Asymmetric Spatial Competition

RUBY TOH and SOUGATA PODDAR
Department of Economics, National University of Singapore
E-mail: ecssp@nus.edu.sg
July 2006

Abstract

This paper considers price and location decisions of competing duopolists through an approach that integrates the traditional inside location and outside location model. One firm locates inside a linear city along with consumers while the other locates outside it. We analyze a location-price simultaneous game as well as a location-then-price sequential game and characterize the equilibria in pure strategies. The transport cost are assumed to be linear-quadratic and borne by the consumers. We find the results are contrasting to the traditional inside and outside location models and the stability of the proposed model is intermediate between the two.

Keywords: Inside-outside location model, spatial competition, product differentiation, transportation costs, cross-border shopping

JEL Classification: C72, D43, L13
1. INTRODUCTION

Competition in space arises because market activities occur at dispersed points in space. The study of spatial competition has been well established since Hotelling (1929)’s pioneering work, with notable contributions by Prescott and Visscher (1977), d’Aspremont et al. (1979), Gabszewicz and Thisse (1986; 1992), de Palma et al. (1985), and Anderson (1988), *inter alia*. From the product differentiation perspective a location model can be used as a framework to depict horizontal product differentiation as well as vertical product differentiation by locating the firms in appropriate places. For example, in a linear city model, where consumers are uniformly distributed along the city, when firms are located within the city limits (e.g. within a unit interval) it is a model of horizontal product differentiation. In spatial economics, this framework is also known as inside location model. On the other hand, if firms are consecutively located outside the city limit on either side, it is a model of vertical product differentiation. This framework is also known as outside location model. The literature on spatial competition and product differentiation has widely used these two established frameworks in different contexts. However, a location model (in one dimension) where both the horizontal as well as vertical product differentiation characteristics can co-exist did not get much attention (except Tabuchi and Thisse (1995) and Lambertini (1997), see latter for a discussion on this). Thus, here we introduce a model that has the feature of both horizontal as well as vertical product differentiation by modifying Hotelling’s linear city paradigm. We call this an inside-outside location model (or IO model). The proposed inside-outside location model integrates the pure inside location model and the pure outside location model in a specific manner.

We have two firms, firm 1 and firm 2 located in separate linear markets of length $[0,1]$ and $[1,\infty[$ respectively. The market boundary is located at the point 1. Consumers are located only within one of the markets which constitute the linear city in this model. Here we assume consumers are located within $[0,1]$ which constitutes the city. Consumers may travel to either of the two firms to purchase the product by incurring transportation cost. We call firm 1 the inside firm (as it locates within city) and firm 2 the outside firm. Firm entry into rival market, however, is closed. It will be shown that the only viable option for the outside firm is to locate in the vicinity of the market boundary. Intuitively, proximity to the market boundary is crucial for the
outside firm to remain in competition with the inside firm by reducing the transportation costs incurred by the consumers, *ceteris paribus*.

The inside-outside location model constructed here in this manner is reflective of many real world situations in which physical entry by firms into rival markets is either too costly or legally prohibitive, but product entry is not. The outside firm either sells the product to consumers by transporting the good to them and charging them the delivered price, or synonymously, consumers travel across the market boundary to purchase the good. In both instances, consumers pay the mill price plus transportation cost. The first situation is reflective of trading nations or cities in which firms produce goods within their own precincts and ship them to neighbouring markets to be sold, while the second is reflective of cross-border shoppers who travel out of their domestic market to shop. Our suggested model here is of the second type. For example, this model is applicable between two cities (or neighborhoods) in the study of cross-border shopping, a common phenomenon in the border regions of US and Canada, US and Mexico, several European countries, and Singapore and Malaysia in Southeast Asia (e.g., see Bode *et al.* 1994; Brodowsky and Anderson 2003; Timothy and Butler 1995; Toh 1999). This framework may also be adapted to the context of workers or tourists who travel to a neighbouring country or city to work or tour and return back after doing some shopping. In the context of a single city, our proposed framework is applicable in the study of consumer shopper behaviour and retail outlet location, such as the competition between hyper marts (usually located just outside the city limits), and small and medium retail stores (often located within the city where the majority of the population lives). This framework is, therefore, able to capture certain types of markets and consumer shopping behaviour.

As we mentioned before in this model when the markets are segregated geographically, we observe both horizontal and vertical product differentiation characteristics coexist. For example, at one extreme, when the inside firm locates at the market boundary at point 1 (i.e., closest to the outside firm), the model reduces to one that mainly exhibits strong vertical product differentiation characteristics. At the other extreme, when the inside firm locates at 0 (i.e., furthest from the outside firm), strong horizontal product differentiation characteristics predominate. At locations away from the endpoints of the inside firm, the model naturally displays both horizontal and vertical differentiation attributes.
From the literature of spatial competition, it is important to mention here that Tabuchi and Thisse (1995) and Lambertini (1997) also considered a modified Hotelling duopoly framework where the firms are permitted to locate inside or beyond the city boundaries. However, we would like to point out that their modelling and main purposes of their analyses are quite different from the present problem. In Tabuchi and Thisse (1995) the main purpose is to study the impact of consumer concentration around the market centre on the equilibrium locations of location-price games. In Lambertini (1997) the main purpose is to endogenize the timing of moves with respect to the choice of location and price of the two competing firms. Another difference is, both Tabuchi and Thisse, and Lambertini in their analysis considered only quadratic transport cost for the consumers. In contrast, we consider here a linear-quadratic transport cost for the consumers which naturally enable us to address the case of quadratic and linear transport costs as special cases of the main analysis.

From the literature of marketing we can also draw a connection to this model in the following sense. Our model is essentially equivalent to a model used to capture brand loyalty. For certain location, the inside firm has brand loyalty as more consumers would prefer to buy from it compared to the outside firm at equal prices. Thus, this framework can also be used to study questions relating to brand loyalty.

From a different point of view, it is also worth noting that the proposed IO model is directly applicable to adjoining market areas segmented economically and (or) geographically at the border. It also highlights, as we will see in our analysis, the distinction between an economic boundary and geographical boundary between two regions, which in most of the cases do not necessarily coincide.

In this hybrid model, we find that price and location competition do not necessarily lead to the same results as in the pure inside or outside location model. We study all the contrasting findings and pure-strategy equilibria in prices and locations under linear, quadratic and linear-quadratic transportation costs. We also find that the proposed inside-outside location model possesses stability that is intermediate between the two pure location models.

