Understanding Stable Matchings:
A Non-Cooperative Approach

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Abstract

We present a series of non-cooperative games with monotone best replies whose set of Nash equilibria coincides with the set of stable matchings. Key features of stable matchings are established as familiar properties of games with monotone best replies. Then we present a sense in which our method is necessary for the monotonicity approach. We also establish the connection of our approach with other monotone methods in the literature.

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1 Introduction

To be written.

2 Framework

This section presents the model.

2.1 Two-Sided Matching Model

A market is tuple \( \Gamma = (S, C, P) \). We denote by \( S \) and \( C \) finite and disjoint sets of students and colleges, respectively. Let \( N = S \cup C \) be the set of all agents. For each \( s \in S \), \( P_s \) is a strict preference relation over \( 2^C \). For each \( c \in C \), \( P_c \) is a strict preference relation over \( 2^S \). The non-strict counterpart of \( P_i \) is denoted by \( R_i \), so we write \( XR_iX' \) if and only if \( XP_iX' \) or \( X = X' \).

For each \( s \in S \) and \( C' \subseteq C \), the chosen set \( Ch_s(C') \) is a set such that (1) \( Ch_s(C') \subseteq C' \), and (2) \( C'' \subseteq C' \) implies \( Ch_s(C'R_sC'') \). That is, \( Ch_s(C') \) is the subset of \( C' \) that is most preferred by \( s \). In other words, \( Ch_s(C') \) the set of colleges that \( s \) would choose if she can choose partners freely from \( C' \). Note that \( Ch_s(C') \) is uniquely specified in the above definition since preferences are strict. For each \( c \in C \) and \( S' \subseteq S \), the set \( Ch_c(S') \) is similarly defined as the most preferred subset of \( S' \) by \( c \).

A matching \( \mu \) is a (possibly empty) correspondence from \( C \cup S \) to \( C \cup S \) such that \( \mu(s) \subseteq C \) for every \( s \in S \), \( \mu(c) \subseteq S \) for every \( c \in C \) and, for every \( i, j \in N \), \( i \in \mu(j) \) if and only if \( j \in \mu(i) \). We abuse notation and, for any \( i \in N \), write \( \mu P_i \nu \) if \( \mu(i) P_i \nu(i) \) and \( \mu R_i \nu \) if \( \mu(i) R_i \nu(i) \).

Given a matching \( \mu \), we say that it is blocked by \( (s, c) \in S \times C \) if \( s \notin \mu(c) \), \( s \in Ch_c(\mu(c) \cup s) \) and \( c \in Ch_s(\mu(s) \cup c) \). In words, a student-college pair is a blocking pair if both of its members have incentives to deviate from the current matching \( \mu \) by matching with each other (while potentially rejecting some of their current partners). A matching \( \mu \) is individually rational if \( Ch_i(\mu(i)) = \mu(i) \) for every \( i \in N \). A matching \( \mu \) is pairwise stable if it is individually rational and is not blocked. We simply refer to pairwise stability as stability when there is no confusion.\(^1\)

\(^1\)A pairwise-stable matching may be vulnerable to blocks by larger groups in many-to-many matching problems (see Roth and Sotomayor (1990) and Sotomayor (1999, 2004)). However, Konishi and Ünver (2006), Echenique and Oviedo (2006), and Klaus and Walzl (2009) offer various preference domains under which pairwise stable matchings are immune to certain group deviations, suggesting that pairwise stability is a reasonable solution concept.
For each $i \in S$ (respectively $i \in C$), her preference relation $P_i$ is **substitutable** (Kelso and Crawford, 1982) if for any $X, X' \subseteq S$ (respectively $X, X' \subseteq C$) with $X \subseteq X'$, we have $Ch_i(X') \cap X \subseteq Ch_i(X)$.\(^2\) That is, a partner who is chosen from a larger set of potential partners is always chosen from a smaller set of potential partners.\(^3\) If every agent has substitutable preferences, then there exists a pairwise-stable matching (Roth, 1984).

