) > q^* \), then \( W(m - 1; n) > W(m; n) \) for all \( n \leq m - 1 \). This implies \( q(m) = 1 \), a contradiction. Thus \( q(m - 1) = q^* \). Next suppose \( m - 1 \notin T \). Let \( \overline{m} \) be the largest signal in \( T \) that is smaller than \( m \). Since \( q(m') = q^* \) for all \( m' \in T \) and \( m' \geq m \), we have \( W(t_{\overline{m} + 1}; \overline{m} + 1) = W(m'; \overline{m} + 1) \) for any \( t_{\overline{m} + 1} \in T_{\overline{m} + 1} \). For all \( n > \overline{m} + 1 \), since \( W(\overline{m} + 1; n) \geq W(t_n; n) \) for \( t_n \in T_n \) implies \( W(\overline{m} + 1; \overline{m} + 1) > W(t_n; \overline{m} + 1) = W(t_{\overline{m} + 1}; \overline{m} + 1) \), it follows from the refinement that \( \hat{q}(\overline{m} + 1) = 0 \). Given this, if \( q(\overline{m}) > q^* \), then \( W(\overline{m} + 1; n) \geq W(t_n; n) \) for any \( n \leq \overline{m} \) and any \( t_n \in T_n \) implies \( q(\overline{m} + 1) > q^* \). It then follows that \( W(\overline{m} + 1; \overline{m} + 1) > W(t_{\overline{m} + 1}; \overline{m} + 1) \), and therefore \( q(\overline{m} + 1) = 1 \) by the refinement, a contradiction. Thus, \( q(\overline{m}) = q^* \).

#### A.3. Proof of Lemma 3

(i), (ii) Suppose \( p(m'; m) > 0 \) for some \( m, m' \in T \) and \( m' > m \geq l \). By optimality we have \( W(m'; m) - W(m; m) \geq 0 \). Since \( q(m) > q(m') \) by Lemma 2, part (ii) of Lemma 1 implies \( W(m'; n) - W(m; n) > 0 \) and hence \( p(m; n) = 0 \) for all \( n < m \). Part (i) of the lemma follows by setting \( m = l \) and noting that \( p(l; l) = 0 \) implies \( l \notin T \), a contradiction. Part (ii) follows by noting that for any \( m \in T \) and \( m > l \), \( p(m; m) < 1 \) implies that \( q(m) = 1 \), contradicting Lemma 2.

(iii) By optimality \( W(\min T_m; m) \geq W(n; m) \) for all \( n \geq \min T_m \). Since from Lemma 2 we have \( q(\min T_m) > q(n) \), part (ii) of Lemma 1 implies that \( W(\min T_m; m') > W(n; m') \) for all \( m' \) such that \( m < m' \leq l \leq \min T_m \). Hence \( p(n; m') = 0 \), and max \( T_m \leq \min T_m \).

(iv) Let \( x > 0 \) be the largest signal such that \( x \notin T \). Note that in any inflationary equilibrium \( x < N \). We first show by contradiction that \( \hat{q}(x) = 0 \). This claim follows from the refinement if \( W(x; n) \geq W(t_n; n) \) for any \( n > x \) and \( t_n \in T_n \) implies that \( W(x; x) > W(t_x; x) \) for any \( t_x \in T_x \). To establish the latter claim, note that for any \( n > x + 1 \), by optimality \( W(x; n) \geq W(t_n; n) \) implies that \( W(x; n) \geq W(x + 1; n) \). Since
\(\hat{q}(x) > 0\) and \(\hat{q}(x+1) = 0\), part (iii) of Lemma 1 implies that \(W(x; x+1) > W(x+1; x+1)\). Since \(x+1 \in T_{x+1}\) by (i) and (ii) above, we have \(W(x; x+1) > W(t_{x+1}; x+1)\) for all \(t_{x+1} \in T_{x+1}\), which by optimality implies \(W(x; x+1) > W(t_x; x+1)\). Since \(t_x \geq x+1\), by part (i) of Lemma 1, we have \(W(x; x) > W(t_x; x)\).

Next, we claim that \(q(x) = 1\). To see this, note that for each \(n < x\) and any \(t_n \in T_n\), by optimality \(W(x; n) \geq W(t_n; n)\) implies \(W(x; n) \geq W(x+1; n)\). Since \(\hat{q}(x) = \hat{q}(x+1) = 0\), if \(W(x; n) \geq W(x+1; n)\) then \(q(x) > q(x+1)\). Since \(t_x \geq x+1\), from Lemma 2 we have \(q(x) > q(t_x)\). It then follows from part (ii) of Lemma 1 that \(W(x; x) > W(t_x; x)\). By the refinement, \(q(x) = 1\). Thus there is no \(m < x\) such that \(m \in T\).

(v) First, consider \(\hat{q}(0)\). By (ii) and (iii) above, \(p(N; 0) > 0\); otherwise, \(q(N) = 1\), which is a contradiction. For any \(n > 0\) and \(t_n \in T_n\), if \(W(0; n) \geq W(t_n; n)\) then by optimality \(W(0; n) \geq W(N; n)\). By part (i) of Lemma 1, we have \(W(0; 0) > W(N; 0)\). It follows from the refinement that \(\hat{q}(0) = 0\).

Next, we show that \(\hat{q}(m) = 0\) for any \(m = \{1, \ldots, l - 1\}\). Suppose instead \(\hat{q}(m) > 0\). We will show that \(W(m; n) \geq W(t_n; n)\) for any \(m > n\) and \(t_n \in T_n\) implies \(W(m; m) > W(t_m; m)\) for any \(t_m \in T_m\), which leads to a contradiction by the refinement. First, for any \(n > l\), by optimality \(W(m; n) \geq W(t_n; n)\) implies \(W(m; n) \geq W(l; n)\). Since \(\hat{q}(m) > 0\) and \(\hat{q}(l) = 0\), by part (iii) of Lemma 1 \(W(m; l) > W(l; l)\). Second, for any \(n\) such that \(m < n \leq l\), we have \(t_m \geq n\). If \(W(m; n) \geq W(t_n; n)\), optimality implies that \(W(m; n) \geq W(t_m; n)\). It then follows from part (i) of Lemma 1 that \(W(m; m) > W(t_m; m)\). Combining these two cases, we have \(\hat{q}(m) = 0\), as desired.