The plan of the paper is as follows. In section 2, we present the inside-outside location model. In section 3, we solve for the equilibrium prices under parametric locations of firms. In section 4, we consider the simultaneous price-location game between the two firms. In section 5, we consider the location-then-price sequential game. Section 6 concludes.
2. THE INSIDE-OUTSIDE LOCATION MODEL

We examine the duopolistic competition between firms selling a homogeneous product in two adjoining markets with entry-barrier to the rival’s market. Two contiguous straight lines represent two markets $i \in \{1, 2\}$ that sell the product with no storage, distribution or production costs. Market 1 is denoted by the bounded unit interval $[0, 1]$ along which firm 1 (the inside firm) and all consumers are located. Market 2 is denoted by the unbounded interval $[1, +\infty]$ along which firm 2 (the outside firm) locates. The two markets meet at the market boundary situated at point 1 and together constitute a continuous straight line of infinite length (although an upper bound in location is necessary for firm 2 to remain viable, as will be shown in Section 3). Consumers are uniformly distributed along $[0, 1]$ with density one. Firm 1 is located at distance $x_1$ from the left endpoint of the line, i.e., $x_1 \in [0, 1]$, while firm 2 is located at distance $x_2$ outside the unit interval with $x_2 \in [1, +\infty]$. In the next section we are going to assume that the two firms have fixed locations and compete only in prices. This assumption of fixed location will be relaxed in Sections 4 and 5. Each consumer buys one unit of the product from the firm charging the lower full price, i.e., mill price plus transportation costs. Price ties are resolved in favour of the nearer firm. Consumers are assumed to bear the transportation costs. Let $c(d)$ denote the transportation cost function which is continuous, increasing and convex (weakly or strongly) in distance $d$ and presents itself as one of three forms: linear, quadratic and linear-quadratic, with $c(0) = 0$. Let $p_1$ and $p_2$ denote the mill price of firm 1 and firm 2 respectively. Figure 1 gives a graphic representation of the model.

![Geographical configuration of the marginal consumer and firms](image)

**Figure 1** Geographical configuration of the marginal consumer and firms
2.1 Market and Demand

Let $m(p_1, p_2)$ be the “marginal consumer” $y \in [0,1]$ who is perfectly indifferent between travelling to firm 1 or firm 2 satisfying

$$p_1 + c|y - x_1| = p_2 + c|x_2 - y|$$

and is unique whenever he exists.

The market is segmented at $m(p_1, p_2)$: consumers located in $[0, m(p_1, p_2)]$ buy from firm 1 while those in $[m(p_1, p_2), 1]$ buy from firm 2. If $m(p_1, p_2)$ does not exist, then either of the following two conditions holds:

1. $p_1 + c|y - x_1| < p_2 + c|x_2 - y|$ for all $y \in [0,1], \quad$ or
2. $p_1 + c|y - x_1| > p_2 + c|x_2 - y|$ for all $y \in [0,1].$

In the first case, firm 1 serves the whole market at price $p_1$ while in the second case, the whole market is served by firm 2 at price $p_2$. The strategies of this two-player game are $p_1, p_2 \in [0, +\infty[$ with the payoff function of firm 1 is given by

$$\Pi_1(p_1, p_2; x_1, x_2) = p_1 \int_{m(p_1, p_2)}^{m(p_1, p_1)} f(z)dz \quad \text{if } m(p_1, p_2) \text{ exists,}$$

$$= p_1 \quad \text{if equation (1) holds,}$$

$$= 0 \quad \text{if equation (2) holds}$$

while the payoff function of firm 2 is given by

$$\Pi_2(p_1, p_2; x_1, x_2) = p_2 \int_{m(p_1, p_2)}^{1} f(z)dz \quad \text{if } m(p_1, p_2) \text{ exists,}$$

$$= p_2 \quad \text{if equation (2) holds,}$$

$$= 0 \quad \text{if equation (1) holds.}$$
Assuming linear transportation costs, figure 2 illustrates the full price of the good at various locations of the consumer given the cost schedule ABC if he buys from firm 1 and DF if he buys from firm 2. The bold line ABEF depicts the lowest full price at any given location. The intersection of the two cost schedules at $y$ denotes the location of the marginal consumer. It is obvious from the figure that for the marginal consumer to exist, he must locate in $[x_1,1]$.

Under the type of cost structures we are considering here, we take the most general form of cost function (i.e. the linear-quadratic form) $c(d) = td + sd^2$ where $c(0) = 0$ and $t,s > 0$ for our analysis. The linear cost case (when $s = 0$) and the quadratic cost case (when $d = 0$) will be special cases of it. We will report the results from these two special cases wherever appropriate.

3. Price Equilibrium Under Parametric Locations
If $m(p_1, p_2)$ exists, it must be the solution of the equation

$$p_1 + t(y-x_1) + s(y-x_1)^2 = p_2 + t(x_2-y) + s(x_2-y)^2.$$
(3) \[ m_1(p_1, p_2) = \frac{p_2 - p_1}{2[t + s(x_2 - x_1)]} + \frac{(x_1 + x_2)}{2} \]
and
(4) \[ m_2(p_1, p_2) = \frac{p_1 - p_2}{2[t + s(x_2 - x_1)]} + \frac{(2 - x_1 - x_2)}{2}. \]

with the payoff functions given by \( \Pi_1(p_1, p_2) = p_1 \cdot m_1(p_1, p_2) \) and \( \Pi_2(p_1, p_2) = p_2 \cdot m_2(p_1, p_2) \) respectively. Maximizing profits on the part of firm 1 and firm 2 by setting \( \frac{\partial \Pi_i}{\partial p_i} = 0 \) where \( i = [1, 2] \), gives the following best response functions:

(5) \[ p_1^* = \frac{1}{2} \left[ p_2 + (t + s(x_2 - x_1))(x_1 + x_2) \right] \]
and
(6) \[ p_2^* = \frac{1}{2} \left[ p_1 + (t + s(x_2 - x_1))(2 - x_1 - x_2) \right]. \]

Solving equations (5) and (6) gives the non-cooperative Bertrand-Nash price equilibrium in pure strategies

(7) \[ (p_1^*, p_2^*) = \left( \frac{t + s(x_2 - x_1)}{3}(x_1 + x_2 + 2), \frac{t + s(x_2 - x_1)}{3}(4 - x_1 - x_2) \right). \]

For non-zero \( p_2^* \), we assume that \( x_1 + x_2 < 4 \). In other words, the upper bound on the location of firm 2 for it to remain viable is \( x_2 < 4 - x_1 \). If \( x_1 + x_2 = 4 \), an equilibrium exists at

\[ (p_1^*, p_2^*) = \left( \frac{t + s(x_2 - x_1)}{2}(x_1 + x_2), 0 \right). \]

Intuitively, this means that when the distance between the two firms becomes too large, firm 1 becomes a monopoly and gains the whole market while firm 2 drops out of the competition. The distribution of market demand between firm 1 and firm 2 at Nash equilibrium is obtained by substituting equation (7) into equations (3) and (4) giving

(8) \[ (m_1^*, m_2^*) = \left( \frac{1}{6}(x_1 + x_2 + 2), \frac{1}{6}(4 - x_1 - x_2) \right). \]

Equation (8) shows that the equilibrium demand is dependent only on the location of the two firms.