We note that our model assumes that an agent can be matched with an arbitrary number of partners if they want to do so. However, there are many economic situations where an agent has a capacity (quota) constraint, that is, an agent cannot be matched with more than a certain number of partners. Such feasibility constraints can be accommodated in our model by assuming preferences appropriately. For example, we can simply assume that the agent prefers the null set of partners to any set of partners whose cardinality exceeds her capacity. A particular case with quota constraint has attracted much attention in the literature. Preference relation $P_s$ of student $s$ is **responsive with quota** $q_s$ if

1. $(C' \cup c) P_s (C'') \iff c P_s c'$ for any $C' \subset C$ with $|C'| < q_s$ and any $c, c' \in C \setminus C'$,

2. $(C' \cup c) P_s C' \iff c P_s \emptyset$ for any $C' \subset C$ with $|C'| < q_s$, and

3. $\emptyset P_s C'$ for any $C' \subseteq C$ with $|C'| > q_s$.

Symmetric definition applies to colleges. Clearly any responsive preference with a quota is substitutable, implying that a stable matching exists if all agents have responsive preferences.

### 2.2 Lattice Theory

Let $L$ be a set and $\geq$ be a partial order on $L$.\(^4\) The pair $(L, \geq)$ is said to be a **complete lattice** if any subset of $L$ has a supremum and an infimum. An immediate consequence of the definition is that any complete lattice $(L, \geq)$ has the largest and the smallest elements, that is, elements $\vee, \wedge \in L$ such that $\vee \geq x \geq \wedge$ for every $x \in L$.\(^5\) Let $(L, \geq)$ be a complete lattice. Function $f : L \rightarrow L$ is said to be **monotone increasing** if, for all $x, y \in L$, $x \geq y$ implies $f(x) \geq f(y)$.

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\(^2\)To be more precise, Kelso and Crawford (1982) first define substitutability in their model of matching with wages. The current definition is due to Roth and Sotomayor (1990).

\(^3\)Recall that the mapping $Ch_i(\cdot)$ depends on preference $P_i$ although the dependence is suppressed in the current notation.

\(^4\)We will use the term “order” to mean a partial order when there is no risk of confusion.

\(^5\)To see this, simply apply the definition of a complete lattice to the set $L$ itself.
Result 1 (Tarski’s Fixed Point Theorem) Let \((L, \geq)\) be an nonempty complete lattice and function \(f : L \to L\) is monotone increasing. Then the set of all fixed points of \(f\) is a nonempty complete lattice with respect to \(\geq\).

When \(L\) is a finite set, it is easy to find the largest and smallest fixed points of increasing function \(f\). To find the largest fixed point, for instance, consider a sequence \(\bar{x}, f(\bar{x}), f^2(\bar{x}), \ldots\), where \(\bar{x}\) is the largest point in \(L\). It is well known that the sequence converges in a finite number of steps, say \(T\), and the limit \(f^T(\bar{x})\) is the largest fixed point of \(f\). The smallest fixed point can be found by a similar iteration of function \(f\) starting from the smallest point in \(L\).

3 Two-Stage Game

We consider the following two-stage game. The set of players is \(C\), while colleges in \(S\) are passive players. In the first stage, each \(c \in C\) simultaneously makes offers to a subset of students \(x_c \subseteq S\). In the second stage, each student \(s \in S\) chooses her most preferred subset of colleges among those that made offers to her, i.e., to \(Ch_s(F_s(x_C))\) where \(F_s(x_C) := \{c \in C | s \in x_c\}\) is the set of colleges that make an offer to \(s\) at strategy profile \(x_C\). The outcome of the game is a matching where student \(s \in S\) is matched to \(Ch_s(F_s(x_C))\). We write \(x_C' := (x_c)_{c \in C'}\) for any \(C' \subseteq C\), \(x_c = x_C \setminus c\), and \(\phi(x_C)\) to be the matching that results in the end of this game when colleges announce \(x_C\).\(^6\)

This two-stage game is a simplified version of games analyzed by Alcalde and Romero-Medina (2000), Sotomayor (2004) and Echenique and Oviedo (2006). These papers define a two-stage game in which each student is an active player who chooses which of the offers made to it to accept in the second stage. Since the best response for students in the second stage is straightforward, the analysis of the current paper is unchanged if we follow this setup.

