Exhaustible Resources and Economic Growth

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Abstract

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1 Introduction

Can the presence of an exhaustible natural resource prevent a poor country from being caught in a poverty trap? The natural resource is a source of income, which, together with the income coming from domestic production, can be used to buy consumption goods, or to accumulate capital, or directly to buy capital abroad. The idea is that a poor country with abundant natural resources could extract and sell an amount of resource which would enable it to buy a quantity of foreign capital sufficient to overcome the weakness of its initial domestic stock of capital. The first question we want to answer is on what circumstances would such a scenario occur. The second aim of this paper is the reexamination of the natural resource curse (see for instance Sachs and Warner [5], and Gylfason et al. [2]), which states that an abundance of natural resources tends to hamper economic growth.

2 The model

We consider a country which possesses a stock of a non-renewable natural resource $S$. This resource is extracted at a rate $R_t$, and then sold abroad at a price $P_t$, in terms of the numeraire, which is the domestic good. The inverse demand function for the resource is $P(R_t)$. The revenue from the sale of the natural resource, $\phi(R_t) = P(R_t)R_t$, is used to buy a foreign good, which is supposed to be a perfect substitute of the domestic good, used for consumption and capital accumulation. The domestic production function is $F(k_t)$, supposed to be convex for low levels of capital and then concave. The depreciation rate is $\delta$. We define the function $f(k_t) = F(k_t) + (1 - \delta)k_t$, and we shall, in the following, name it for simplicity the production function. We are interested in the optimal growth of this country which, if its initial capital is low, can be doomed to a poverty trap. Will the revenues coming from the extraction of the natural resource allow it to overcome the poverty trap? Or will the natural resource be, on the contrary, a curse destroying any incentive to accumulate?

We have to solve problem $(P)$

$$\max_{t=0}^{+\infty} \beta^t u(c_t), \beta \in (0, 1)$$
under the constraints
\[
\forall t, c_t \geq 0, k_t \geq 0, R_t \geq 0, \\
c_t + k_{t+1} \leq f(k_t) + \phi(R_t), \\
\sum_{t=0}^{+\infty} R_t \leq \mathcal{S}, \\
\mathcal{S} > 0, \ k_0 \geq 0 \text{ are given}
\]

We denote by \( Z(k_0, \mathcal{S}) \) the Value-function of Problem \((P)\). We make the following assumptions:

**H1** The utility function \( u \) is strictly concave, strictly increasing, continuously differentiable in \( R_+ \), and satisfies \( u(0) = 0, \ u'(0) = +\infty \).

**H2** The production function \( f \) is continuously differentiable in \( R_+ \), strictly increasing, strictly convex from 0 to \( k_I \), strictly concave for \( k \geq k_I \). Moreover, it satisfies \( f(0) = 0 \).

**H3** The function \( \phi \) is continuously differentiable, concave, strictly increasing from 0 to \( \hat{R} \leq +\infty \), and strictly decreasing for \( R > \hat{R} \). It also satisfies \( \phi(0) = 0 \).

Throughout this paper, an infinite sequence \( (x_t)_{t=0}^{+\infty} \) will be denoted by \( x \). An optimal solution to Problem \((P)\) will be denoted by \((c^*, k^*, R^*)\). We say that the sequences \( c, k, R \) are feasible from \( k_0 \) if they satisfy the constraints:

\[
\forall t, c_t \geq 0, k_t \geq 0, R_t \geq 0 \\
c_t + k_{t+1} \leq f(k_t) + \phi(R_t), \\
\sum_{t=0}^{+\infty} R_t \leq \mathcal{S}, \ k_0 \text{ is given.}
\]

Let \( \Omega(k_0, \mathcal{S}) \) denote the set of \((k, R)\) feasible from \( k_0 \) and \( \mathcal{S} \), i.e.,

\[
\forall t, 0 \leq k_{t+1} \leq f(k_t) + \phi(R_t), \ 0 \leq R_t \\
\sum_{t=0}^{+\infty} R_t \leq \mathcal{S}, \ k_0 \geq 0 \text{ is given.}
\]

We first list some preliminary results necessary for the main results of our paper.

**Lemma 1** The Value-function \( Z \) is continuous in \( k_0 \), given \( \mathcal{S} \).

**Proof:** We first prove that the correspondence \( \Omega \) is compact-valued and continuous in \( k_0 \), for the product topology, given \( \mathcal{S} \).

To prove that \( \Omega(k_0, \mathcal{S}) \) is compact, take a sequence \( \{k^n, R^n\} \) which converges to \( \{k, R\} \) for the product topology. First, observe that for any feasible \( k \) we have

\[
\forall t, 0 \leq k_{t+1} \leq f(k_t) + \phi(R_t) \leq f(k_t) + \max\{\phi(\hat{R}), \phi(\mathcal{S})\}.
\]
Therefore, \( k \) will be in a compact set for the product topology (see e.g. Le Van and Dana [3]). Second,
\[
\forall n, \forall t, 0 \leq k_{t+1}^n \leq f(k_t^n) + \phi(R_t^n),
\]
hence, by taking the limits we get
\[
\forall t, 0 \leq k_{t+1} \leq f(k_t) + \phi(R_t).
\]
We have proved that the set of feasible \( k \) is closed for the product topology. It is obvious that the set of feasible \( R \) belongs to a fixed compact set. To prove that this set is closed, observe that \( \forall N, \forall n \sum_{t=0}^{N} R_t^n \leq \overline{S} \). Taking the limit we get
\[
\forall N, \forall n \sum_{t=0}^{N} R_t \leq \overline{S}.
\]
That implies \( \sum_{t=0}^{+\infty} R_t \leq \overline{S} \). Summing up, we have proved that \( \Omega(k_0, \overline{S}) \) is compact.

It is easy to check that \( \Omega \) is upper hemi-continuous in \( k_0 \). It is less easy for the lower hemi-continuity of \( \Omega \). We will prove that, actually, \( \Omega \) is lower hemi-continuous. Let \( k_0^n \to k_0 \) as \( n \) goes to \( +\infty \) and \((k, R) \in \Omega(k_0, \overline{S})\). We have to show there exists a subsequence still denoted by \((k^n, R^n)\), for short, which converges to \((k, R)\) and satisfies \((k^n, R^n) \in \Omega(k_0^n, \overline{S}), \forall n\). We have three cases.

Case 1:
\[
0 \leq k_{t+1} < f(k_t) + \phi(R_0), \quad \forall t < T - 1 \\
0 \leq k_t \leq f(k_{t-1}) + \phi(R_{t-1}), \quad \forall t \geq T.
\]
There exists \( N \) such that for any \( n \geq N \), we have \( k_1 < f(k_0^n) + \phi(R_0) \). Define, for any \( n \geq N \), any \( t, k_t^n = k_t, R_t^n = R_t \) and the proof is done.