Finally, consider \(q(m)\) for any \(m = \{1, \ldots, l - 1\}\). Suppose that \(W(m; n) \geq W(t_n; n)\) for some \(n < m\) and \(t_n \in T_n\). By optimality \(W(m; n) \geq W(t_m; n)\), which implies \(q(m) > q(t_m)\) as \(\hat{q}(m) = 0\). It then follows from part (ii) of Lemma 1 that \(W(m; m) > W(t_m; m)\). By the refinement, \(q(m) = 1\).

A.4. Proof of Proposition 1

Denote as \(p_n\) a reporting strategy in state \(n\) (i.e. the vector \((p(0; n), \ldots, p(N; n))\)), and let \(P^n\) be a sequence \((p_n, \ldots, p_N)\). For a given \(p_n\), we use \(t_n\) (\(t_n\)) to denote the largest (smallest) \(m\) such that \(p(m; n) > 0\). Although \(P^n\) is not a complete strategy, we construct
the market beliefs under the assumption that \( p(m; n') = 0 \) for all \( m \) and all \( n' < n \) and denote it \( q(m|P^n) \). When \( q(m|P^n) \) cannot be obtained using Bayes’ rule, it is defined as 1, while \( \hat{q}(m|P^n) \) is defined as 0. Finally, we denote as \( W(m; n|P^k) \) the expected payoff to the agency that issues \( m \) good ratings and has \( n \) clients of good quality when the market beliefs are given by \( q(m|P^k) \) and \( \hat{q}(m|P^k) \) for each \( m = 0, ..., N \).

**Definition A.1.** We say \( p_n \) for some \( n > 0 \) is compatible with \( l \) given \( P^{n+1} \) if it satisfies:

(i) \( p(n; n) = 1 \) in case \( n > l \);

(ii) in case \( n = l \): (a) \( p(m; n) = 0 \) for all \( m < l \); (b) \( W(n; n|P^n) = W(t; n|P^n) \) for all \( n < t < \bar{t}_n \); (c) \( W(n; n|P^n) \leq W(\bar{t}_n; n|P^n) \) if \( \bar{t}_n < N \); and (d) \( p(N; n) = 1 - \sum_{m < N} p(m; n) \);

(iii) in case \( n < l \): (a) \( \bar{t}_{n+1} + 1 \geq \bar{t}_n \geq \bar{t}_{n+1} \); (b) \( W(t_n; n|P^n) = W(t; n|P^n) \) for all \( t_n < t < \bar{t}_n \); (c) \( W(\bar{t}_n; n|P^n) \leq W(\bar{t}_n; n|P^n) \) if \( \bar{t}_n < N \); (d) \( p(N; n) = 1 - \sum_{m < N} p(m; n) \);

(e) \( W(\bar{t}_n - 1; n'|P^n) \leq W(\bar{t}_n - 1; n'|P^n) \) for all \( n < n' \) if \( \bar{t}_n < N \); (f) \( W(\bar{t}_n + 1; n+1|P^n) \)

\(W(\bar{t}_n; n + 1|P^n) \) if \( \bar{t}_n > \bar{t}_n \); and (g) \( W(\bar{t}_n; n'|P^n) = W(\bar{t}_n; n'|P^n) \) for all \( n' > n \) if \( \bar{t}_n > \bar{t}_n \).

We say \( P^n \) is compatible with \( l \) if \( p_k \) is compatible with \( l \) given \( P^{k+1} = (p_{k+1}, ..., p_N) \) for each \( k \) such that \( N > k \geq n \), and \( P^n \) is compatible if it is compatible with some \( l = 0, ..., N \).

**Lemma A.1.** If \( P = (p_0, ..., p_N) \) is the reporting strategy in an equilibrium with threshold \( l \), then \( P^1 = (p_1, ..., p_N) \) is compatible with \( l \).

**Proof of Lemma A.1.** We show that, if \( P^{n+1} \) is compatible with \( l \) and \( P \) is an equilibrium strategy, then \( p_n \) is compatible with \( l \) given \( P^{n+1} \). The lemma then follows from observing that there is only way of constructing an equilibrium strategy \( p_N \) and it is compatible with all \( l \).

Part (i) of Definition A.1 follows immediately from the definition of equilibrium.

Part (ii). (a) follows immediately from the definition of equilibrium. Next, note that part (iii) of Lemma 3 implies that in any equilibrium with threshold \( l \) and each \( n \leq l \), \( W(t; n|P^n) = W(t; n) \) for all \( t < \bar{t}_n \) and \( W(\bar{t}_n; n|P^n) \geq W(\bar{t}_n; n) \). Thus by Lemma 3, (b) and (c) must hold in equilibrium. (d) is clearly necessary.
Part (iii). (a) follows from part (iii) and (iv) of Lemma 3. The proof of (b), (c) and (d) is the same as for part (ii) above. (e) is necessary for equilibrium because \( W(t_n' - 1; n'|P_n) = W(t_n' - 1; n') \) and \( W(t_n'; n'|P_n) \geq W(t_n'; n') \) for all \( n \leq n' \leq l \). (f) and (g) follow because \( W(m; n'|P_n) = W(m; n') \) for all \( n' \) and all \( m < t_n \). Q.E.D.

Lemma A.1 establishes that compatibility of \( P_1 \) is a necessary condition for \( P = (p_0, P_1) \) to be an equilibrium. However, it is not sufficient since no restriction is imposed on \( p_0 \). Next we give a definition of compatibility of a vector \( p_0 = (p(0; 0), ..., p(N; 0)) \) given a compatible \( P_1 \). This definition ensures that all the equilibrium conditions are satisfied but does not require \( \sum_m p(m; 0) = 1 \) or \( p(m; 0) > 0 \) for all \( m = 0, ..., N \), so that \( p_0 \) may not be a valid reporting strategy.