A strategy profile \(x_C\) is a Nash equilibrium if \(\phi(x_C) R_c \phi(x'_c, x_{-c})\) for every \(c \in C\) and \(x'_c \subseteq C\). We introduce a (partial) order \(\succeq\) on the set of strategies by \(x'_c \succeq x_c\) if \(x_c \subseteq x'_c\). For any \(C' \subseteq C\), we define the order by the product order, that is, \(x'_c \succeq x_C\) if \(x_c \subseteq x'_c\) for every \(c \in C'\).

For any \(x_{-c}\), let \(A_c(x_{-c}) := \{s \in S | c \in Ch_s(F_s(x_C) \cup c)\}\) (recall that \(F_s(x_C) = \{c' \in C | s \in x_{c'}\}\)). This is the set of students who, given strategy profile \(x_{-c}\), will accept an offer from \(c\) if it makes an offer. Given any strategy profile \(x_C\) and \(c \in C\), it is easy to see that

\(^6\)We often write \(x\) for a singleton set \(\{x\}\) for simplicity.
$x'_c$ is a best response of $c$ to $x_{-c}$ if and only if

$$Ch_c(A_c(x_{-c})) \subseteq x'_c \subseteq Ch_c(A_c(x_{-c})) \cup (C \setminus A_c(x_{-c})).$$

For any such $x'_c$, the college $c$ will be matched to the same set $Ch_c(A_c(x_{-c}))$ of students, which is the best possible outcome for her given strategies of other colleges. Let $BR_c(\cdot)$ be the best response correspondence. We say that a function $br_c(\cdot)$ is a best response selection if $br_c(x_{-c}) \in BR_c(x_{-c})$ for all $x_{-c}$. The following lemma shows that there is a sense in which it is without loss to focus on best response selections.

**Lemma 1** Let $br_c$ be an arbitrary best response selection from $BR_c$ for each $c$. The set of matchings at fixed points of $br = (br_c)_{c \in C}$ is identical to the set of matchings at fixed points of $BR = (BR_c)_{c \in C}$.

**Proof.*** to be written. ***■

Substitutability is a condition that is of particular interest for our purposes. In many applications, preferences may violate responsiveness but satisfy substitutability. For instance, it was a common practice in the United Kingdom that a hospital residency program demanded a certain gender balance in the resident composition (Roth, 1991). In the context of school choice, a school (or the school district that sets admission criteria for the schools) may prefer to achieve balance in ethnic distribution (Abdulkadiroğlu and Sonmez, 2003) or in academic achievement (Abdulkadiroğlu, Pathak, and Roth, 2005). In the New York City public school system, for instance, each Educational Option school must allocate 16% of its seats to top performers in a standardized exam, 68% to middle performers, and 16% to bottom performers. Moreover, substitutability is a necessary condition for guaranteeing the non-emptiness of the core.\(^7\) Finally, if preferences are substitutable, then pairwise stable matchings are immune to certain group deviations as well (Echenique and Oviedo, 2004; Klaus and Walzl, 2009). For these reasons, in the sequel, we assume that every agent has substitutable preferences unless stated otherwise.

We define the largest best response $\overline{br} : (2^S)^{|C|} \to (2^S)^{|C|}$ as a best response selection such that, for each $c \in C$ and $x_C$,

$$\overline{br}_c(x_C) = Ch_c(A_c(x_{-c})) \cup (S \setminus A_c(x_{-c})).$$

The following proposition establishes a link between stable matchings and Nash equilibria of our game.