Case 2:
\[
k_{t+1} = f(k_t) + \phi(R_t), \quad \forall t \leq T; \\
k_{T+1} < f(k_T) + \phi(R_T), \\
k_{t+1} \leq f(k_t) + \phi(R_t), \quad \forall t \geq T + 1.
\]
Define, for \( t = 0, \ldots, T - 1 \) and for any \( n, k_{t+1}^n = f(k_t^n) + \phi(R_t) \). Obviously, \( k_t^n \to k_t \) for \( t = 0, \ldots, T - 1 \). Hence, there exists \( N \) such that for any \( n \geq N \), \( k_{T+1} < f(k_T^n) + \phi(R_T) \). The sequences \((k_0^n, k_1^n, \ldots, k_{T+1}^n, k_{T+2}, \ldots)\) and \( R^n = R \), for every \( n \), satisfy the required conditions.

Case 3:
\[
\forall t, k_{t+1} = f(k_t) + \phi(R_t).
\]
It suffices to take \( k_{t+1}^n = f(k_t^n) + \phi(R_t) \) for every \( t \), every \( n \).
The second step is to prove that the intertemporal utility function is continuous on the feasible set for the product topology. But the proof is standard (see e.g. Le Van and Dana [3]).

The third step is to apply the Maximum Theorem to conclude that \( Z \) is continuous in \( k_0 \).

**Lemma 2** There exists a constant \( A \) which depends on \( k_0, \hat{R}, \text{ and } S \), such that for any feasible sequence \((c, k, R)\), we have \( \forall t, 0 \leq c_t \leq A, 0 \leq k_t \leq A \).

Moreover, Problem \((P)\) has an optimal solution. If \( k_I = 0 \), then the solution is unique.

**Proof:** It is obvious that \( R_t \leq S, \forall t \). Now, if \( \hat{R} < +\infty \) then for any \( t \), we have \( c_t + k_{t+1} \leq f(k_t) + \phi(\hat{R}) \). And if \( \hat{R} = +\infty \) then for all \( t \), \( c_t + k_{t+1} \leq f(k_t) + \phi(S) \). Since \( f'(+\infty) < 1 \), from Le Van and Dana [3], page 17, there exists a constant \( A \) which depends on \( k_0, \hat{R} \) (if \( \hat{R} < +\infty \)) or on \( k_0, S \) such that \( \forall t, 0 \leq c_t \leq A, 0 \leq k_t \leq A \).

It is easy to check that the set of feasible sequences is compact for the product topology and the intertemporal utility function is continuous on the feasible set for the same topology. Hence, there exists a solution to Problem \((P)\). When \( k_I \) equals 0, because of the strict concavity of the technology and the utility function \( u \), the solution will be unique.

## 3 Properties of the optimal paths

**Proposition 1** For any \( t \), \( c_t^* > 0 \) and \( R_t^* < \hat{R} \). If \( \phi'(0) = +\infty \), then \( R_t^* > 0 \) for all \( t \). Obviously, \( R_t^* \to 0 \) as \( t \to +\infty \).

**Proof:** Let \( V \) denote the Value-function. Observe that \( V(k_0) > 0 \) for any \( k_0 \geq 0 \), since the sequence \( c \) defined by \( c_0 = f(k_0) + \phi(S) \) and \( c_t = 0 \) for any \( t > 0 \) is feasible. Hence \( V(k_0) \geq u(c_0) > 0 \). That implies \( c_t^* > 0, \forall t \), by the Inada condition \( u'(0) = +\infty \).

Let us prove that \( R_t^* < \hat{R} \) for all \( t \). If \( \hat{R} = +\infty \), the proof is obvious. So, assume \( \hat{R} < +\infty \). We cannot have \( R_t^* > \hat{R} \) for some \( t \), since \( u \) is strictly increasing and \( \phi \) is strictly decreasing for \( R > \hat{R} \). We cannot have \( R_t^* = \hat{R} \) for all \( t \) since \( \sum_{t=0}^{+\infty} R_t^* = S \). If there exists \( T \) with \( R_T^* = \hat{R} \), we can suppose \( R_{T+1}^* < \hat{R} \). Without loss of generality, take \( T = 0 \). So

\[
\begin{align*}
c_0^* + k_1^* &= f(k_0) + \phi(\hat{R}) \\
c_1^* + k_2^* &= f(k_1^*) + \phi(R_1^*), \text{ with } R_1^* < \hat{R}.
\end{align*}
\]
Choose $\varepsilon > 0$ small enough such that $R_1^* + \varepsilon < \bar{R}$ and $\bar{R} - \varepsilon > 0$. Let
\begin{align*}
c_0 + k_1^* &= f(k_0) + \phi(\bar{R} - \varepsilon) \\
c_1 + k_2^* &= f(k_1^*) + \phi(R_1^* + \varepsilon)
\end{align*}
\[\text{and } c_t = c_t^*, \forall t \geq 2.\]

Let $\triangle_{\varepsilon} = \sum_{t=0}^{+\infty} \beta^t u(c_t) - \sum_{t=0}^{+\infty} \beta^t u(c_t^*)$. We have
\begin{align*}
\triangle_{\varepsilon} &= u(c_0) - u(c_0^*) + \beta[u(c_1) - u(c_1^*)] \\
&\geq u'(c_0)\phi'(\bar{R} - \varepsilon)(-\varepsilon) + \beta u'(c_1)[\phi'(R_1^* + \varepsilon)(\varepsilon)] \\
&\geq \varepsilon[\beta u'(c_1)\phi'(R_1^* + \varepsilon) - u'(c_0)\phi'(\bar{R} - \varepsilon)].
\end{align*}

Let $\varepsilon \to 0$. Then $\lim_{\varepsilon \to 0} \triangle_{\varepsilon} \geq \beta u'(c_1^*)\phi'(R_1^*) > 0$. Thus $\triangle_{\varepsilon} > 0$ for $\varepsilon$ small enough. That is a contradiction to the optimality of $c^*$.

Now consider the case $\phi'(0) = +\infty$. First assume $R_1^* = 0$, $\forall t$. Then let
\begin{align*}
c_0 &= f(k_0) - k_1^* + \bar{S} > c_0^* \\
c_t &= f(k_t^*) - k_{t+1}^* = c_t^*, \text{ for } t \geq 1.
\end{align*}

Then $\sum_{t=0}^{+\infty} u(c_t) > \sum_{t=0}^{+\infty} u(c_t^*)$: a contradiction. Hence if $R_T^* = 0$ we can assume that $R_{T+1}^* > 0$. Without loss of generality, take $T = 0$. So
\begin{align*}
c_0^* &= f(k_0) - k_1^* \\
c_1^* &= f(k_1^*) - k_2^* + \phi(R_1^*), \text{ with } 0 < R_1^* < \bar{R}.
\end{align*}

Let $\varepsilon \in (0, R_1^*)$. Define
\begin{align*}
c_0 &= f(k_0) - k_1^* + \phi(\varepsilon) \\
c_1 &= f(k_1^*) - k_2^* + \phi(R_1^* - \varepsilon) \\
c_t &= c_t^*, \forall t \geq 2.
\end{align*}

Then
\begin{align*}
\triangle_{\varepsilon} &= \sum_{t=0}^{+\infty} \beta^t u(c_t) - \sum_{t=0}^{+\infty} \beta^t u(c_t^*) \\
&= u(c_0) - u(c_0^*) + \beta[u(c_1) - u(c_1^*)] \\
&\geq u'(c_0)\phi(\varepsilon) + \beta u'(c_1)[\phi(R_1^* - \varepsilon) - \phi(R_1^*)] \\
&\geq [u'(c_0)\phi'(\varepsilon) - \beta u'(c_1)\phi'(R_1^* - \varepsilon)]\varepsilon.
\end{align*}