A similar exposition can be conducted for the cases in which transportation costs are linear of the form \( c(d) = td \) where \( c(0) = 0 \) and \( t > 0 \), and quadratic of the form \( c(d) = sd^2 \) where \( c(0) = 0 \) and \( s > 0 \). The results for these two cases are reported
in Table 1. In both instances, $p_2^* > 0$ whenever $x_1 + x_2 < 4$. Under linear transportation costs, however, a unique equilibrium exists if and only if $(x_1 + x_2 + 2)/3 \geq 4(2 + x_1 - 2x_2)/3$ and $(4 - x_1 - x_2)/3 \geq 4(2x_1 + x_2 - 1)/3$ when $x_1 + x_2 \leq 4$. (The complete analysis of these two cases is available upon request).

In Table 1, we report the equilibrium prices and equilibrium demands for non-zero $p_2^*$, for all possible transport costs considered for the IO model along with the contrasting results for the pure inside location and outside location models.

It is obvious from the results that the inside-outside (IO) model shares some of the features of the pure inside location model as well as the pure outside location model. The equilibrium price and demand are the same for the IO model and the inside location model for all transportation costs considered, and are identical for the IO model and the outside location model under quadratic transportation costs.
Table 1
Equilibrium prices and demands of the inside, outside and IO models under various transportation cost structures when location is parametric.

<table>
<thead>
<tr>
<th>Inside Location Model</th>
<th>Price Equilibrium</th>
<th>Demand Equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c(d) = td$</td>
<td>$(p_1^<em>, p_2^</em>) = \left( \frac{1}{3} (x_1 + x_2 + 2) \frac{1}{2} (4 - x_1 - x_2) \right)$</td>
<td>$(m_1^<em>, m_2^</em>) = \left( \frac{1}{6} (x_1 + x_2 + 2) \frac{1}{6} (4 - x_1 - x_2) \right)$</td>
</tr>
<tr>
<td>$c(d) = sd^2$</td>
<td>$(p_1^<em>, p_2^</em>) = \left( \frac{2}{3} (x_1 - x_1)(x_1 + x_2 + 2) \frac{2}{3} (x_2 - x_1)(4 - x_1 - x_2) \right)$</td>
<td>$(m_1^<em>, m_2^</em>) = \left( \frac{1}{6} (x_1 + x_2 + 2) \frac{1}{6} (4 - x_1 - x_2) \right)$</td>
</tr>
<tr>
<td>$c(d) = td + sd^2$</td>
<td>$(p_1^<em>, p_2^</em>) = \left( \frac{t + s(x_2 - x_1)}{3} (x_1 + x_2 + 2) + \frac{s(x_2 - x_1)}{3} (4 - x_1 - x_2) \right)$</td>
<td>$(m_1^<em>, m_2^</em>) = \left( \frac{1}{6} (x_1 + x_2 + 2) \frac{1}{6} (4 - x_1 - x_2) \right)$</td>
</tr>
</tbody>
</table>

Outside Location Model

<table>
<thead>
<tr>
<th>Outside Location Model</th>
<th>Price Equilibrium</th>
<th>Demand Equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c(d) = td$</td>
<td>$(p_1^<em>, p_2^</em>) = \left( t(x_2 - x_1) \right)$</td>
<td>$(m_1^<em>, m_2^</em>) = \left( 1, 0 \right)$</td>
</tr>
<tr>
<td>$c(d) = sd^2$</td>
<td>$(p_1^<em>, p_2^</em>) = \left( \frac{2}{3} (x_1 - x_1)(x_1 + x_2 + 2) \frac{2}{3} (x_2 - x_1)(4 - x_1 - x_2) \right)$</td>
<td>$(m_1^<em>, m_2^</em>) = \left( \frac{1}{6} (x_1 + x_2 + 2) \frac{1}{6} (4 - x_1 - x_2) \right)$</td>
</tr>
<tr>
<td>$c(d) = td + sd^2$</td>
<td>$(p_1^<em>, p_2^</em>) = \left( \frac{t + s(x_2 - x_1)}{3} (x_1 + x_2 + 2) + \frac{s(x_2 - x_1)}{3} (4 - x_1 - x_2) \right)$</td>
<td>$(m_1^<em>, m_2^</em>) = \left( \frac{1}{6} (x_1 + x_2 + 2) \frac{1}{6} (4 - x_1 - x_2) \right)$</td>
</tr>
</tbody>
</table>

Inside-Outside Location Model

<table>
<thead>
<tr>
<th>Inside-Outside Location Model</th>
<th>Price Equilibrium</th>
<th>Demand Equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c(d) = td$</td>
<td>$(p_1^<em>, p_2^</em>) = \left( \frac{1}{3} (x_1 + x_2 + 2) \frac{1}{2} (4 - x_1 - x_2) \right)$</td>
<td>$(m_1^<em>, m_2^</em>) = \left( \frac{1}{6} (x_1 + x_2 + 2) \frac{1}{6} (4 - x_1 - x_2) \right)$</td>
</tr>
<tr>
<td>$c(d) = sd^2$</td>
<td>$(p_1^<em>, p_2^</em>) = \left( \frac{2}{3} (x_1 - x_1)(x_1 + x_2 + 2) \frac{2}{3} (x_2 - x_1)(4 - x_1 - x_2) \right)$</td>
<td>$(m_1^<em>, m_2^</em>) = \left( \frac{1}{6} (x_1 + x_2 + 2) \frac{1}{6} (4 - x_1 - x_2) \right)$</td>
</tr>
<tr>
<td>$c(d) = td + sd^2$</td>
<td>$(p_1^<em>, p_2^</em>) = \left( \frac{t + s(x_2 - x_1)}{3} (x_1 + x_2 + 2) + \frac{s(x_2 - x_1)}{3} (4 - x_1 - x_2) \right)$</td>
<td>$(m_1^<em>, m_2^</em>) = \left( \frac{1}{6} (x_1 + x_2 + 2) \frac{1}{6} (4 - x_1 - x_2) \right)$</td>
</tr>
</tbody>
</table>

Note:
When transportation costs are linear, a unique price equilibrium exists if and only if $(x_1 + x_2 + 2)^2 \geq 4(2 + x_1 - x_2)/3$ and $(4 - x_1 - x_2)^2 \geq 4(1 + 2x_1 - x_2)/3$ when $x_1 + x_2 < 1$, and $p_1^* = p_2^* = 0$ when $x_1 + x_2 = 1$ for the inside location model (see d’Aspremont et al. 1979); the same condition is true for the IO model when $x_1 + x_2 \leq 4$. A unique price equilibrium exists for all location pairs $(x_1, x_2)$ of the outside location model.

With regard to transportation costs structures, we have the following result in the IO model whenever a solution exists in pure strategies with non-zero prices (see Appendix 1 for the proofs).