**Definition A.2.** Given \( P_1 \) compatible with \( l \), we say that \( p_0 \) is compatible with \( P_1 \) if, when \( l > 0 \):

(i) \( p(m; 0) = 0 \) for all \( m < \bar{l}_1 \);

(ii) \( p(\bar{l}_1; 0) \) is such that \( W(t_n; n|p_0, P_1) = W(\bar{l}_1; n|p_0, P_1) \) and \( W(\bar{l}_n - 1; n|p_0, P_1) \leq W(\bar{l}_n; n|p_0, P_1) \), for all \( n \leq l \);

(iii) \( p(m; 0) \) for \( \bar{l}_1 < m \leq N \) satisfies: a) if \( p(\bar{l}_1; 0) > 0 \), then \( p(m; 0) \) is the value such that \( W(\bar{l}_1; 0|p_0, P_1) = W(m; 0|p_0, P_1) \); and b) if \( p(\bar{l}_1; 0) = 0 \), then \( p(\bar{l}_1 + 1; 0) \) satisfies \( W(\bar{l}_1; 0|p_0, P_1) \leq W(\bar{l}_1 + 1; 0|p_0, P_1) \) and \( W(\bar{l}_1; 1|p_0, P_1) \geq W(\bar{l}_1 + 1; 1|p_0, P_1) \), and \( p(m; 0) \) is such that \( W(\bar{l}_1 + 1; 0|p_0, P_1) = W(m; 0|p_0, P_1) \) for \( m > \bar{l}_1 + 1 \);

and, when \( l = 0 \), \( p(0; 0) \in [0, 1] \) and for each \( m > 0 \), \( p(m; 0) \) satisfies (iii) with \( \bar{l}_1 = 0 \).

By definition, if \( P_1 \) is compatible with \( l \) and \( p_0 \) is compatible with \( P_1 \) we have that

\[
W(\bar{l}_n; n|p_0, P_1) = W(t; n|p_0, P_1) \geq W(m; n|p_0, P_1), \ \forall n \leq l, \ m \geq l, \ t_n \leq t \leq \bar{l}_n. \tag{A.1}
\]

Moreover, for any vector \( p_0 \) that is not compatible with \( P_1 \), (A.1) is violated and hence \( P = (p_0, P_1) \) is not an equilibrium. By Definition A.1 and A.2, only \( p(N; 0) \) can be negative in a compatible vector \( p_0 \) and, in that case \( p(m; 0) = 0 \) for each \( m < N \). Thus, if \( \sum_m p(m; 0) = 1 \), the vector \( p_0 \) is a reporting strategy and \( P = (p_0, P_1) \) satisfies all properties of Lemma 3. It follows that \( P = (p_0, P_1) \) is an equilibrium only if \( p_0 \) is compatible with \( P_1 \) and
The following lemma verifies that these two conditions are also sufficient for an equilibrium.

**Lemma A.2.** Let $P^1 = (p_1, ..., p_N)$ be compatible. Then $P = (p_0, P^1)$ is an equilibrium if and only if $p_0$ is compatible with $P^1$ and $\sum_m p(m; 0) = 1$.

**Proof.** Since (A.1) holds, we only need to argue that: (a) given $P$ there are no profitable deviation to out of equilibrium messages; and (b) in all states $n > l$, the agency has no incentive to deviate from the reporting strategy $p(n; n) = 1$.

(a) The most profitable deviation among the out-of-equilibrium signals is always $m = l - 1$. Part (iii) (e) of Definition A.1 and part (ii) of Definition A.2 imply that

$$W(l; l - 1) \geq W(l - 1; l - 1) \quad (A.2)$$

because either $t_{l-1} = l$ or $q(l) = 1$. By part (i) of Lemma 1 (A.2) implies that $W(l; l) > W(l - 1; l)$, and, since $q(l - 1) \geq q(l)$, by (ii) of Lemma 1 it implies that $W(l; n) \geq W(l - 1; n)$ for all $n < l - 1$. Finally, since $W(l; l) > W(l - 1; l)$ and since $W(l; n) - W(l - 1; n) = W(l; l) - W(l - 1; l)$ for all $n > l$ because $\hat{q}(l - 1) = \hat{q}(l) = 0$, we have that $W(l; n) > W(l - 1; n)$ for all $n > l$.

(b) The claim is trivial if $l = 0$ since in that case $p_0$ compatible with $P^1$ implies that $q(m) = q^*$ for all $m = 1, ..., N$. If $P^1$ is not compatible with $l = 0$, then $q(l) > q^*$ because either $l > 1$ and (A.2) holds, or $l = 1$ and $p(l; l) < 1$, which implies $q(l) = 1$. It follows that $q(m) > q(n) > q^*$ for all $n > m \geq l$. First we show that $W(n; 0) - W(m; 0) \geq 0$ implies $W(n; n) - W(m; n) \geq 0$. This is trivially true for any $m$ if $n = m$. Now, the condition $W(n; 0) - W(m; 0) \geq 0$ is equivalent to:

$$\frac{U(B, g, q(m)) - U(B, g, q(n))}{U(B, g, q(n)) - U(B, b, 0)} \leq \frac{n - m}{m} \quad (A.3)$$

Similarly, the condition $W(n; n) - W(m; n) \geq 0$ is equivalent to:

$$\frac{U(G, g, q(m)) - U(G, g, q(n))}{U(G, g, q(n)) - U(G, b, 0)} \leq \frac{n - m}{m} \quad (A.4)$$

When Assumption 4 holds, $[U(G, g, q) - U(G, b, 0)]/[U(B, g, q) - U(B, b, 0)]$ is decreasing in $q \in (q^*, 1)$. So,

$$\frac{U(G, g, q(m)) - U(G, b, 0)}{U(B, g, q(m)) - U(B, b, 0)} \leq \frac{U(G, g, q(n)) - U(G, b, 0)}{U(B, g, q(n)) - U(B, b, 0)}.$$
This condition implies
\[
\frac{U(B, g, q(m)) - U(B, g, q(n))}{U(B, g, q(n)) - U(B, b, 0)} \leq \frac{U(G, g, q(m)) - U(G, g, q(n))}{U(G, g, q(n)) - U(G, b, 0)}.
\]

Hence, (A.3) implies (A.4). It then follows that if \( p(m; n) > 0 \) for some \( n > m \geq l \), then \( W(m; 0) > W(n; 0) \). By part (ii) of Lemma 1, \( W(m; k) > W(n; k) \) for any \( k \leq m \), which contradicts (A.1).

Q.E.D.

For two reporting strategies \( p_n \) and \( p'_n \) in state \( n \), we denote as \( p_n >_F p'_n \) that \( p_n \neq p'_n \) and \( p_n \) first order stochastically dominates \( p'_n \). We define a binary relation on the set of all sequences of reporting strategies in state \( 1, \ldots, N \).

**Definition A.3.** For any two sequences of reporting strategies \( P^1 = (p_1, \ldots, p_N) \) and \( \tilde{P}^1 = (\tilde{p}_1, \ldots, \tilde{p}_N) \), we say that \( P^1 >_{LF} \tilde{P}^1 \) if \( p_n >_F \tilde{p}_n \) for some \( n = 1, \ldots, N \) and \( \tilde{p}_{n'} \not>_F p_{n'} \) for all \( n' > n \).