\(^7\)Formally, suppose that there are at least two students $s$ and $s'$, and that a preference of $s$ is not substitutable. Then there exists a responsive preference profile for other agents such that the core is empty. The first version of this result is demonstrated by Sönmez and Ünver (2010) in the context of course allocation. The present statement is shown by Hatfield and Kojima (2008).
**Proposition 1** The following sets of matchings coincide:

1. The set of stable matchings,
2. The set of Nash equilibrium outcomes, and
3. The set of Nash equilibrium outcomes in largest best responses.

**Proof.** Sotomayor (2004) and Echenique and Oviedo (2006) show that the set of stable matchings and the set of subgame perfect equilibrium outcomes coincide in their two-stage games. The equivalence of statements (1) and (2) follows from their results because the set of the Nash equilibrium outcomes in our game clearly coincides with the set of subgame perfect equilibrium outcomes in their games (as noted in a previous remark, their games are essentially equivalent to ours). The equivalence between statements (2) and (3) is immediate from Lemma ***. ■

**Remark 1** The set of Nash equilibrium outcomes in largest strategies may not coincide with the set of stable matchings when student preferences are not substitutable. See Example 6 in Appendix B.

Now we are ready to present the first main result of the paper.

**Theorem 1** The colleges’ final offers game has

(i) Strategic complementarity: That is, the largest best response function \( \overline{br} \) is monotone increasing.

(ii) Negative externality for colleges: That is, for any \( c \in C, x_C \) and \( x'_C \) with \( x'_{-c} \succeq x_{-c} \), if \( x_c = \overline{br}_c(x_C) \) then \( \phi(x_C) R_c \phi(x'_C) \).

(iii) Positive externality for students: That is, for any \( s \in S, x_C \) and \( x'_C \succeq x_C \), \( \phi(x'_C) R_s \phi(x_C) \).

**Proof.** (i) Since \( x'_{-c} \succeq_{-c} x_{-c} \) and colleges have substitutable preferences, we have \( A_c(x'_{-c}) \subseteq A_c(x_{-c}) \). Then, we have

\[
\overline{br}_c(x_C) = Ch_c(A_c(x_{-c})) \cup (S \setminus A_c(x_{-c})) \\
\subseteq [Ch_c(A_c(x_{-c})) \cap A_c(x'_{-c})] \cup (S \setminus A_c(x'_{-c})).
\]
The second line holds because (1) \( Ch_c(A_c(x_{-c})) \subseteq [Ch_c(A_c(x_{-c})) \cap A_c(x'_{-c})] \cup (S \setminus A_c(x'_{-c})) \) and (2) \((S \setminus A_c(x_{-c})) \subseteq (S \setminus A_c(x'_{-c}))\). Now, note further that college \( c \)'s substitutable preferences implies \( Ch_c(A_c(x_{-c})) \cap A_c(x'_{-c}) \subseteq Ch_c(A_c(x'_{-c})) \). This establishes

\[
\overline{br_c(x_C)} = Ch_c(A_c(x_{-c})) \cup (S \setminus A_c(x_{-c})) \\
\subseteq Ch_c(A_c(x'_{-c})) \cup (S \setminus A_c(x'_{-c})) = \overline{br_c(x'_C)}.
\]

(ii) Since student preferences are substitutable, \( x'_C \succeq x_{-c} \) implies \( A_c(x'_{-c}) \subseteq A_c(x_{-c}) \). Thus \( \phi(x_C)(c) = Ch_c(A_c(x_{-c})))R_cCh_c(A_c(x'_{-c})) = \phi(x'_C)(c) \).

(iii) Since \( x'_C \succeq x_C \), \( C'_s \subseteq C_s \), where \( C'_s \) is the set of colleges that apply to \( s \) under \( x'_C \). Thus \( \phi(x'_C)(s) = Ch_s(C'_s)R_sCh_s(C_s) = \phi(x_C)(s) \), completing the proof.

This fact and Result 1 imply the following result.

**Corollary 1 (Existence of a Stable Matching)** In the two-stage game, the set of Nash equilibria in largest strategies forms a non-empty lattice with respect to the partial order \( \succeq \). Therefore the set of stable matchings is nonempty.