Proposition 1

When firm locations are fixed, the equilibrium relative price $p_2^*/p_1^*$ is independent of the transportation cost structure.
**Corollary 1**

*In equilibrium, relative demand is equivalent to relative prices.*

The equilibrium relative price of the good offered by the outside firm to the inside firm is an indication of the “exchange rate” of the good at the two sources. Intuitively, Proposition 1 means that when firm locations are fixed, the outside (inside) firm is able to attract the consumer by offering the good at the same relative price (relative to that offered by the inside (outside) firm) regardless of the nature of the transportation costs faced by the consumers.

Corollary 1 can be interpreted as follows. The relative market demand, which reflects the market area of the two firms, delineates the market boundary that is determined solely by the relative price in the IO model.

Looking at Table 1 we see that these results also apply to the inside location model. In the inside location model, the relative price and relative demand under the three transportation cost structures are also equivalent. The result, however, are not valid for the outside location model.

Now we relax the assumption of parametric location and allow the firms to choose locations as well. In the next section, we will consider a price-location simultaneous move game and in the following section we consider a location-then-price sequential move game in between the two firms.

**4. THE SIMULTANEOUS PRICE-LOCATION GAME**

When firms choose price and location together, we have a simultaneous price-location game. The choice of both price and location simultaneously in one period is reflective of situations in which players commit to a price for a period as long as the product lifetime. For example, firms like chain stores publish a catalogue and stick to it for a while.

In the simultaneous move game, the strategy pairs are \((p_1, x_1)\) for the inside firm 1 and \((p_2, x_2)\) for the outside firm 2, with \(p_1, p_2 \in [0, +\infty[, x_1 \in [0,1]\) and \(x_2 \in ]0, +\infty[\). The Nash equilibrium in prices and locations is one in which no firm wishes to change its price and/or location given the price and location it anticipates the other firm will choose. The payoff function for firm 1 at equilibrium satisfies
for all $x_1 \in [0,1]$ and $p_1 \geq 0$, while the payoff function for firm 2 satisfies
\begin{equation}
\Pi_2 \left((p_1^*, x_1^*), (p_2^*, x_2^*) \right) \geq \Pi_2 \left((p_1^*, x_1^*), (p_2, x_2) \right)
\end{equation}
for all $x_2 \in ]1, +\infty[$ and $p_2 \geq 0$.

It is readily verified that the only pure strategy equilibrium for the simultaneous game involves $x_1^* \in [0,1]$ and $x_2^* = 1 + \varepsilon$ where $\varepsilon > 0$ is a small constant close to zero representing a physical divide.\(^\dagger\) In other words, the dominant location strategy for firm 2 is to locate at $x_2^* = 1 + \varepsilon$. It is easy to note that any other feasible location of firm 2 will result in lower profit.

Under linear-quadratic transportation costs, the profit functions of firm 1 and firm 2 are given by the following equations respectively:
\begin{equation}
\Pi_1 \left((p_1, x_1), (p_2, x_2) \right) = \frac{p_1 p_2 - p_1^2}{2(t + \bar{s}(x_2 - x_1))} + \frac{(x_1 + x_2)}{2} p_1
\end{equation}
and
\begin{equation}
\Pi_2 \left((p_1, x_1), (p_2, x_2) \right) = \frac{p_1 p_2 - p_2^2}{2(t + \bar{s}(x_2 - x_1))} + \frac{(2 - x_1 - x_2)}{2} p_2.
\end{equation}

Firm 1 maximizes profit by choosing $x_1^*$ with the first order condition given by
\begin{equation}
\frac{\partial \Pi_1 \left((p_1, x_1), (p_2, x_2) \right)}{\partial x_1} = \frac{p_1}{2} \left( \frac{s(p_2^* - p_1^*)}{t + s(x_2^* - x_1^*)} + 1 \right) = 0.
\end{equation}

Substituting $p_1^* = \left[ t + s(x_2 - x_1) \left| x_1^* + x_2^* + 2 \right| \right]/3$ and $p_2^* = \left[ t + s(x_2 - x_1) \left| 4 - x_1^* - x_2^* \right| \right]/3$ (obtained from equation (7) by maximising the respective firm’s profit with respect to price) into the above gives
\begin{equation}
\frac{s}{18} \left( x_1^* + x_2^* + 2 \right) \left( 2 - 5x_1^* + x_2^* + \frac{3t}{s} \right) = 0.
\end{equation}

Since $s > 0$ and $(x_1^* + x_2^* + 2) > 0$, this implies that $(2 - 5x_1^* + x_2^* + 3t/s) = 0$. In other words, the equilibrium location of firm 1 is at
\begin{equation}
x_1^* = \frac{2 + x_2^*}{5} + \frac{3t}{5s}
\end{equation}
which gives the response function in location of firm 1.

In the case of firm 2, it maximises profit by choosing $x_2^*$ such that

\(^\dagger\) For example, a physical divide between countries or cities could be a sea, a mountain etc or legal restrictions. When $\varepsilon = 0$, the market boundary at 1 represents a seamless economic and (or) geographical border between the two markets. In the ensuing discussions, we assume that $\varepsilon > 0$.\)
\[ \frac{\partial \Pi_2 ( (p_1, x_1), (p_2, x_2) )}{\partial x_2} = -\frac{p_2^*}{2} \left( \frac{s(p_1^* - p_2^*)}{t + s(x_2^* - x_1^*)} \right)^2 + 1 < 0 \]

since \( p_1^* > p_2^* \) from equation (7) for all \( x_1 + x_2 < 4 \). This implies that firm 2 increases profit by moving towards the market border, i.e., \( x_2^* = 1 + \varepsilon \), \( \varepsilon > 0 \). Solving for \( x_1^* \) by substituting \( x_2^* = 1 + \varepsilon \) into equation (11) gives \( x_1^* = 3/5(1 + t/s) + \varepsilon/5 \). For a unique equilibrium in location to exist, \( x_1^* \leq 1 \) or \( t/s \leq 2/3 \). The equilibrium prices are obtained by substituting \( x_1^* \) and \( x_2^* \) into equation (7) so that

(12) \[ p_1^* = \frac{2}{25} \left( 6s + t \left( \frac{7}{s} + \frac{1}{t} \right) \right) + 4\varepsilon/25 \left( 2t + 7s + 2\varepsilon s \right); \quad \text{and} \]

\[ p_2^* = \frac{2}{25} \left( 4s + t \left( 3 - \frac{t}{s} \right) \right) - 4\varepsilon/25 \left( 2t - 3s + 2\varepsilon s \right). \]

**Proposition 2**

The simultaneous price-location equilibrium in pure strategies under linear-quadratic transportation costs is given by

\[
\left( (p_1^*, x_1^*), (p_2^*, x_2^*) \right) = \left( \left[ \frac{2}{25} \left( 6s + t \left( \frac{7}{s} + \frac{1}{t} \right) \right) + 4\varepsilon/25 \left( 2t + 7s + 2\varepsilon s \right) \right], \left[ \frac{3}{5} \left( 1 + \frac{t}{s} \right) + \varepsilon \right], \left[ \frac{2}{25} \left( 4s + t \left( 3 - \frac{t}{s} \right) \right) - 4\varepsilon/25 \left( 2t - 3s + 2\varepsilon s \right) \right], \left( 1 + \varepsilon \right) \right)
\]

where \( \varepsilon > 0 \).