That implies $\lim_{\varepsilon \to 0} \triangle_{\varepsilon} = +\infty$. Thus $\triangle_{\varepsilon} > 0$ for $\varepsilon$ small enough: a contradiction.  ■
Proposition 2 Let $k_0 \geq 0$. Assume $\phi'(0) < +\infty$. Then we have the following Euler conditions:

(i) $\forall t, \ f'(k_{t+1}^*) \leq \frac{u'(c_t^*)}{\beta u'(c_t^*)}$ (E1)

with equality if $k_{t+1}^* > 0$,

(ii) $\forall t, \ \beta^t u'(c_t^*) \phi'(R_t^*) = \beta^t u'(c_t^*) \phi'(R_t^*)$, (E2)

if $R_t^* > 0$, $R_t^* > 0$, and

(iii) $\forall t, \ \beta^t u'(c_t^*) \phi'(R_t^*) \leq \beta^t u'(c_t^*) \phi'(R_t^*)$, (E2')

if $R_t^* = 0$, $R_t^* > 0$.

Proof: (i) Given $t$, $k_{t+1}^*$ solves:

$$\max_y \ 0 \leq y \leq f(k_t^*) + \phi(R_t^*)$$

Then we have the following

Since $c_t^* = f(k_t^*) + \phi(R_t^*) - k_{t+1}^* > 0$, one easily gets (E1).

(ii) Since $\overline{S} > 0$, there exists $t$ with $R_t^* > 0$. Fix some $T$ such that there exists $t \leq T$ with $R_t^* > 0$. Then $(R_0^*, ..., R_T^*)$ solve

$$\max_{(R_0, ..., R_T)} \sum_{t=0}^{T} \beta^t u(f(k_t^*) + \phi(R_t) - k_{t+1}^*)$$

s.t. $0 \leq R_t \leq \overline{S} - \sum_{\tau=0}^{+\infty} R_t^*$

$0 \leq R_t, \forall t = 0, ..., T$

$k_{t+1}^* - f(k_t^*) \leq \phi(R_t), \forall t = 0, ..., T$

Since $\phi$ is concave and $u$ is strictly concave, $(R_0^*, ..., R_T^*)$ will be the unique solution. Moreover, since $c_t^* > 0$ for every $t$, the third constraints system will not be binding. There exist therefore $\lambda \geq 0$ and $\mu_t \geq 0, t = 0, ..., T$ such that $(R_0^*, ..., R_T^*)$ maximize

$$\sum_{t=0}^{T} \beta^t u(f(k_t^*) + \phi(R_t) - k_{t+1}^*) - \lambda \left[ \sum_{t=0}^{T} R_t - \overline{S} + \sum_{\tau=0}^{+\infty} R_t^* \right] + \sum_{t=0}^{T} \mu_t R_t$$

with $\mu_t R_t^* = 0, \forall t = 0, ..., T$. One easily obtains (E2) and (E2').

The following proposition shows that, even if the initial capital stock equals 0, if the marginal productivity at the origin of the production function $f$ is high enough, then, thanks to the exhaustible resource, the country will accumulate from some date on. More precisely, the marginal productivity at the origin of the initial production function $F$ must be larger than $\delta$. 

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Proposition 3 Let \( k_0 \geq 0 \). Assume \( \phi'(0) < +\infty \). If \( f'(0) > 1 \), then there exists \( T \geq 1 \) with \( k^*_t > 0 \) for any \( t \geq T \).

Proof: Assume \( k^*_t = 0 \), \( \forall t \). Then we have the FOC

\[
(uo\phi)'(R^*_t) = \beta(uo\phi)'(R^*_{t+1}), \ \forall t.
\]

Since \( uo\phi \) is strictly concave, we have \( R^*_t > R^*_{t+1}, \forall t \), and the decreasing sequence \( R^* \) converges to 0. But the FOC can be written as

\[
\frac{u'(\phi(R^*_t))}{\beta u'(\phi(R^*_{t+1}))} = \frac{\phi'(R^*_{t+1})}{\phi'(R^*_t)}.
\]

Choose \( \epsilon > 0 \) small, such that \( f'(0) > 1 + \epsilon \). For \( t \) large enough, say \( t \geq T \), we have \( \frac{\phi'(R^*_{t+1})}{\phi'(R^*_t)} < 1 + \epsilon \). Choose \( \epsilon_1 \leq \epsilon \) which satisfies \( \phi(R^*_t) - \epsilon_1 > 0 \). Let

\[
c_t + \epsilon_1 = \phi(R^*_t)
\]

\[
c_{t+1} = f(\epsilon_1) + \phi(R^*_{t+1}).
\]

Let

\[
\Delta_{\epsilon_1} = \sum_{t=0}^{+\infty} \beta^t u(c_t) - \sum_{t=0}^{+\infty} \beta^t u(c^*_t)
\]

\[
= \beta^T \left[ u(\phi(R^*_T) - \epsilon_1) - u(\phi(R^*_T)) + \beta(u(f(\epsilon_1) + \phi(R^*_T)) - u(\phi(R^*_{T+1}))) \right]
\]

\[
\geq \beta^T \left[ -u'(\phi(R^*_T) - \epsilon_1) + \beta u'(f(\epsilon_1) + \phi(R^*_T)) \frac{f(\epsilon_1)}{\epsilon_1} \right] \epsilon_1
\]

\[
\geq \beta^T\frac{\epsilon_1}{u'(f(\epsilon_1) + \phi(R^*_T))} \left[ \frac{f(\epsilon_1)}{\epsilon_1} - \frac{u'(\phi(R^*_T) - \epsilon_1)}{\beta u'(f(\epsilon_1) + \phi(R^*_T))} \right].
\]

Let

\[
Y(\epsilon_1) = \left[ \frac{f(\epsilon_1)}{\epsilon_1} - \frac{u'(\phi(R^*_T) - \epsilon_1)}{\beta u'(f(\epsilon_1) + \phi(R^*_T))} \right].
\]

Then

\[
Y(\epsilon_1) \to f'(0) - \frac{u'(\phi(R^*_T))}{\beta u'(\phi(R^*_T))} = f'(0) - \frac{\phi'(R^*_T)}{\phi'(R^*_T)} \text{ from relation (1),}
\]

\[
> 1 + \epsilon - \frac{\phi'(R^*_T)}{\phi'(R^*_T)}> 0.
\]

Hence \( \Delta_{\epsilon_1} > 0 \) when \( \epsilon_1 \) is small enough: a contradiction. So, \( k^*_t > 0 \). Suppose \( k^*_T = 0 \). We then have

\[
c^*_{T+1} = f(k^*_T + \phi(R^*_T))
\]

\[
c^*_{T+2} = \phi(R^*_T) > 0.
\]
We have the following Euler conditions:

\[-u'(c_{T+1}^*) + \beta u'(c_{T+2}^*)f'(0) \leq 0\]
\[u'(c_{T+1}^*)\phi'(R_{T+1}^*) - \beta u'(c_{T+2}^*)\phi'(R_{T+2}^*) \leq 0\]

with equality if \(R_{T+2}^* > 0\).