Let \( \mathcal{P}^1 \) denote the collection of all sequences of reporting strategies \( P^1 = (p_1, \ldots, p_N) \) that are compatible. The next two lemmas establish that \( \mathcal{P}^1 \) is non-empty, closed and connected, and that the binary relation \( >_{LF} \) on \( \mathcal{P}^1 \) is a strict order.

**Lemma A.3.** Let \( P^{n+1} = (p_{n+1}, \ldots, p_N) \) be compatible with \( l \). Then, the set of reporting strategies in state \( n \) compatible with \( l \) given \( P^{n+1} \) is non-empty, closed and connected. Further, the binary relation \( >_F \) on such set is a strict order.

**Proof.** (i) For \( n > l \) the claim is trivially true since by Definition A.1 there is a unique \( p_n \) compatible with \( l \) given \( P^{n+1} \) and it has \( p(n; n) = 1 \).

(ii) For \( n = l \), the definition of compatibility implies that \( p_l \) is compatible if and only if:

a) \( p_n(m) = 0 \) for all \( m < l \); b) \( p(l; l) \in [0, 1] \); c) for each \( l < m < N \), \( p(m; l) \) is equal to the minimum of \( 1 - \sum_{k<l} p(k; l) \) and the value such that \( W(l; l|p_l, P^{l+1}) = W(m; l|p_l, P^{l+1}) \); and d) \( p(N; l) = 1 - \sum_{k<N} p(k; l) \). Note that \( W(l; l|p_l, P^{l+1}) \) is independent of \( p_l \). Further, for all \( m > l \) we have: \( W(m; l|p_l, P^{l+1}) \) depends only on \( p(m; l) \), is strictly decreasing in \( p(m; l) \), and satisfies \( W(l; l|p_l, P^{l+1}) < W(m; l|p_l, P^{l+1}) \) when \( p(m; l) = 0 \). Thus, the set of compatible \( p_l \) is non-empty and it is closed because each compatible \( p_l \) is uniquely determined by the value of \( p(l; l) \) and there exists a compatible \( p_l \) for each value of \( p(l; l) \in \)
[0, 1]. Moreover, for two compatible strategies $p_l$ and $p'_l$ such that $p(l; l) < p'(l; l)$ we have $p_l >_{Fr} p'_l.$ Finally, if $p_l$ is compatible and \{p'_l\}_{i=1,2,...}$ is a sequence of compatible strategies with $\lim_{i\to \infty} p'(l; l) = p(l; l)$ then $\lim_{i\to \infty} p'(m; l) = p(m; l)$ for each $m = 0, ..., N,$ hence the set of compatible strategies is connected.

(iii) For $n < \ell$, the definition of compatibility implies that there is exactly one $p_n$ compatible with $\ell$ given $P^{n+1}$ if $\tilde{t}_{n+1} = N$ or $W(\tilde{t}_{n'}; n'|P^{n+1}) < W(\tilde{t}_{n'}; n'|P^n)$ for some $n < n'$. Otherwise, $p_n$ is compatible if and only if: a) for each $m < \tilde{t}_{n+1}$, $p(m; n) = 0$; b) $p(\tilde{t}_{n+1}; n)$ is non-negative and not larger that the minimum of 1 and the largest value such that $W(\tilde{t}_{n'} - 1; n'|P^n) \leq W(\tilde{t}_{n'}; n'|P^n) = W(\tilde{t}_{n'}; n'|P^n)$ for all $n < n'$; c) if $p(\tilde{t}_{n+1}; n) > 0$ then: ci) $p(m; n)$ is the minimum of $1 - \sum_{k < m} p(k; n)$ and the value such that $W(\tilde{t}_{n+1}; n|P^n) = W(m; n|P^n)$ for each $\tilde{t}_{n+1} < m < N$; and ci) $p(N; m) = 1 - \sum_{k < N} p(k; n)$; d) if $p(\tilde{t}_{n+1}; n) = 0$ then: di) $p(\tilde{t}_{n+1+1}; n)$ is at least as large as the minimum of 1 and the value such that $W(\tilde{t}_{n+1+1}; n + 1|P^n) = W(\tilde{t}_{n+1} + 1; n + 1|P^n)$ and it is at most as large as the minimum of 1 and the value such that $W(\tilde{t}_{n+1}; n|P^n) = W(\tilde{t}_{n+1+1}; n + 1|P^n)$; dii) for each $\tilde{t}_{n+1} + 1 < m < N$, $p(m; n)$ is the minimum of $1 - \sum_{k < m} p(m; n)$ and the value such that $W(\tilde{t}_{n+1+1}; n|P^n) = W(\tilde{t}_{n+1}; n|P^n)$; and diii) $p(N; n) = 1 - \sum_{k < N} p(k; n)$.

By compatibility of $P^{n+1}$, $p(\tilde{t}_{n+1}; n) = 0$ satisfies b), hence the set of all values of $p(\tilde{t}_{n+1}; n)$ that satisfy b) is non-empty, closed and connected. Further, for each $p(\tilde{t}_{n+1}; n) > 0$ that satisfies b), $p(m; n)$ is uniquely determined by c) for each $m > \tilde{t}_{n+1}$. When $p(\tilde{t}_{n+1}; n) = 0$ instead, if $\tilde{t}_{n+1} + 1 = N$ $p(\tilde{t}_{n+1} + 1; n)$ is uniquely determined equal to 1 by diii). Otherwise, note that $q(m|P^n)$ depends only on $p(m; n)$, is strictly decreasing in $p(m; n)$, and satisfies $q(m|P^n) = 1$ when $p_n(m) = 0$ for each $m > \tilde{t}_{n+1}$. Further, by Lemma 1, the value such that $W(\tilde{t}_{n+1}; n + 1|P^n) = W(\tilde{t}_{n+1} + 1; n + 1|P^n)$ is smaller than the value such that $W(\tilde{t}_{n+1}; n|P^n) = W(\tilde{t}_{n+1+1}; n + 1|P^n)$. It follows that the set of values of $p(\tilde{t}_{n+1} + 1; n)$ that satisfy di) is non-empty, closed and connected, and for each such value, $p(m; n)$ is uniquely determined by dii) and diii) for all $m > \tilde{t}_{n+1} + 1$. Finally, when $p(\tilde{t}_{n+1} + 1; n)$ assumes the maximum value that respects condition di), the resulting $p_n$ also satisfies condition c). This concludes the proof of the first part of the claim. To establish that the set of $p_n$ compatible with $l$ given $P^{n+1}$ is ordered by stochastic dominance, note that $p(\tilde{t}_{n+1}; n) < \tilde{p}(\tilde{t}_{n+1}; n)$ implies $q(\tilde{t}_{n+1}|p_n, P^{n+1}) > q(\tilde{t}_{n+1}|\tilde{p}_n, P^{n+1})$ and hence
\[ p(m; n) < \hat{p}(m; n) \text{ for each } \tilde{t}_{n+1} < m < \tilde{t}_n \text{ by ci). A similar argument also holds for the case } p(\tilde{t}_{n+1}; n) = \hat{p}(\tilde{t}_{n+1}; n) = 0 \text{ and } p(\tilde{t}_{n+1} + 1; n) < \hat{p}(\tilde{t}_{n+1} + 1; n). \] Q.E.D.