The simultaneous price-location equilibrium in pure strategies under quadratic transportation costs is reported in Table 2. (The complete analysis is available upon request). Moreover, we find that there is a non-existence problem of the simultaneous price-location game under linear transport costs (see appendix 2 for details).

**4.1 Comparative Analysis**

The results of the simultaneous move game for the IO model are summarised in Table 2, along with the comparative equilibrium strategies for the pure inside location and outside location models. We know that no simultaneous price-location equilibrium in pure strategies can exist in the inside location model while the simultaneous price-location equilibrium in pure strategies for the outside location model is for the two firms to always locate at \( x_1^* = x_2^* = 1 \) with prices \( p_1^* = p_2^* = 0 \) (see Gabszewicz and Thisse 1992). The IO model, with the horizontal differentiation characteristics of the

\[2\] de Palma et al. (1985), however, showed that a simultaneous price-location equilibrium exists in the inside location model if the product is heterogeneous enough. Anderson et al. (1992) further showed
inside location model, has the same instability problem as the inside location model under linear transportation costs. Incorporating the vertical differentiation characteristics of the outside location model, however, has rendered the IO model greater stability than the pure inside location model in that an equilibrium in pure strategies exists when the transportation cost structure is quadratic and linear-quadratic.

**Table 2**
Simultaneous price-location equilibrium of the inside, outside and IO models under various transportation cost structures

<table>
<thead>
<tr>
<th>Location Equilibrium</th>
<th>Price Equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inside Location Model</td>
<td></td>
</tr>
<tr>
<td>$c(d) = td$</td>
<td>No equilibrium exists</td>
</tr>
<tr>
<td>$c(d) = sd^2$</td>
<td>No equilibrium exists</td>
</tr>
<tr>
<td>$c(d) = td + sd^2$</td>
<td>No equilibrium exists</td>
</tr>
<tr>
<td>Outside Location Model</td>
<td></td>
</tr>
<tr>
<td>$c(d) = td$</td>
<td>$\left( x_1^<em>, x_2^</em> \right) = (1,1)$</td>
</tr>
<tr>
<td>$c(d) = sd^2$</td>
<td>$\left( x_1^<em>, x_2^</em> \right) = (1,1)$</td>
</tr>
<tr>
<td>$c(d) = td + sd^2$</td>
<td>$\left( x_1^<em>, x_2^</em> \right) = (1,1)$</td>
</tr>
<tr>
<td>Inside-Outside Location Model</td>
<td></td>
</tr>
<tr>
<td>$c(d) = td$</td>
<td>No equilibrium exists</td>
</tr>
<tr>
<td>$c(d) = sd^2$</td>
<td>$\left( x_1^<em>, x_2^</em> \right) = \left( \frac{1}{5} \left( 3 + \frac{t}{s} \right), 1 + \frac{t}{s} \right)$</td>
</tr>
<tr>
<td>$c(d) = td + sd^2$</td>
<td>$\left( x_1^<em>, x_2^</em> \right) = \left( \frac{3}{5} \left( 1 + \frac{t}{s} \right) + \frac{e}{5}, 1 + \frac{e}{5} \right)$</td>
</tr>
</tbody>
</table>

**Note:**
When transportation costs are linear-quadratic, a unique equilibrium in location exists whenever $t/s \leq 2/3$ for the IO model.

that the only symmetric pure strategy equilibrium occurs when both firms agglomerate at the market centre.
5. The Sequential Location-price Game

Now we consider a two-stage process in which the location is chosen first in full anticipation of the ensuing price equilibrium, followed by the price in the second stage, where prices are decided based on the location choice made in the first stage. The solution to the sequential game is worked out using backward induction. In a subgame consisting of the second stage, a non-cooperative price equilibrium in pure strategies with prices $p_1^*(x_1,x_2)$ and $p_2^*(x_1,x_2)$ are chosen for given locations $x_1$ and $x_2$. The pure strategy equilibrium to the first-stage location game is the pair of locations $(x_1^*, x_2^*)$ which maximises the profit function $\Pi(p_1^*(x_1,x_2), p_2^*(x_1,x_2), x_1,x_2)$ where $i = \{1,2\}$. This profit function is well defined whenever the price equilibrium exists and is unique. The full (subgame perfect) equilibrium to the game is then given by the quadruple $(p_1^*, p_2^*, x_1^*, x_2^*)$ where $p_1^* = p_1^*(x_1^*, x_2^*)$ and $p_2^* = p_2^*(x_1^*, x_2^*)$.

We have shown earlier that when $x_1 + x_2 < 4$, the unique price equilibrium in pure strategies is given by equation (7), i.e.,

$$(13) \quad (p_1^*(x_1,x_2), p_2^*(x_1,x_2)) = \left( \frac{t + s(x_2 - x_1)}{3} (x_1 + x_2 + 2), \frac{t + s(x_2 - x_1)}{3} (4 - x_1 - x_2) \right).$$

The profit function of the inside firm 1 is given by

$$\Pi_1(p_1(x_1,x_2), p_2(x_1,x_2), x_1,x_2) = \frac{p_1 p_2 - p_1^2}{2[t + s(x_2 - x_1)]} + \frac{x_1 + x_2}{2} - p_1.$$

Substituting equation (13) gives

$$\Pi_1(p_1^*(x_1,x_2), p_2^*(x_1,x_2), x_1,x_2) = \frac{t + s(x_2 - x_1)}{18} (x_1 + x_2 + 2)^2.$$

Optimising with respect to $x_1$ gives

$$\frac{\partial \Pi_1(p_1^*(x_1,x_2), p_2^*(x_1,x_2), x_1,x_2)}{\partial x_1} = \frac{x_1^* + x_2^* + 2}{18} (2t + s(x_2^* - 3x_1^* - 2)).$$

Since $(x_1^* + x_2^* + 2) > 0$, one possible scenario is that $2t + s(x_2^* - 3x_1^* - 2) < 0$ or

$$(14) \quad \frac{t}{s} < \frac{3x_1^* + x_2^*}{2} + 1.$$

If equation (14) holds, then $\frac{\partial \Pi_1(p_1^*, p_2^*, x_1,x_2)}{\partial x_1} < 0$ which implies that as $x_1$ decreases, firm 1’s profit increases so that firm 1’s optimal location is at the point 0. If the converse of equation (14) holds, then two instances can arise: either
\( \partial \Pi_1(p_1^*, p_2^*, x_1, x_2) / \partial x_1 > 0 \) or \( \partial \Pi_1(p_1^*, p_2^*, x_1, x_2) / \partial x_1 = 0 \). If the former holds, then firm 1’s profit is maximised by locating at point 1.