Thus

\[f'(0) \leq \frac{u'(c_{T+1}^*)}{\beta u'(c_{T+2}^*)} \leq \frac{\phi'(R_{T+2}^*)}{\phi'(R_{T+1}^*)} < 1 + \epsilon.\]

But \(f'(0) > 1 + \epsilon\): a contradiction. Hence, \(k_{T+2}^* > 0\). By induction, \(k_t^* > 0, \forall t > T\). ■

In the following proposition, we show that thanks to the exhaustible resource, the country will accumulate at any period, provided that the marginal productivity at the origin of the initial production function \(F\) is larger than the investment cost, \(\frac{1}{\beta} - 1 + \delta\). Notice that when the initial capital stock is equal to 0, the same economy without natural resources never takes-off (Dechert and Nishimura [1]).

**Proposition 4** Let \(k_0 \geq 0\). Assume \(f'(0) > \frac{1}{\beta}\). Then \(k_t^* > 0\) for any \(t \geq 1\).

**Proof:** Assume \(k_1^* = 0\). Then we have

\[c_0^* = f(k_0) + \phi(R_0^*)\]
\[c_1^* + k_2^* = \phi(R_1^*).\]

The following Euler conditions hold:

\[-u'(c_0^*) + \beta u'(c_1^*)f'(0) \leq 0\]
\[u'(c_0^*)\phi'(R_0^*) - \beta u'(c_1^*)\phi'(R_1^*) \leq 0.\]

This implies

\[1 < \frac{1}{\beta} < f'(0) \leq \frac{u'(c_0^*)}{\beta u'(c_1^*)} \leq \frac{\phi'(R_1^*)}{\phi'(R_0^*)}.\]

From these inequalities, we get \(u'(c_0^*) > u'(c_1^*), \phi'(R_1^*) > \phi'(R_0^*)\) or equivalently, \(c_1^* > c_0^*\) and \(R_0^* > R_1^*\). A contradiction arises:

\[\phi(R_1^*) \geq \phi(R_1^*) - k_2^* = c_1^* > c_0^* = f(k_0) + \phi(R_0^*) \geq \phi(R_0^*) > \phi(R_1^*).\]

Therefore, \(k_1^* > 0\). By induction, \(k_t^* > 0\) for all \(t \geq 1\). ■

We want to show that the natural resource will be exhausted in finite time if the marginal productivity at the origin of the production function is high enough. Before proving that, let us introduce an intermediary step.
Consider Problem \((Q)\)

\[
U(k_0) = \max \sum_{t=0}^{+\infty} \beta^t u(c_t), \quad \beta \in (0, 1)
\]

under the constraints

\[
\forall t, c_t \geq 0, k_t \geq 0,
\]
\[
c_t + k_{t+1} \leq f(k_t),
\]
\[
k_0 \geq 0 \text{ is given.}
\]

Let \(\varphi\) denote the optimal correspondence of \((Q)\), i.e., \(k_1 \in \varphi(k_0)\) iff we have \(k_1 \in [0, f(k_0)]\) and

\[
U(k_0) = u(f(k_0) - k_1) + \beta U(k_1)
\]

\[
= \max \{u(f(k_0) - y) + \beta U(y) : y \in [0, f(k_0)]\}.
\]

Next consider Problem \((Q_a)\) where \(a\) is a sequence of non-negative real numbers which satisfies \(\sum_{t=0}^{+\infty} a_t < +\infty\):

\[
W(k_0, (a_t)_{t \geq 0}) = \max \sum_{t=0}^{+\infty} \beta^t u(c_t), \quad \beta \in (0, 1)
\]

under the constraints

\[
\forall t, c_t \geq 0, k_t \geq 0,
\]
\[
c_t + k_{t+1} \leq f(k_t) + a_t,
\]
\[
k_0 \geq 0 \text{ is given.}
\]

Obviously, \(W(k_0, 0) = U(k_0)\), and \(W(k_0, (a_t)_{t \geq 0}) \geq U(k_0)\). We also have the Bellman equation: for all \(k_0\),

\[
W(k_0, (a_t)_{t \geq 0}) = \max \{u(f(k_0) - y + a_0) + \beta W(y, (a_t)_{t \geq 1}) : y \in [0, f(k_0) + a_0]\}.
\]

Let \(\psi(., (a_t)_{t \geq 0})\) denote the optimal correspondence associated with \((Q_a)\), i.e., \(k_1 \in \psi(k_0, (a_t)_{t \geq 0})\) iff \(W(k_0, (a_t)_{t \geq 0}) = u(f(k_0) - k_1 + a_0) + \beta W(k_1, (a_t)_{t \geq 1})\) and \(k_1 \in [0, f(k_0) + a_0]\). We have the following lemma.

**Lemma 3** Let \(k_0^n \to k_0\) and \(a^n \to 0\) in \(l^\infty\) when \(n\) converges to infinity. If, for any \(n\), \(k_1^n \in \psi(k_0^n, a^n)\) and \(k_1^n \to k_1\) as \(n \to +\infty\), then \(k_1 \in \varphi(k_0)\).
Proof: We first prove that \( W(k_0^n, a^n) \rightarrow U(k_0) \) as \( n \rightarrow +\infty \). We have:

\[
\forall n, W(k_0^n, (a_t^n)_{t \geq 0}) \geq U(k_0^n),
\]

hence

\[
\liminf_{n \rightarrow +\infty} W(k_0^n, (a_t^n)_{t \geq 0}) \geq \lim_{n \rightarrow +\infty} U(k_0^n) = U(k_0).
\]

We now prove that \( \limsup_{n \rightarrow +\infty} W(k_0^n, (a_t^n)_{t \geq 0}) \leq U(k_0) \). Let \( \alpha > 0 \). There exists \( N \) such that, for any \( n \geq N \), we have \( f(k_0^n) + a_0^n \leq f(k_0) + \alpha \) and \( k_0^n \leq \alpha \). Let \( \tilde{k}^\alpha \) be the largest value of \( k \) which satisfies \( f(\tilde{k}^\alpha) + \alpha = \tilde{k}^\alpha \). Using the same argument as in Le Van and Dana [3], page 17, one can show that, for any feasible sequences from \( k_0^n, c^n, k^n \) of \( (Q_{an}) \), for any \( n \geq N \), any \( t \), we have \( c_t^n \leq \max\{\tilde{k}^\alpha, k_0 + \alpha\} \), \( k_t^n \leq \max\{\tilde{k}^\alpha, k_0 + \alpha\} \). Let \( c^{*n}, k^{*n} \) be the optimal sequences from \( k_0^n \) of Problem \( (Q_{an}) \). Let \( \epsilon > 0 \). There exists \( T \) such that

\[
\forall n, W(k_0^n, (a_t^n)_{t \geq 0}) \leq \sum_{t=0}^{t=T} \beta^t u(c_t^{*n}) + \epsilon.
\]

For \( t = 0, ... T \), we can suppose that \( c_t^{*n} \rightarrow \bar{c}_t \) and \( k_t^{*n} \rightarrow \bar{k}_t \). Since for \( t = 0, ... T \), we have \( c_t^{*n} + k_t^{*n} = f(k_t^{*n}) + a_t^n \), we obtain \( \bar{c}_t + \bar{k}_t = f(\bar{k}_t) \) for \( t = 0, ..., T \). Define \( \bar{c} = (\bar{c}_0, ..., \bar{c}_T, 0, 0, ..., 0, ...) \). We get

\[
\limsup_{n \rightarrow +\infty} W(k_0^n, (a_t^n)_{t \geq 0}) \leq \sum_{t=0}^{t=T} \beta^t u(\bar{c}_t) + \epsilon = \sum_{t=0}^{t=+\infty} \beta^t u(\bar{c}_t) + \epsilon \leq U(k_0) + \epsilon.
\]

This inequality holds for any \( \epsilon > 0 \). We have proved \( \limsup_{n \rightarrow +\infty} W(k_0^n, a^n) \leq U(k_0) \).