**Lemma A.4.** The set \( \mathcal{P}^1 \) is non-empty, closed and connected. Further, \( >_{LF} \) on \( \mathcal{P}^1 \) is a strict order.

**Proof.** An immediate implication of Lemma A.3 and the definition of compatibility is that, for each \( l \), the collection of all \( \mathcal{P}^1 \) compatible with \( l \) is non-empty, closed, connected and is completely ordered by \( >_{LF} \). We will show that for each \( 0 < l < N \), the smallest \( \mathcal{P}^1 \) compatible with \( l \) is also compatible with \( l - 1 \). The claim then follows from observing that the largest \( \mathcal{P}^1 \) compatible with \( N - 1 \) has \( p(N, n) = 1 \) for all \( n = 1, ..., N \) which is the only sequence of reporting strategies compatible with \( N \).

If \( \mathcal{P}^1 \) is the smallest sequence of reporting strategies compatible with \( l \), by the proof of Lemma A.3 and the definition of \( >_{LF} \), \( p(l; l) = 1 \) and \( p(l; l - 1) \) is the minimum of 1 and the value such that \( W(l; l - 1|\mathcal{P}^{l-1}) = W(l - 1; l - 1|\mathcal{P}^{l-1}) \). This implies that \( \mathcal{P}^{l-1} \) is also compatible with \( l - 1 \). The claim follows by noting that for all \( n < l - 1 \) the definition of \( p_n \) compatible with \( l \) is identical to the definition of \( p_n \) compatible with \( l - 1 \). Q.E.D.

Next, consider the correspondence \( \phi: \mathcal{P}^1 \to \mathbb{R} \) defined as

\[
\phi(\mathcal{P}^1) = \{ x \in \mathbb{R} \mid x = \sum_m p(m; 0) \text{ for some } p_0 \text{ compatible with } \mathcal{P}^1 \}.
\]

For each compatible \( \mathcal{P}^1 \), the set \( \phi(\mathcal{P}^1) \) is obtained by first finding all vectors \( p_0 \) compatible with \( \mathcal{P}^1 \) and then by summing over all entries for each such vector. By Lemma A.2, for any equilibrium \( P = (p_0, \mathcal{P}^1) \), we have \( \phi(\mathcal{P}^1) \ni 1 \). Further, if \( \phi(\mathcal{P}^1) \ni 1 \) for some \( \mathcal{P}^1 \in \mathcal{P}^1 \), then there exists a \( p_0 \) such that \( P = (p_0, \mathcal{P}^1) \) is an equilibrium. The following properties \( \phi \) will be used to conclude the proof of the proposition.

**Lemma A.5.** For all \( \mathcal{P}^1 \in \mathcal{P}^1 \), the set \( \phi(\mathcal{P}^1) \) is closed and convex. Further, \( \phi \) is upper hemicontinuous, and monotone in that \( \max \phi(\mathcal{P}^1) < \min \phi(\tilde{\mathcal{P}}^1) \) for all \( \mathcal{P}^1, \tilde{\mathcal{P}}^1 \in \mathcal{P}^1 \) such that \( \mathcal{P}^1 >_{LF} \tilde{\mathcal{P}}^1 \).

**Proof.** First we establish that \( \phi(\mathcal{P}^1) \) is closed and convex. Let \( \mathcal{P}^1 \) be compatible with \( l \). We distinguish three cases.
(i) If \( l = 0 \), then \( p(n; n) = 1 \) for all \( n = 1, \ldots, N \) and by Definition A.2, \( p_0 \) is compatible if and only if \( p(0; 0) \in [0, 1] \) and \( p(m; 0) = (1 - q^*)\pi_m / (q^*\pi_0) \) for all \( m = 1, \ldots, N \).

(ii) If \( l = N \), then \( p(N; n) = 1 \) for all \( n = 1, \ldots, N \) and by Definition A.2, \( p_0 \) is compatible if and only if \( p(m; 0) = 0 \) for all \( m = 0, \ldots, N - 1 \), and \( p(N; 0) \) is not larger that the value such that \( W(N - 1; N - 1|p_0, P^1) = W(N; N - 1|p_0, P^1) \). Such value exists and is unique since \( W(N; N - 1|p_0, P^1) \) is continuous and strictly decreasing in \( p(0; 0) \), \( W(N; N - 1|p_0, P^1) > W(N - 1; N - 1|p_0, P^1) \) in the limit as \( p(N; 0) \to -\infty \), and \( W(N; N - 1|p_0, P^1) < W(N - 1; N - 1|p_0, P^1) \) in the limit as \( p(N; 0) \to +\infty \).

(iii) If \( 0 < l < N \), an argument analogous to that in part (iii) of the proof of Lemma A.3 can be used to establish that the set of vectors \( p_0 \) compatible with \( P^1 \) is non-empty, closed, connected and compact. Thus, the function \( \sum_m p(m; 0) \) on the set of compatible vectors \( p_0 \) has a minimum and maximum and assumes all values in between.