Now consider the profit function for the outside firm 2 which is given by

\[
\Pi_2(p_1(x_1, x_2), p_2(x_1, x_2), x_1, x_2) = \frac{p_1 p_2 - p_2^2}{2[t + s(x_2 - x_1)]} + \frac{(2 - x_1 - x_2)}{2} p_2.
\]

Substituting equation (13) gives

\[
\Pi_2(p_1^*(x_1, x_2), p_2^*(x_1, x_2), x_1, x_2) = \frac{t + s(x_2 - x_1)}{18}(4 - x_1 - x_2)^2.
\]

Optimising with respect to \( x_2 \) gives

\[
\frac{\partial \Pi_2(p_1^*(x_1, x_2), p_2^*(x_1, x_2), x_1, x_2)}{\partial x_2} = \left(4 - x_1^* - x_2^*\right) \left(s(x_1^* - 3x_2^* + 4) - 2t\right).
\]

Since \( x_1^* + x_2^* < 4 \), this implies that for \( \partial \Pi_2(p_1^*, p_2^*, x_1, x_2) / \partial x_2 = 0 \),

\[ s(x_1^* - 3x_2^* + 4) - 2t = 0 \]

or

\[ x_1^* - 3x_2^* + 4 = \frac{2t}{s}. \]

Suppose that equation (14) holds, i.e., \( \partial \Pi_1(p_1^*, p_2^*, x_1, x_2) / \partial x_1 < 0 \) and \( x_1^* = 0 \).

Substituting \( x_1^* = 0 \) into equation (15) and solving gives \( x_2^* = 4/3 - 2t/3s \). Since \( x_2 \in [1, +\infty[ \), the condition for \( x_2^* \) to hold is that \( t/s < \frac{1}{2} \).

We will now show that the converse of equation (14) is never valid. Suppose that \( t/s > 1 + (3x_1^* + x_2^*)/2 \) holds so that \( \partial \Pi_1(p_1^*, p_2^*, x_1, x_2) / \partial x_1 > 0 \) and \( x_1^* = 1 \).

Substituting \( x_1^* = 1 \) into equation (15) and solving gives \( x_2^* = 5/3 - 2t/3s \). This solution, however, cannot exist because it contradicts the assumed condition that \( t/s > 1 + (3x_1^* + x_2^*)/2 \). Substituting \( x_1^* = 1 \) and \( x_2^* = 5/3 - 2t/3s \) gives \( t/s > 5/2 \). This condition, however, suggests that \( x_2^* = 5/3 - 2t/3s < 0 \) which cannot hold since \( x_2 \in [1, +\infty[ \).

We will now show that \( \partial \Pi_1(p_1^*, p_2^*, x_1, x_2) / \partial x_1 = 0 \) also cannot hold. Suppose on the contrary that \( \partial \Pi_1(p_1^*, p_2^*, x_1, x_2) / \partial x_1 = 0 \). In that case, equation (14) becomes the equality \( 3x_1^* - x_2^* + 2 = 2t/s \). Solving this equation together with equation (15) gives \( x_1^* = t/2s - 1/4 \) and \( x_2^* = 5/4 - t/2s \). By assumption of the model, we have \( x_1 \in [0, 1] \) and \( x_2 \in [1, +\infty[ \). Consequently, \( x_1^* = t/2s - 1/4 \) implies that \( t/s \in [\frac{1}{2}, \frac{3}{2}] \) and \( x_2^* = 5/4 - t/2s \) implies that \( t/s < \frac{1}{2} \), which contradicts \( t/s \in [\frac{1}{2}, \frac{3}{2}] \).
The only solution in pure strategies to the first-stage of the sequential game is, therefore, 
\((x_1^*, x_2^*) = (0, 4/3 - 2t/3s)\) for \(x_1 + x_2 < 4\) and \(t/s < 1/2\). The second-stage game is then solved by substituting \(x_1^*\) and \(x_2^*\) into equation (13). The equilibrium price pair in pure strategies is given by

\[
(p_1^*, p_2^*) = \left( \frac{2(20s + t(1-t/s))}{27}, \frac{2(16s + t(8 + t/s))}{27} \right)
\]

**Proposition 3**

The subgame perfect equilibrium to the sequential game in pure strategies under linear-quadratic transportation costs is given by

\[
(p_1^*, p_2^*, x_1^*, x_2^*) = \left( \frac{2}{27} \left( 20s + t \left( 1 - \frac{t}{s} \right) \right), \frac{2}{27} \left( 16s + t \left( 8 + \frac{t}{s} \right) \right), 0, \frac{4}{3} - \frac{2t}{3s} \right)
\]

where \(x_1 + x_2 < 4\) and \(t/s < 1/2\).

The equilibrium of the sequential game in pure strategies under quadratic transportation costs is reported in Table 3 (analysis available upon request). There is also a non-existence problem of the sequential game under linear transport costs (see appendix 3 for details).

5.1 **Comparative Analysis**

The results of the sequential game for the IO model are summarized in Table 3, along with the comparative equilibrium strategies for the pure inside and outside location models. When transportation costs are linear or linear-quadratic, no equilibrium in pure strategies can exist in the inside location model, while the equilibrium in pure strategies for the outside location model always exists (see Gabszewicz and Thisse 1992). As in the pure inside location model, it is shown that an equilibrium in pure strategies also fails to exist for the sequential game of the IO model when transportation costs are linear. However, unlike the inside location model which possesses an equilibrium for the sequential game when transportation costs are quadratic but not when they are linear or linear-quadratic (d’Aspremont et al. 1979; Anderson 1988), an equilibrium always exists for the IO model whenever transportation costs are strictly convex.

When transportation costs are quadratic, the Principle of Maximum Differentiation is established in the inside location model as well as the IO model.
Simply put, the Principle of Maximum Differentiation states that two firms have a tendency to locate in opposite directions towards the end points of the linear city as a result of competition. By locating at \((x_1^*, x_2^*) = (0.1, 0.4/3)\) respectively, firms in the inside and IO model exhibit greater tendency of differentiation under quadratic transportation costs.

When faced with linear-quadratic transportation costs, firms in the outside location model and IO models make their location decisions based on all the parameters of the model. In the case of the outside location model, some tendency of increasing differentiation is observed although its intensity is lower than that under quadratic transportation costs. Similarly, the IO model reflects a tendency toward increasing differentiation which loses its intensity because of the linear-quadratic transportation cost function.