Now, let \( k_1^n \in \psi(k_0^n, a^n) \) and suppose \( k_1^n \rightarrow k_1 \) as \( n \rightarrow +\infty \). We have

\[
W(k_0^n, (a^n)_{t \geq 0}) = u(f(k_0^n) - k_1^n + a_0^n) + \beta W(k_1^n, (a^n)_{t \geq 1}),
\]

and \( k_1^n \in [0, f(k_0^n) + a_0^n] \). Taking the limits we get

\[
U(k_0) = u(f(k_0) - k_1) + \beta U(k_1),
\]

with \( k_1 \in [0, f(k_0)] \). That proves \( k_1 \in \varphi(k_0) \). \( \blacksquare \)

The next proposition states that if the marginal revenue of the resource at the origin is finite, then the stock of resource will be exhausted in finite time.

Proposition 5 Assume \( \phi'(0) < +\infty \) and \( f'(0) > 1 \). Then there exists \( T_\infty \) such that, for all \( t \geq T_\infty \), we have \( R_t^s = 0 \).

Proof: From Proposition 3, there exists \( T \) such that \( \forall t \geq T, k^*_t > 0 \).
Step1 We will show that there exists $T'$ such that $R_{T'v}^* = 0$. If not, for any $t \geq T$ we have the Euler conditions:

$$\beta u'(c_{t+1}^*) f'(k_{t+1}^*) = u'(c_t^*), \quad \beta u'(c_{t+1}^*) \phi'(R_{t+1}^*) = u'(c_t^*) \phi'(R_t^*).$$

Hence

$$f'(k_{t+1}^*) = \frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = \frac{\phi'(R_{t+1}^*)}{\phi'(R_t^*)}.$$ 

Since $\frac{\phi'(R_{t+1}^*)}{\phi'(R_t^*)} \to 1$, we have $f'(k_{t+1}^*) \to 1$, as $t \to +\infty$. Under our assumptions there exists a unique $\hat{k}$ which satisfies $f'(\hat{k}) = 1$. Thus $k_{t+1}^* \to \hat{k}$. In this case, for $t$ large enough, $u'(c_{t+1}^*) > u'(c_t^*) \iff c_t^* > c_{t+1}^*$. The sequence $c^*$ converges to $\bar{c}$. If $\bar{c} > 0$, we have $f'(\hat{k}) = \frac{1}{\beta}$: a contradiction. So, $\bar{c} = 0$. Since

$$\forall t, c_{t+1}^* + k_{t+2}^* = f(k_{t+1}^*) + \phi(R_{t+1}^*),$$

we have $\hat{k} = f(\hat{k})$ with $f'(\hat{k}) = 1$, and that is impossible. Hence, there must be $T'$ with $R_{T'v}^* = 0.$

Step2 Assume there exists three sequences $(c_{t^*}^*), (k_{t^*}^*), (R_{t^*}^*)$ which satisfy

$$\forall \nu, c_{t^*}^* + k_{t^*}^* = f(k_{t^*}^*), \quad c_{t^*}^* + k_{t^*+1}^* = f(k_{t^*}^*) + \phi(R_{t^*}^*), \text{ with } R_{t^*}^* > 0.$$ 

Hence

$$\forall \nu, f'(k_{t^*}^*) = \frac{u'(c_{t^*}^* - 1)}{\beta u'(c_{t^*}^*)} \leq \frac{\phi'(R_{t^*}^*)}{\phi'(0)} < 1.$$ 

Therefore, $\forall \nu, k_{t^*}^* > \hat{k}$. Observe that there exists $\lambda > 0$ such that

$$\forall \nu, \beta^\nu u'(c_{t^*}^*) \phi'(R_{t^*}^*) = \lambda.$$ 

This implies $c_{t^*}^* \to 0$ as $\nu \to +\infty$. From Lemma 2, $k_{t^*}^* \leq A, \forall \nu$. One can suppose $k_{t^*}^* \to \hat{k} \geq \hat{k} > 0$ and $k_{t^*+1}^* \to \hat{k} = f(\hat{k})$. From Lemma 3, $\bar{k} \in \varphi(\bar{k})$. This implies $c_{t^*}^* \to \bar{c} = f(\hat{k}) - \hat{k} = 0$. But, since $\hat{k} > 0$, we must have $\bar{c} > 0$ (see Le Van and Dana [3]). This contradiction implies the existence of $T_\infty$ such that for all $t \geq T_\infty$, we have $R_t^* = 0.$

Remark 1 When the function $f$ is concave, the proof will be very short. Indeed, from Le Van and Saglam [4], the sequence $(\beta^t u'(c_t^*))$ satisfies $\sum_{t=0}^{+\infty} \beta^t u'(c_t^*) < +\infty$. Since there exists $\lambda > 0$ such that $\beta^\nu u'(c_{t^*}^*) \phi'(R_{t^*}^*) = \lambda$, we have $\beta^\nu u'(c_{t^*}^*) \to \frac{\lambda}{\phi'(0)} > 0$. This excludes $\sum_{t=0}^{+\infty} \beta^t u'(c_t^*)$ to be bounded from above.
In the following corollary, we prove that, even the initial capital equals 0, thanks to the natural resource, the country may take-off if the marginal productivity at the origin of the initial production function is larger than the investment cost. But, when this productivity is low, the natural resource cannot prevent the country to ”collapse” in the long term. More precisely, we have

**Corollary 1** Let $k_0 \geq 0$. Assume $\phi'(0) < +\infty$.

(a) If $f'(0) > \frac{1}{\beta}$, then $k_t^* \to k^*$ where $k^*$ is defined by $f'(k^*) = \frac{1}{\beta}$.

(b) If $f$ is concave and $1 < f'(0) \leq \frac{1}{\beta}$, then $k_t^* \to 0$.

**Proof**: From the previous proposition, we know that $R_t^* = 0$ for $t \geq T_\infty$. The optimal sequence $(k_t^*)_{t \geq T_\infty}$ solves problem $(Q)$ with initial data $k_0^* > 0$. Assertion (a) follows from Dechert and Nishimura [1], while assertion (b) follows, e.g., from Le Van and Dana [3].