Next we establish the monotonicity of \( \phi \). First note that if \( P^n = (p_n, p_{n+1}, \ldots, p_N) \) and \( \hat{P}^n = (\hat{p}_n, p_{n+1}, \ldots, p_N) \) are both compatible with \( l \) and \( p_n \succeq \hat{p}_n \), then \( q(m|P^n) \geq q(m|\hat{P}^n) \) for all \( m < \hat{t}_n \). Further, either \( \bar{t}_n = \hat{t}_n \) and \( q(\bar{t}_n|P^n) < q(\bar{t}_n|\hat{P}^n) \), or \( \bar{t}_n > \hat{t}_n \) and \( q(\bar{t}_n|P^n) < q(\bar{t}_n|\hat{P}^n) = 1 \). This property can be used together with the definition of compatibility A.1 to establish that if \( n - 1 > 0 \) for any \( p_{n-1} \) compatible with \( P^n \) and any \( \hat{p}_{n-1} \) compatible with \( \hat{P}^n \), \( p_{n-1} \succeq \hat{p}_{n-1} \). This in turn implies that \( q(m|P^{n-1}) \geq q(m|\hat{P}^{n-1}) \) for all \( m < \hat{t}_{n-1} \) and either \( \bar{t}_{n-1} = \hat{t}_{n-1} \) and \( q(\bar{t}_{n-1}|P^{n-1}) < q(\bar{t}_{n-1}|\hat{P}^{n-1}) \), or \( \bar{t}_{n-1} > \hat{t}_{n-1} \) and \( q(\bar{t}_{n-1}|P^{n-1}) < q(\bar{t}_{n-1}|\hat{P}^{n-1}) = 1 \). Iterating the argument leads to: i) \( q(m|P^1) \geq q(m|\hat{P}^1) \) for each \( m < \hat{t}_1 \); and ii) either \( \bar{t}_1 = \hat{t}_1 \) and \( q(\bar{t}_1|P^1) < q(\bar{t}_1|\hat{P}^1) \), or \( \bar{t}_1 > \hat{t}_1 \) and \( q(\bar{t}_1|P^1) = q(\bar{t}_1|\hat{P}^1) = 1 \). Properties i) and ii) together with the definition of compatibility A.2 imply that for any \( p_0 \) compatible with \( P^1 \) and any \( \hat{p}_0 \) compatible with \( \hat{P}^1 \), \( p(m; 0) \leq \hat{p}(m; 0) \) for all \( m \) and \( p(\bar{t}_1; 0) < \hat{p}(\bar{t}_1; 0) \). This establishes the claimed monotonicity property.

To prove that \( \phi \) is upper hemicontinuous we consider two separate cases. Suppose \( P^1 \) is such that, for some \( n > 0 \), \( W(\bar{t}_n; n|P^1) \neq W(\bar{t}_n; n|P^1) \). By Definition A.2 there exists a unique \( p_0 \) compatible with \( P^1 \). Take any sequence of \( \{P_i^1 \in P^1\}_{i=1, 2, \ldots} \) such that \( \lim_{i \to \infty} p_i(m; n) = p(m; n) \), for all \( m \) and \( n = 1, \ldots, N \). By definition we have that \( \lim_{i \to \infty} W(m; n|P_i^1) = W(m; n|P^1) \) for all \( m \) and \( n = 1, \ldots, N \) as well as \( \lim_{i \to \infty} \bar{t}_{n,i} = \bar{t}_n \) and \( \lim_{i \to \infty} \bar{t}_{n,i} = \bar{t}_n \) for each \( n = 1, \ldots, N \). Moreover, since \( P_i \in P^1 \) there is at most one \( n \) such
that \( W(\bar{t}_{n,i}; n|P^1) \neq W(t_{n,i}; n|P^1) \) and, in the limit, it coincides with the state \( n \) such that \( W(\bar{t}_n; n|P^1) \neq W(t_n; n|P^1) \). It follows that, for \( P^1 \) sufficiently close to \( P^1 \), there is a unique \( p_{i,0} \) compatible with \( P^1 \) and \( p_i(m; 0) \) converges to \( p(m; 0) \) for each \( m \). Thus, near \( P^1 \), \( \phi \) is a function and it is continuous.

If \( P^1 \) is such that \( W(\bar{t}_n; n|P^1) = W(t_n; n|P^1) \), we show that for any sequence \( \{P^i \in P^1\}_{i=1,2,...} \) such that \( \lim_{i \to \infty} p_i(m; n) = p(m; n) \), for all \( m \) and \( n = 1, ..., N \): a) if \( P^1 <_{LF} P^1 \) for all \( i \), then \( \lim_i \max \phi(P^i) = \lim_i \min \phi(P^i) = \max \phi(P^1) \); and b) if \( P^1 <_{LF} P^1 \) for all \( i \), then \( \lim_i \max \phi(P^i) = \lim_i \min \phi(P^i) = \max \phi(P^1) \). To prove a) first note that since \( \lim_{i \to \infty} \bar{t}_{1,i} = \bar{t}_1 \), from the proof of the monotonicity claim above we have that for all \( i \) large enough, \( q(m|P^1) \geq q(m|P^1) \) for each \( m < \bar{t}_1 \) and \( q(\bar{t}_1|P^1) < q(\bar{t}_1|P^1) \). Hence, for all \( i \) sufficiently large, there is some \( n \geq 1 \) such that \( W(\bar{t}_n; n|P^1) \neq W(t_n; n|P^1) \). Then, by Definition A.2, there exists a unique \( p_0, 1 \) compatible with \( P^1 \). Since \( \lim_i W(m; n|P^1) = W(m; n|P^1) \) for all \( m \) and \( n = 1, ..., N \) and \( \lim_{i \to \infty} \bar{t}_{n,i} = \bar{t}_n \) and \( \lim_{i \to \infty} t_{n,i} = t_n \) for each \( n = 1, ..., N \), the definition of compatibility implies that \( \lim_i p_i(\bar{t}_1; 0) = 0 \) and for each \( m > \bar{t}_1 \), \( p_i(m; 0) \) converges to the value such that \( W(\bar{t}_1; 0|p_0, P^1) = W(m; 0|p_0, P^1) \), which concludes the proof of part a). A similar argument establishes b).

Q.E.D.