**Table 3**

<table>
<thead>
<tr>
<th>Location Equilibrium</th>
<th>Price Equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Inside Location Model</strong></td>
<td></td>
</tr>
<tr>
<td>(c(d) = td)</td>
<td>no equilibrium exists</td>
</tr>
<tr>
<td>(c(d) = sd^2)</td>
<td>((x_1^<em>, x_2^</em>) = (0.1, 0.4))</td>
</tr>
<tr>
<td>(c(d) = td + sd^2)</td>
<td>no equilibrium exists</td>
</tr>
<tr>
<td><strong>Outside Location Model</strong></td>
<td></td>
</tr>
<tr>
<td>(c(d) = td)</td>
<td>((x_1^<em>, x_2^</em>) = (1, 1))</td>
</tr>
<tr>
<td>(c(d) = sd^2)</td>
<td>((x_1^<em>, x_2^</em>) = (1, 5/3))</td>
</tr>
<tr>
<td>(c(d) = td + sd^2)</td>
<td>((x_1^<em>, x_2^</em>) = (1, 5/3 - t/3s))</td>
</tr>
<tr>
<td><strong>Inside-Outside Location Model</strong></td>
<td></td>
</tr>
<tr>
<td>(c(d) = td)</td>
<td>no equilibrium exists</td>
</tr>
<tr>
<td>(c(d) = sd^2)</td>
<td>((x_1^<em>, x_2^</em>) = (0, 4/3))</td>
</tr>
<tr>
<td>(c(d) = td + sd^2)</td>
<td>((x_1^<em>, x_2^</em>) = (0, 4/3 - 2t/3s))</td>
</tr>
</tbody>
</table>

**Note:**
When transportation costs are linear-quadratic, a unique equilibrium in location exists whenever the following conditions hold: (a) outside location model: \(t/s \leq 2\); and (b) IO model: \(t/s < \frac{1}{2}\).
5.2 Stability
One interesting observation that surfaced under parametric firm location is that the results of the IO model and the inside location model are identical for all transportation costs considered, and for the IO model and the outside location model when transportation costs are quadratic. On the other hand, under variable firm locations and linear transportation costs, the results of the IO model and the inside location model are identical to a certain extent, i.e., there is a non-existence of equilibrium problem. However, the non-existence problem dissipates in the IO model when price and location decisions are made simultaneously under quadratic and linear-quadratic transportation costs while it persists in the inside location model. Moreover, the IO model does not suffer from the non-existence problem as the inside location model when the game is played sequentially under linear-quadratic transportation costs. These results contrast with the outside location model where a solution always exists when location is variable. Thus, the stability of the IO model can be said to be intermediate between the inside location model and the outside location model. This is not that surprising since the IO location model is an integration of the two models.

6. Conclusions
Product differentiation due to firms’ locations at the boundary regions of countries or cities is of pertinent significance and interest to various segments of society because of its attendant economic benefits and trickle down effects on the rest of the economy. The inside-outside location model presented here offers a simple framework for understanding and analysing the location and pricing decisions of firms situated on either side of the border, as well as the purchase and travel decisions of consumers between the inside firm and the outside firm. The IO model is readily applicable for analysing cross-border shopping behaviour which is a common phenomenon in various parts of the world. From the marketing discipline this is also a framework in which economics of brand loyalty can be studied.

As such, the IO model is directly applicable to situations in which adjoining markets are segmented geographically and (or) economically. It also highlights the distinction between an economic boundary and geographical boundary between two regions, which in most cases do not necessarily coincide.
REFERENCES
APPENDIX 1

PROOF OF PROPOSITION 1 AND COROLLARY 1

Proposition 1

When the transportation cost structure is linear, i.e., when $c(d) = td$, $t > 0$, the unique price equilibrium in pure strategies exists at the pair of prices

$$\left(p_1^*, p_2^*\right) = \left(\frac{t}{3}(x_1 + x_2 + 2), \frac{t}{3}(4 - x_1 - x_2)\right)$$

The equilibrium relative price of the good is given by

$$(A1) \quad \frac{p_2^*}{p_1^*} = \frac{t(4 - x_1 - x_2)/3}{t(x_1 + x_2 + 2)/3} = \frac{4 - x_1 - x_2}{x_1 + x_2 + 2}$$

When transportation costs are quadratic, i.e., when $c(d) = sd^2$, $s > 0$, the unique equilibrium in pure strategies exists at the pair of prices

$$\left(p_1^*, p_2^*\right) = \left(\frac{s}{3}(x_2 - x_1)(x_1 + x_2 + 2), \frac{s}{3}(x_2 - x_1)(4 - x_1 - x_2)\right)$$

The equilibrium relative price of the good is given by

$$(A2) \quad \frac{p_2^*}{p_1^*} = \frac{s(x_2 - x_1)(4 - x_1 - x_2)/3}{s(x_2 - x_1)(x_1 + x_2 + 2)/3} = \frac{4 - x_1 - x_2}{x_1 + x_2 + 2}$$

In the case of linear-quadratic transportation costs, i.e., when $c(d) = td + sd^2$, $t > 0$ and $s > 0$, the unique equilibrium in pure strategies exists at the pair of prices

$$\left(p_1^*, p_2^*\right) = \left(\frac{t + s(x_2 - x_1)}{3}, (x_1 + x_2 + 2), \frac{t + s(x_2 - x_1)}{3}, (4 - x_1 - x_2)\right)$$

The equilibrium relative price of the good is given by

$$(A3) \quad \frac{p_2^*}{p_1^*} = \frac{[t + s(x_2 - x_1)][4 - x_1 - x_2]/3}{[t + s(x_2 - x_1)][x_1 + x_2 + 2]/3} = \frac{4 - x_1 - x_2}{x_1 + x_2 + 2}$$

Equations (A1), (A2) and (A3) are all equivalent.

QED.

Corollary 1

The equilibrium demand is the same for all transportation costs and is given by

$$\left(m_1^*, m_2^*\right) = \left(\frac{1}{6}(x_1 + x_2 + 2), \frac{1}{6}(4 - x_1 - x_2)\right)$$
Thus for all transportation cost functions, the equilibrium relative demand as

\[
\frac{m_2^*}{m_1^*} = \frac{1}{6} \left(4 - x_1 - x_2\right) = \frac{4 - x_1 - x_2}{x_1 + x_2 + 2}
\]

Since (A4) is equivalent to (A3), we have \( \frac{m_2^*}{m_1^*} = \frac{p_2^*}{p_1^*} \) for all transportation cost functions. \( \text{QED.} \)

**APPENDIX 2**

I. **Equilibrium Prices under Parametric Locations with Linear Transportation Costs**

Assume that the transportation cost function is linear of the form \( c(d) = td \), where \( c(0) = 0 \) and \( t > 0 \). If \( m(p_1, p_2) \) exists, it must be the solution of the equation

\[
p_1 + t(y - x_1) = p_2 + t(x_2 - y).
\]