We now show that the country may never accumulate in physical capital if the marginal productivity is very low, and the initial capital stock is small.

**Proposition 6** Assume $\phi'(0) < +\infty$ and $f'(k_I) < 1$.

(a) Let $k_0 \geq 0$. Then there exists $T$ with $k_t^* = 0$, $\forall t \geq T$.

(b) There exists $\varepsilon > 0$ such that, if $k_0 \leq \varepsilon$, then $k_t^* = 0$, $\forall t$.

**Proof**: (a) There must be $t_0$ with $R_{t_0}^* > 0$. We claim that $R_{t_0 + 1}^* > 0$. By induction, $R_t^* > 0$, $\forall t > t_0$. Thus, for $t \geq t_0$, we have the FOC:

$$f'(k_{t+1}^*) = \frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} \geq \frac{\phi'(0)}{\phi(R_t^*)} > 1,$$

which is impossible. Hence $R_{t_0 + 1}^* > 0$. By induction, $R_t^* > 0$, $\forall t > t_0$. Thus, for $t \geq t_0$, we have the FOC:

$$f'(k_{t+1}^*) = \frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = \frac{\phi'(R_{t+1}^*)}{\phi'(R_t^*)}, \text{ if } k_{t+1}^* > 0.$$

If there exists an infinite sequence $(k_{t_0 + 1}^*)_{\nu}$ with $k_{t_0 + 1}^* > 0$, $\forall \nu$, then from the previous FOC we have $\lim_{\nu \to +\infty} f'(k_{t_0 + 1}^*) = 1$: a contradiction since $\forall \nu$, $f'(k_{t_0 + 1}^*) \leq f'(k_I) < 1$. Therefore, $k_t^* = 0$ for any $t$ large enough.

(b) Consider Problem $(R)$:

$$S(k_0, S) = \max \sum_{t=0}^{+\infty} \beta^t u(c_t)$$

under the constraints

$$0 \leq c_0 \leq f(k_0) + \phi(R_0)$$
$$\forall t \geq 1, \ 0 \leq c_t \leq \phi(R_t), \ 0 \leq R_t$$
$$\sum_{t=0}^{+\infty} R_t \leq S.$$
We first prove the claim for \( k_0 = 0 \). Let \((R_t^*, c_t^*)\) be the solution. We have \( \sum_{t=0}^{+\infty} R_t^* = \overline{S} \) and \( c_t^* = \phi(R_t^*), \forall t \). There exists \( \lambda \) such that \( \forall t, \beta' u'(\phi(R_t^*)) \phi'(R_t^*) = \lambda \). Let \((k_t, R_t)\) be a solution to the initial problem. We have \( \sum_{t=0}^{+\infty} R_t = \overline{S} \).

Consider
\[
\Delta = \sum_{t=0}^{+\infty} \beta^t \left[ u(\phi(R_0^*)) - u(\phi(R_t) + f(k_t) - k_{t+1}) \right].
\]

We have
\[
\Delta \geq \sum_{t=0}^{+\infty} \beta^t u' \phi(\phi(R_t^*)) \phi'(R_t^*) (R_t^* - R_t) + \sum_{t=0}^{+\infty} \beta^t u'(\phi(R_t^*))(k_{t+1} - f(k_t))
\]
\[
\geq \lambda \sum_{t=0}^{+\infty} (R_t^* - \sum_{t=0}^{+\infty} R_t) + \sum_{t=0}^{+\infty} \beta^t u'(\phi(R_t^*))(k_{t+1} - f(k_t))
\]
\[
\geq \sum_{t=0}^{+\infty} \beta^t u'(\phi(R_t^*))(k_{t+1} - f(k_t)).
\]

Recall that \( \forall t, R_{t+1}^* < R_t^* \). Since \( u'(\phi(R_t^*)) \phi'(R_t^*) = \beta u'(\phi(R_{t+1}^*)) \phi'(R_{t+1}^*) \) we have \( u'(\phi(R_t^*)) > \beta u'(\phi(R_{t+1}^*)) \), \forall t \). From part (a), there exists \( T \) such that \( k_t = 0, \forall t \geq T + 1 \). Therefore
\[
\Delta \geq \sum_{t=1}^{T} \beta^t u'(\phi(R_t^*))(k_t - f(k_t)).
\]

Since \( f'(k) < 1 \) and \( f(0) = 0 \), we have \( f(k) < k \). Thus, \( \Delta > 0 \). This is a contradiction.

Now, let \( k_0 > 0 \). Then we have \( \beta^t u'(\phi(R_t^*)) \phi'(R_t^*) = \lambda \) for some \( \lambda > 0 \) and \( t \geq 1 \), and \( u'(\phi(R_0^*)) \phi'(R_0^*) \leq \beta u'(\phi(R_1^*)) \phi'(R_1^*) \) with equality if \( R_0^* > 0 \). The same computations as above give
\[
\Delta \geq \sum_{t=2}^{T} \beta^t u'(\phi(R_t^*))(k_t - f(k_t)) + u'(f(k_0) + \phi(R_0^*)) k_1 - \beta u'(\phi(R_1^*)) f(k_1).
\]

When \( k_0 = 0 \) we have \( u'(\phi(R_0^*)) > \beta u'(\phi(R_1^*)) \). But \( R_0^* \) and \( R_1^* \) are continuous in \( k_0 \). Hence, when \( k_0 \leq \varepsilon \) with \( \varepsilon \) small enough, we still have \( u'(f(k_0) + \phi(R_0^*)) > \beta u'(\phi(R_1^*)) \). A contradiction arises as before.

We expect that, when the productivity of the technology is low, the country will never accumulate if the stock of non-renewable resource is very large. The result is true if we assume \( \hat{R} = +\infty \). To simplify the proof we will use some explicit forms of the functions \( u \) and \( \phi \).

**Proposition 7** Assume \( \phi'(0) < +\infty \) and \( f'(k_t) < 1 \). Assume also \( \hat{R} = +\infty \), \( \phi(R) = aR, a > 0 \) and \( u(c) = c^\theta / \theta \) with \( 0 < \theta < 1 \). Let \( k_0 > 0 \) be given. Then we have \( k_t^* = 0, \forall t \), when \( \overline{S} \) is large enough.
Proof: Let \((R^*_t)_{t=0,...,\infty}\) be the solution. We have

\[
\forall t \geq 1, \quad \mu_t + a \left( f(k_0) + aR^*_0 \right)^{\theta-1} = \beta^t a (aR^*_t)^{\theta-1}
\]

where \(\mu_0 \geq 0\), and \(\mu_0 R^*_0 = 0\).

We obtain

\[
\forall t \geq 1, \quad R^*_t = \frac{\beta^{(1-\theta)t}}{a} \left[ \frac{\mu_t + a \left( f(k_0) + aR^*_0 \right)^{\theta-1}}{a} \right]^{\frac{1}{\theta-1}}
\]

\[
\sum_{t=0}^{\infty} R^*_t = R^*_0 + \frac{\beta^{1-\theta}}{a(1-\beta^{1-\theta})} \left[ \frac{\mu_0 + a \left( f(k_0) + aR^*_0 \right)^{\theta-1}}{a} \right]^{\frac{1}{\theta-1}},
\]

and \(\mu_0 R^*_0 = 0\).