To complete the proof of the proposition, first note that the smallest compatible \( P^1 \) has \( p(n; n) = 1 \) for all \( n = 1, ..., N \). By Definition A.2, there is a vector \( p_0 \) compatible with \( P^1 \) such that \( p(0; 0) = 1 \) and \( p(m; 0) > 0 \) for all \( m > 1 \). Thus, \( \max \phi(P^1) > 1 \). Next, the largest compatible \( P^1 \) has \( p(N; n) = 1 \) for all \( n = 1, ..., N \). By Definition A.2, any vector \( p_0 \) with \( p(m; 0) = 0 \) for all \( m < N \) and \( p(N; 0) \) sufficiently small is compatible with \( P^1 \). Since \( \phi \) is upper hemicontinuous and monotone, there exists a unique \( P^1 \) such that \( \phi(P^1) \geq 1 \). Moreover, from the proof of Lemma A.5, we have that for any two vectors \( p_0, p'_0 \) compatible with \( P^1 \), \( \sum_m p(m; 0) = \sum_m p'(m; 0) \) if and only if \( p_0 = p'_0 \), hence, there exists a unique equilibrium. To establish when the equilibrium has threshold \( l > 0 \), note that the only \( P^1 \) compatible with \( l = 0 \) has \( p(n; n) = 1 \) for all \( n = 1, ..., N \), and the smallest vector \( p_0 \) compatible with \( P^1 \) has \( p(0; 0) = 0 \) and \( p(m; 0) = (1-q^*)\pi_m/(q^*\pi_0) \). Thus, at the smallest \( p_0 \) compatible with \( P^1 \), \( \sum_m p(m; 0) > 1 \) if and only if \( \pi_0 < 1 - q^* \). This concludes the proof of the proposition.
A.5. Lemma A.6 and proof

**Lemma A.6.** Suppose that for some $\epsilon > 0$ there exists a subsequence in $N$ such that

$$
\lim_{N \to \infty} \sum_{n \in [N(\pi_\gamma), N(\pi_\gamma+\epsilon)]} \sum_{m > N(\pi_\gamma+\epsilon)} \pi_n^N p_n^N(m; n) > 0.
$$

Then, for some $N$ sufficiently large, there exist $n \in [N(\pi_\gamma), N(\pi_\gamma+\epsilon)]$ and $m > N(\pi_\gamma+\epsilon)$ such that $p_n^N(m; n) > 0$ and $W(n; n) > W(m; n)$.

**Proof.** Since $\sum_{n \notin [N(\pi_\gamma), N(\pi_\gamma+\epsilon)]} \pi_n^N$ is 0 in the limit for any $\gamma > 0$,

$$
\lim_{N \to \infty} \sum_{n \in [N(\pi_\gamma), N(\pi_\gamma+\epsilon)]} \sum_{m > N(\pi_\gamma+\epsilon)} \pi_n^N p_n^N(m; n) > 0.
$$

Define the following collection of messages

$$
\mathcal{M}_N = \{m > (\pi + \epsilon)N \mid p_n^N(m; n) > 0 \text{ for some } n \in [N(\pi_\gamma), N(\pi_\gamma+\epsilon)]\}.
$$

The set $\mathcal{M}_N$ contains all messages larger than $(\pi + \epsilon)N$ that are sent with positive probability in some state $n$ close to $\pi N$. We have

$$
\lim_{N \to \infty} \sum_{n \in [N(\pi_\gamma), N(\pi_\gamma+\epsilon)]} \sum_{m \in \mathcal{M}_N} \pi_n^N p_n^N(m; n) > 0.
$$

Since

$$
\lim_{N \to \infty} \sum_{n \notin [N(\pi_\gamma), N(\pi_\gamma+\epsilon)]} \sum_{m \in \mathcal{M}_N} \pi_n^N p_n^N(m; n) = 0,
$$

it follows that

$$
\lim_{N \to \infty} \sup_{m \in \mathcal{M}_N} \frac{\sum_{n \in [N(\pi_\gamma), N(\pi_\gamma+\epsilon)]} \pi_n^N p_n^N(m; n)}{\sum_n \pi_n^N p_n^N(m; n)} = 1.
$$

For any $m \in \mathcal{M}_N$, the equilibrium belief $q(m)^N$ is given by

$$
\frac{\sum_{n \in [N(\pi_\gamma), N(\pi_\gamma+\epsilon)]} \pi_n^N p_n^N(m; n) \frac{m}{\sum_n \pi_n^N p_n^N(m; n)} + \sum_{n \notin [N(\pi_\gamma), N(\pi_\gamma+\epsilon)]} \pi_n^N p_n^N(m; n) \min\{\frac{m}{\sum_n \pi_n^N p_n^N(m; n)}, 1\}}{\frac{\pi_\gamma N}{m} + \sum_{n \notin [N(\pi_\gamma), N(\pi_\gamma+\epsilon)]} \pi_n^N p_n^N(m; n) + \sum_{n \notin [N(\pi_\gamma), N(\pi_\gamma+\epsilon)]} \pi_n^N p_n^N(m; n)}.
$$

Thus,

$$
\lim_{N \to \infty} \inf_{m \in \mathcal{M}_N} q(m)^N \leq \frac{(\pi + \gamma)N}{m}.
$$
Since for all $\mu \in (\pi, 1]$

$$
\pi U(G, g, 1) + (\mu - \pi)U(B, b, 0) > \pi U(G, g, \mu) + (\mu - \pi)U(B, g, \mu),
$$

and since $q^N(n)$ can be made arbitrarily close to 1 for all $n$ close to $\pi N$, we have that for $\gamma$ sufficiently small and $N$ sufficiently large, there exists $m \in \mathcal{M}^N$ such that $W(n; n) > W(m; n)$ for all $n \in [N(\pi - \gamma), N(\pi + \gamma)]$.

### A.6. Proof of Lemma 4

Since $X$ is exchangeable, for any two realizations $x$ and $x'$ of $X$ such that $\sum_i x_i = \sum_i x'_i$, we have $f(x) = f(x')$. Let $f_n$ represent the probability of $X$ such that $\sum_i x_i = n$.

Assume that $f$ is MTP$_2$. Let $h(x_i, z_i)$ be the joint probability function of $(X_i, Z_i)$. For $z_i \geq z'_i$, we have

$$
h(1, z_i)h(0, z'_i) = \left( \begin{array}{c} N - 1 \\ z_i \end{array} \right) \left( \begin{array}{c} N - 1 \\ z'_i \end{array} \right) f_{z_i + 1} f_{z'_i} \\
\geq \left( \begin{array}{c} N - 1 \\ z_i \end{array} \right) \left( \begin{array}{c} N - 1 \\ z'_i \end{array} \right) f_{z_i} f_{z'_i + 1} = h(1, z'_i)h(0, z_i).
$$

Thus, $h(x_i, z_i)$ is also MTP$_2$.