Solving, we obtain the demand functions for firm 1 and firm 2, respectively, as

(A5) \[
m_1(p_1, p_2) = \frac{p_2 - p_1}{2t} + \frac{(x_1 + x_2)}{2}
\]

(A6) \[
m_2(p_1, p_2) = \frac{p_1 - p_2}{2t} + \frac{(2 - x_1 - x_2)}{2}.
\]

The payoff functions are given by \( \Pi_1(p_1, p_2) = p_1 \cdot m_1(p_1, p_2) \) and \( \Pi_2(p_1, p_2) = p_2 \cdot m_2(p_1, p_2) \) respectively. Maximizing profits on the part of firm 1 and firm 2 gives the following response functions:

(A7) \[
p_1^* = \frac{1}{2} \left[p_2^* + t(x_1 + x_2)\right]
\]

(A8) \[
p_2^* = \frac{1}{2} \left[p_1^* + t(2 - x_1 - x_2)\right].
\]

Solving equations A7 and A8 gives the non-cooperative Bertrand-Nash price equilibrium in pure strategies, i.e.,

(A9) \[
(p_1^*, p_2^*) = \left(\frac{t}{3} (x_1 + x_2 + 2), \frac{t}{3} (4 - x_1 - x_2)\right).
\]

II. **Equilibrium Non-Existence – Simultaneous Price-Location game with Linear Transport Costs**

We will now turn to the non-existence problem of the simultaneous price-location game when both firms face linear transportation costs. Assume that transportation
costs are linear of the form \( c(d) = td \) where \( c(0) = 0 \) and \( t > 0 \). The profit functions of firm 1 and firm 2 are given by the following equations respectively:

\[
\Pi_1\left((p_1, x_1), (p_2, x_2)\right) = \frac{p_1 p_2 - p_1^2}{2t} + \frac{(x_1 + x_2)}{2} p_1
\]

and

\[
\Pi_2\left((p_1, x_1), (p_2, x_2)\right) = \frac{p_1 p_2 - p_2^2}{2t} + \frac{(2 - x_1 - x_2)}{2} p_2.
\]

Firm 1 chooses the optimal location \( x_1^* \) that maximises its profit. Since

\[
\frac{\partial \Pi_1\left((p_1, x_1), (p_2, x_2)\right)}{\partial x_1} = \frac{p_1^*}{2} > 0,
\]

firm 1 raises its profit by moving towards firm 2 which gives its equilibrium location as \( x_1^* = 1 \). At the same time, firm 2’s dominant strategy is to choose \( x_2^* = 1 + \varepsilon \). This is obvious from maximising firm 2’s profit with respect to location which gives

\[
\frac{\partial \Pi_2\left((p_1, x_1), (p_2, x_2)\right)}{\partial x_2} = -\frac{p_2^*}{2} < 0.
\]

As a result, firm 2 increases its profit by moving towards the market boundary. The equilibrium location of firm 2 is then given by \( x_2^* = 1 + \varepsilon \), \( \varepsilon > 0 \) is a small constant.

Substituting \( x_1^* \) and \( x_2^* \) into firm 1 and firm 2’s response function (equations (A7) and (A8)) gives \( p_1^* = \frac{t}{3}(4 + \varepsilon) \) and \( p_2^* = \frac{t}{3}(2 - \varepsilon) \). This price configuration is not possible for the two firms located next to each other. Their attempt to undercut each other and relax price competition by moving apart naturally generates instability in the location choice of the two firms. The simultaneous price-location equilibrium in pure strategies, therefore, does not exist when transportation costs are linear.

### APPENDIX 3

**Equilibrium Non-Existence – Sequential Location-Price game with Linear Transport Costs**

We now turn to the non-existence problem in the sequential game which resurfaces under linear transportation costs. Under linear transportation costs when \( x_1 + x_2 < 4 \), the unique price equilibrium in pure strategies is given by equation (A9), i.e.,

\[
\left(p_1^*, p_2^*\right) = \left(\frac{t}{3}(x_1 + x_2 + 2), \frac{t}{3}(4 - x_1 - x_2)\right).
\]

The profit function of firm 1 is given by
\[ \Pi_1(p_1(x_1,x_2), p_2(x_1,x_2), x_1, x_2) = \frac{p_1 p_2 - p_1^2}{2t} + \frac{(x_1 + x_2)}{2} - p_1. \]

Substituting equation (A10) gives
\[ \Pi_1(p_1^*(x_1,x_2), p_2^*(x_1,x_2), x_1, x_2) = \frac{t}{18} (x_1 + x_2 + 2)^2. \]

Optimising with respect to \( x_1 \) gives
\[ \frac{\partial \Pi_1(p_1^*(x_1,x_2), p_2^*(x_1,x_2), x_1, x_2)}{\partial x_1} = \frac{t}{9} (x_1^* + x_2^* + 2) > 0 \]

since \( t > 0 \) and \((x_1^* + x_2^* + 2) > 0\). Since firm 1’s profit increases as \( x_1 \) increases, it maximises profit by locating at point 1.

Now consider the profit function for firm 2 which is given by
\[ \Pi_2(p_1(x_1,x_2), p_2(x_1,x_2), x_1, x_2) = \frac{p_1 p_2 - p_2^2}{2t} + \frac{(2 - x_1 - x_2)}{2} - p_2. \]

Substituting equation (A10) gives
\[ \Pi_2(p_1^*(x_1,x_2), p_2^*(x_1,x_2), x_1, x_2) = \frac{t}{18} (4 - x_1 - x_2)^2. \]

Optimising with respect to \( x_2 \) gives
\[ \frac{\partial \Pi_2(p_1^*(x_1,x_2), p_2^*(x_1,x_2), x_1, x_2)}{\partial x_2} = \frac{t}{9} (4 - x_1^* - x_2^*) < 0 \]

since \( t > 0 \) and \( x_1^* + x_2^* < 4 \). In other words, firm 2’s profit increases as \( x_2 \) decreases so that it maximises profit by locating at point \( 1 + \varepsilon \) where \( \varepsilon \) is a small constant. The solution in pure strategies to the first-stage of the sequential game is, therefore, \((x_1^*, x_2^*)=(1, 1 + \varepsilon)\) for \( x_1 + x_2 < 4 \).

The second-stage game is then solved by substituting \( x_1^* \) and \( x_2^* \) into equation (A10). The equilibrium price pair in pure strategies is then given by \((p_1^*, p_2^*)=((4 + \varepsilon)/3, (2 - \varepsilon)/3)\). This is again not possible as the price differential creates opportunities for both firms that are situated next to each other to engage in a price war and undercut each other, and relax price competition by moving apart, giving rise to instability in the location choice of the two firms. As with the simultaneous game, therefore, an equilibrium of the sequential game in pure strategies does not exist when transportation costs are linear.