\(R^*_0\), and when \(R^*_0 = 0\), \(\mu_0\) will be determined by the constraint \(\mathcal{S} = \sum_{t=0}^{\infty} R^*_t\). It is obvious that \(\mu_0\) is a decreasing function of \(\mathcal{S}\), while \(R^*_0\) is an increasing function of \(\mathcal{S}\). Thus, when \(\mathcal{S}\) is high enough, we have \(\mu_0 = 0\) and \(R^*_0 > 0\). Hence, from relation (3), we get \((f(k_0) + aR^*_0)^{\theta-1} = \beta (aR^*_1)^{\theta-1}\), i.e. \(u'(f(k_0) + aR^*_0) = \beta u'(aR^*_1)\). Using the proof of Proposition 6, we conclude that the optimal path \((k^*_t)\) equals 0. \(\blacksquare\)

We know, from Dechert and Nishimura [1], if \(f'(0) < \frac{1}{\beta} < \max\{ \frac{f'(k)}{k} : k > 0 \}\), then there exists \(k^c > k_I\), such that if \(k_0 < k^c\) then any solution \(k^*\) to Problem \((Q)\) converges to 0, and if \(k_0 > k^c\), then it converges to a high steady state \(k^*\) fulfilling \(f'(k^*) = \frac{1}{\beta}\). In other words, we have a poverty trap. We will show, under some more assumptions, if \(\mathcal{S}\) is high enough, that the poverty trap can be passed over in our model. More precisely,

**Proposition 8** Assume \(\mathcal{R} = +\infty\), \(\phi(R) = aR\), \(a > 0\) and \(1 < f'(0) \leq \frac{1}{\beta} \leq \max\{ \frac{f'(k)}{k} : k > 0 \}\). Let \(k_0 \geq 0\). The optimal sequence, \(k^*_t \rightarrow k^\ast\) as \(t \rightarrow +\infty\), if \(\mathcal{S}\) is high enough.

**Proof:** Let \((\tilde{S}^\nu)\) be a sequence which converges to \(+\infty\). We consider two cases.

**Case 1:** For any \(\nu\), the optimal sequence \((R^*_t^\nu)\) verifies, \(R^*_0^\nu = \tilde{S}^\nu\). Let \(k^\nu_0\) satisfy \(f(k^\nu_0) = f(k_0) + a\tilde{S}^\nu\). For \(\nu\) large enough, \(k^\nu_0 > k^c\). An optimal sequence \((k^*_t^\nu)\) is also an optimal sequence for the convex-concave optimal growth-model with initial endowment equal to \(k^\nu_0\). Since this one is larger than the critical value \(k^c\), the optimal path \((k^*_t^\nu)\) will converge to the high steady state \(k^\ast\). And the proof is over.
Case 2. From Proposition 6, there exists $T_\infty$ such that:

\[
\begin{align*}
 c_{T_\infty-2}^* + k_{T_\infty-1}^* &= f(k_{T_\infty-2}^*) + aR_{T_\infty-1}^* \\
 c_{T_\infty-1} + k_{T_\infty}^* &= f(k_{T_\infty-1}) + aR_{T_\infty-1}^* \\
 c_{T_\infty} + k_{T_\infty+1}^* &= f(k_{T_\infty}) \\
 c_t^* + k_{t+1}^* &= f(k_t^*), \forall t \geq T_\infty + 1.
\end{align*}
\]

We have

\[
f'(k_{T_\infty-1}^*) \leq \frac{u'(c_{T_\infty-2}^*)}{\beta u'(c_{T_\infty-1}^*)} \leq \frac{\phi'(R_{T_\infty-1}^*)}{\phi'(R_{T_\infty-2}^*)} = 1.
\]

We cannot have $k_{T_\infty-1}^* = 0$, since $f'(0) > 1$. Hence $k_{T_\infty-1}^* \geq \tilde{k}$ which is the unique point with $f'(\tilde{k}) = 1$. Moreover, $\tilde{k} > k^c$. Let $k_0$ satisfy $f(k_0) = f(k_{T_\infty-1}^*) + aR_{T_\infty-1}^*$. Then, $k_0 > k^c$. The sequence $(k_t^*)$, $t = T_\infty - 1, ..., +\infty$ is also an optimal solution to the convex-concave optimal growth-model with initial endowment equal to $k_0$. Since $k_0 > k^c$, the path $(k_t^*), t = T_\infty - 1, ..., +\infty$ converges to the high steady state.

4 Summary of the main results

(a) **High productivity of the technology**

Assume $\phi'(0) < +\infty$ and $f'(0) > \frac{1}{\beta}$. Then the optimal capital path $(k_t^*)_t$ converges to the steady state $k^*$ with $f'(k^*) = \frac{1}{\beta}$. The stock of non-renewable resource will be exhausted in finite time.

(b) **Intermediate productivity of the technology**

Assume $\phi'(0) < +\infty$ and $1 < f'(0) \leq \frac{1}{\beta} \leq \max\{\frac{f(k)}{k^2} : k > 0\}$. The stock of non-renewable resource will be exhausted in finite time.

(b.1) If the technology $f$ is concave then the optimal capital path converges to zero.

(b.2) If the technology is convex-concave, $\tilde{R} = +\infty$, $\phi(R) = aR$, $a > 0$ (the price is inelastic with respect to the demand), then the optimal capital stock converges to the high steady state $k^*$ if the stock of non-renewable resource is large enough.

(c) **Low productivity of the technology**

Assume $\phi'(0) < +\infty$ and $f'(k) < 1, \forall k$.

(c.1) There exists $T$ such that $k_t^* = 0, \forall t > T$.

(c.2) The optimal capital path vanishes for any $k_0$ small enough.

(c.3) Assume $\tilde{R} = +\infty$, $\phi(R) = aR$, $a > 0$ (the price is inelastic with respect to the demand), $u(c) = \frac{c^\theta}{\theta}$, $\theta \in (0, 1)$. Given $k_0$, the optimal capital path vanishes when $S$ is large.
5 Competitive Equilibrium

In our intertemporal economy, there is a single firm which produces, with a technology represented by the function \( f \), an aggregate good which can be consumed or used as physical capital. This firm also imports this aggregate good. The importations are covered by the exportation of the natural resource that the firm extracts.

Formally, if \((p_t)\) is the sequence of prices of the aggregate good, and \(r\) is the price of initial capital stock \(k_0\), then firm solves the problem

\[
\Pi = \max_{(k_t),(R_t)} \left[ \sum_{t=0}^{+\infty} p_t (f(k_t) - k_{t+1}) - r k_0 + \sum_{t=0}^{+\infty} p_t \phi(R_t) \right]
\]

under the constraints:

\[
\forall t, 0 \leq k_{t+1} \leq f(k_t) + \phi(R_t), \ 0 \leq R_t,
\]

\[
\sum_{t=0}^{+\infty} R_t \leq S, \text{ and } k_0 \text{ is given.}
\]

There is one representative consumer who owns the firm and the initial capital stock \(k_0\). Her problem is

\[
\max_{(c_t)} \sum_{t=0}^{+\infty} \beta^t u(c_t)
\]

under the constraints

\[
\sum_{t=0}^{+\infty} p_t c_t \leq \Pi + r k_0,
\]

and \(c_t \geq 0, \forall t\).