Now consider the conditional distributions $\phi(y_i \mid x_i)$ and $\psi(\hat{z}_i \mid z_i)$. These are

$$
\phi(y_i \mid x_i) = \begin{cases} 0 & \text{if } y_i < x_i, \\
1 & \text{if } y_i = x_i = 1, \\
\beta(1 - x_i, y_i - x_i, p) & \text{otherwise};
\end{cases}
$$

$$
\psi(\hat{z}_i \mid z_i) = \begin{cases} 0 & \text{if } \hat{z}_i < z_i, \\
1 & \text{if } \hat{z}_i = z_i = N - 1, \\
\beta(N - 1 - z_i, \hat{z}_i - z_i, p) & \text{otherwise}.
\end{cases}
$$

It is straightforward to verify that both $\phi$ and $\psi$ are MTP$_2$. The joint distribution of $(X_i, Y_i, Z_i, \hat{Z}_i)$ is simply $h(x_i, z_i)\phi(y_i \mid x_i)\psi(\hat{z}_i \mid z_i)$. Since each of these component functions is MTP$_2$, the joint distribution is MTP$_2$.

When $f$ is MRR$_2$, using a similar reasoning we can establish that the joint distribution of $(X_i, Y_i, N - 1 - Z_i, N - 1 - \hat{Z}_i)$ is MTP$_2$.
A.7. Proof of Lemma 5

By Lemma 4, if \( f \) is \( \text{MTP}_2 \), then \( (X_i, Y_i, Z_i, \tilde{Z}_i) \) is \( \text{MTP}_2 \). Since the marginal distribution of any subset of a \( \text{MTP}_2 \) vector is itself \( \text{MTP}_2 \) (Karlin and Rinott, 1980, Proposition 3.1), this means that the joint distribution of \((X_i, Y_i, \tilde{Z}_i)\) is \( \text{MTP}_2 \). Suppose \( h^*(\tilde{z}_i \mid x_i, y_i) \) represents the conditional probability function of \( \tilde{z}_i \) given \( X_i \) and \( Y_i \). The \( \text{MTP}_2 \) property of the joint distribution implies that, for any \( \tilde{z}_i \geq \tilde{z}'_i \), we have \( h^*(\tilde{z}_i \mid 1, 1)h^*(\tilde{z}'_i \mid 0, 1) \geq h^*(\tilde{z}_i \mid 0, 1)h^*(\tilde{z}'_i \mid 1, 1) \), and thus the likelihood ratio \( h^*(\cdot \mid 1, 1)/h^*(\cdot \mid 0, 1) \) is monotone increasing, implying that the distribution \( \{r^G(m)\} \) first-order stochastic dominates \( \{r^B(m)\} \).

When \( f \) is \( \text{MRR}_2 \), the joint distribution \( (X_i, Y_i, N - 1 - \tilde{Z}_i) \) is \( \text{MTP}_2 \). This implies that the likelihood ratio \( h^*(\cdot \mid 1, 1)/h^*(\cdot \mid 0, 1) \) is monotone decreasing. Hence, the distribution \( \{r^B(m)\} \) first-order stochastic dominates \( \{r^G(m)\} \).

A.8. Proof of Lemma 6

Suppose \( f \) is \( \text{MTP}_2 \). Let \( h^{**}(x_i \mid y_i, \tilde{z}_i) \) represent the conditional probability function of \( X_i \) given \( Y_i \) and \( \tilde{Z}_i \). By the same argument as in Lemma 5, the joint distribution of these three variables is \( \text{MTP}_2 \). Therefore, for \( m \geq m' \), the conditional distribution satisfies:

\[
h^{**}(1 \mid 1, m - 1)h^{**}(0 \mid 1, m' - 1) \geq h^{**}(0 \mid 1, m - 1)h^{**}(1 \mid 1, m' - 1).
\]

This condition implies that \( q(m) = h^{**}(1 \mid 1, m - 1) \geq h^{**}(1 \mid 1, m' - 1) = q(m') \).

When \( f \) is \( \text{MRR}_2 \), we have \( q(m') \geq q(m) \) for \( m \geq m' \), as

\[
h^{**}(1 \mid 1, m' - 1)h^{**}(0 \mid 1, m - 1) \geq h^{**}(0 \mid 1, m' - 1)h^{**}(1 \mid 1, m - 1).
\]

A.9. Proof of Lemma 7

For each \( m = 1, \ldots, N \),

\[
r^B(m) = \sum_{n=0}^{N-1} \Pr[\tilde{Z}_i = m - 1 \mid X_i = 0, Y_i = 1, Z_i = n] \Pr[Z_i = n \mid X_i = 0, Y_i = 1]
\]

\[
= \sum_{n=0}^{m-1} \beta(N - n - 1, m - n - 1, p) \frac{\pi_n N - n}{1 - \pi N}
\]

\[
= \sum_{n=0}^{m} \pi_n \beta(N - n, m - n, p) \frac{m - n}{N(1 - \pi)p}.
\]

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For each \( m = 1, \ldots, N \), define

\[
N(m) = \sum_{n=0}^{m} \pi_n \beta(N - n, m - n, p)n;
\]
\[
D(m) = \sum_{n=0}^{m} \pi_n \beta(N - n, m - n, p)m.
\]

Since \( \sum_{m=1}^{N} r^B(m) = 1 \), we have

\[
\sum_{m=1}^{N} \left( D(m) - N(m) \right) = N(1 - \pi)p.
\]

Note that since \( r^G(m) = N(m)/(\pi N) \), from \( \sum_{m=1}^{N} r^G(m) = 1 \) we have

\[
\sum_{m=1}^{N} N(m) = N\pi,
\]

and hence

\[
\sum_{m=1}^{N} D(m) = N(\pi + (1 - \pi)p).
\]

Finally, note that from equation (7) we have \( q(m) = N(m)/D(m) \). Since \( r^B(m) = (N(m) - D(m))/(N(1 - \pi)p) \), we have

\[
\sum_{m=1}^{N} r^B(m)q(m) = \frac{1}{N(1 - \pi)p} \sum_{m=1}^{N} D(m)q(m)(1 - q(m)).
\]

Since \( q(1 - q) \) is concave in \( q \), Jensen’s inequality implies the above is less than or equal to

\[
\frac{\sum_{m=1}^{N} D(m)}{N(1 - \pi)p} \left( \frac{\sum_{m=1}^{N} q(m)D(m)}{\sum_{m=1}^{N} D(m)} \right) \left( 1 - \frac{\sum_{m=1}^{N} q(m)D(m)}{\sum_{m=1}^{N} D(m)} \right).
\]

The lemma follows immediately.

**References**