Let us recall that \(l^\infty = \{x : \sup_t |x_t| < +\infty\}\) and \(l^1 = \{p : \sum_{t=0}^{+\infty} |p_t| < +\infty\}\).

**Definition 1** The list \(\{c^*, k^*, R^*, p^*, r^*\}\) is a competitive equilibrium of our economy if:

1. \(c^* \in l^\infty, k^* \in l^\infty, R^* \in l^1, p^* \in l^1, r^* \geq 0, \{p^*, r^*\} \neq \{0, 0\}\).
2. Given \(\{p^*, r^*\}, \{k^*, R^*\}\) solve the problem of the firm, i.e.

\[
\Pi^* = \max_{(k_t),(R_t)} \left[ \sum_{t=0}^{+\infty} p_t^* (f(k_t) - k_{t+1}) - r^* k_0 + \sum_{t=0}^{+\infty} p_t^* \phi(R_t) \right]
\]

\[
= \left[ \sum_{t=0}^{+\infty} p_t^* (f(k_t^*) - k_{t+1}^*) - r^* k_0 + \sum_{t=0}^{+\infty} p_t^* \phi(R_t^*) \right].
\]

3. Given \(\{p^*, r^*\}, c^*\) solve the consumer’s problem:

\[
\sum_{t=0}^{+\infty} \beta^t u(c_t^*) = \max_{(c_t)} \sum_{t=0}^{+\infty} \beta^t u(c_t)
\]
Hence, observe that there exists $\lambda > 0$ under the constraints
\[
\sum_{t=0}^{+\infty} p_t^* c_t \leq \Pi^* + r^* k_0.
\]

(4) Market Clearing:
\[
\forall t, c_t^* + k_{t+1}^* = f(k_t^*) + \phi(R_t^*),
\]

**Proposition 9** Assume $H1, H2, H3$, $k_t = 0$, $f'(0) > 1$, $\phi'(0) < +\infty$ and $k_0 > 0$. Let $\{e^*, k^*, R^*\}$ be the solution to our problem ($P$). Define:
\[
\forall t, p_t^* = \beta^t u'(e_t^*), \text{ and } r^* = p_0^* f'(k_0).
\]

Then the list $\{e^*, k^*, R^*, p^*, r^*\}$ is a competitive equilibrium. Moreover, the equilibrium profit of the firm is positive.

**Proof:** Obviously, $\{e^*, k^*, R^*\} \in l^\infty \times l^\infty \times l^\infty$. We now prove that $\{(\beta^t u'(e_t^*))_t\} \in l^1$.

(a) If $f'(0) > \frac{1}{\beta}$, then, from Corollary 1, statement (a), $c_t^* \to e^* = f(k^*) - k^* > 0$. Hence $\sum_{t=0}^{+\infty} \beta^t u'(e_t^*) < +\infty$.

(b) If $1 < f'(0) \leq \frac{1}{\beta}$, then from Corollary 1, statement (b), $k_t^* \to 0$. Let $1 < \gamma < f'(0)$. Since for any $t$, $\frac{\beta^t u'(e_t^*)}{\beta^{t+1} u'(e_{t+1}^*)} = f'(k_{t+1}^*)$, we have $\frac{p_t^*}{p_{t+1}^*} = \frac{\beta^t u'(e_t^*)}{\beta^{t+1} u'(e_{t+1}^*)} > \gamma$ for $t$ large enough. There exists $T$ such that, for all $\tau \geq 1$, $p_{T+\tau}^* < \left(\frac{1}{\gamma}\right)^\tau p_T^*$. Hence, $\sum_{t=0}^{+\infty} p_t^* < +\infty$.

We now prove that, given $\{p^*, r^*\}, \{k^*, R^*\}$ solve the problem of the firm. Firsts, observe that there exists $\lambda > 0$ and a non-negative sequence $\mu$ such that
\[
\forall t, p_t^* \phi'(R_t^*) = \beta^t u'(e_t^*) \phi'(R_t^*) = \lambda - \mu_t, \text{ and } \mu_t R_t^* = 0.
\]

Now, let
\[
\Delta_T = \left[ \sum_{t=0}^{T} p_t^* (f(k_t^*) - k_{t+1}^*) - r^* k_0 + \sum_{t=0}^{T} p_t^* \phi(R_t^*) \right] - \left[ \sum_{t=0}^{T} p_t^* (f(k_t) - k_{t+1}) - r^* k_0 + \sum_{t=0}^{T} p_t^* \phi(R_t) \right] \geq \sum_{t=0}^{T} p_t^* f'(k_t^*)(k_t^* - k_t) - \sum_{t=0}^{T} p_t^* (k_{t+1}^* - k_t^*) + \sum_{t=0}^{T} p_t^* \phi'(R_t^*) (R_t^* - R_t) \geq -p_T^* (k_{T+1}^* - k_{T+1}) + \lambda \sum_{t=0}^{T} (R_t^* - R_t) - \sum_{t=0}^{T} \mu_t R_t^* + \sum_{t=0}^{T} \mu_t R_t \geq -p_T^* k_{T+1}^* + \lambda \sum_{t=0}^{T} (R_t^* - R_t).
Since $\sum_{t=0}^{\infty} R_t^* = S \geq \sum_{t=0}^{\infty} R_t$, we have $\lim_T \triangle_T \geq -p_{T+1}^* k_{T+1}^*$. Either $k_{T+1}^*$ converges to 0 or to $k^*$. Since $\sum_{t=0}^{\infty} p_t^* < +\infty$, we have $p_t^* \to 0$. Therefore, $\lim_T \triangle_T \geq 0$.

We now prove that, given $\{p^*, r^*\}, c^*$ solves the consumer’s problem. Indeed, let $(c_t)_t$ satisfy

$$\sum_{t=0}^{\infty} p_t^* c_t \leq \Pi^* + r^* k_0.$$ 

We have

$$\sum_{t=0}^{\infty} \beta^t u(c_t^*) - \sum_{t=0}^{\infty} \beta^t u(c_t) \geq \sum_{t=0}^{\infty} \beta^t u'(c_t^*)(c_t^* - c_t) \geq \sum_{t=0}^{\infty} p_t^* (c_t^* - c_t) \geq 0$$

since $\sum_{t=0}^{\infty} p_t^* c_t^* = \Pi^* + r^* k_0$.

Finally, market clearing condition is obviously satisfied.

To prove that the profit, at equilibrium, of the firm is positive, observe that

$$\Pi^* \geq \left[ \sum_{t=0}^{\infty} p_t^*(f(k_t) - k_{t+1}) - r^* k_0 \right]$$

for any feasible sequence $(k_0, k_1, ..., k_t, ...)$. The sequence $(k_0, 0, ..., 0, ..)$ is feasible. Therefore, $\Pi^* \geq p_0^* f(k_0) - r^* k_0 > (p_0^* f'(k_0) - r^*) k_0 = 0.$ 

**References**