A stable matching \( \mu \) is a **student-optimal stable matching** if \( \mu R_s \nu \) for every \( s \in S \) and every stable matching \( \nu \). A stable matching \( \mu \) is a **college-optimal stable matching** if \( \mu R_c \nu \) for every \( c \in C \) and every stable matching \( \nu \). A stable matching \( \mu \) is a **student-pessimal stable matching** if \( \nu R_s \mu \) for every \( s \in S \) and every stable matching \( \nu \). A stable matching \( \mu \) is a **college-pessimal stable matching** if \( \nu R_c \mu \) for every \( c \in C \) and every stable matching \( \nu \). Our analysis of colleges' final offers game can be used to establish notable structural properties of the set of stable matchings, as shown below.

**Corollary 2 (Side-Optimal/Pessimal Stable Matchings)** There exist a student-optimal stable matching and a college-optimal stable matching. The student-optimal stable matching coincides with the college-pessimal stable matching and the college-optimal stable matching coincides with the student-pessimal stable matching.

**Proof.** By Theorem 1, there exist Nash equilibria \( \bar{x}_C \) and \( x_C \) in largest strategies, such that \( \bar{x}_C \succeq x_C \succeq x_C \) for every Nash equilibrium \( x_C \) in largest strategies. By items (ii) and (iii) of Proposition 1, this implies that \( \phi(\bar{x}_C)R_c\phi(x_C)R_c\phi(\bar{x}_C) \) for every \( s \in S \) and \( \phi(\bar{x}_C)R_s\phi(x_C)R_s\phi(\bar{x}_C) \) for every \( c \in C \), completing the proof.

The monotonicity and externality properties as stated in Theorem 1 can be used to show many other representative properties of stable matchings. Examples include comparative statics regarding the addition or removal of some agents on others, the “vacancy chain dynamics” (Blum, Roth, and Rothblum, 1997) and the “rural hospital theorem” (Roth, 1986). Appendix C presents them.
3.1 Necessity of “Overbooking”

Theorem 1 (i) is our key result, which enables us to connect a given matching model to a corresponding non-cooperative game with strategic complementarity. This property in turn enables us to show many results in matching theory. To obtain it, allowing colleges to overbook, i.e., to make offers to more students than those which they have serious interests in, plays a crucial role. To illustrate this point most clearly, in this section let us focus on the so-called many-to-one “college admission” model, in which each student can be matched to at most one college, and every college has responsive preferences with a quota.

Consider a game in which, in contrast to our two-stage game, each college \( c \) makes offers to at most \( q_c \) students, i.e., overbooking is prohibited. For illustrative purposes, consider a special case in which \( q_c = 1 \) for each \( c \). Then the set of strategies is equal to the set of students and the empty set (being unmatched). In this case, perhaps the most natural strategy ordering one may think of is to define applying to a more preferred student as a larger strategy for each college. For example, if a college prefers student \( s_i \) to \( s_j \), then strategy \( s_i \) is said to be larger than \( s_j \) for the college. Unfortunately, the following example shows that this seemingly natural ordering fails to establish strategic complementarity.

**Example 1** Consider a market with two students \( s_1, s_2 \), and two colleges \( c_1, c_2 \), each of which has a quota of one. Assume that all students (resp. colleges) are acceptable to all colleges (resp. students), and that both students prefer \( c_1 \) to \( c_2 \), and both colleges prefer \( s_1 \) to \( s_2 \). Since \( s_1 \) is preferred to \( s_2 \), \( s_1 \) is a larger strategy than \( s_2 \) for each college. Then \( c_2 \)'s best reply function is not increasing with respect to \( c_1 \)'s strategy: To see this, note that \( c_2 \)'s best reply is \( s_1 \) when \( c_1 \) takes the smaller strategy \( s_2 \), while it decreases to \( s_2 \) if \( c_1 \) increases her strategy to \( s_1 \). Therefore, this ordering does not yield increasing best replies.

The above example suggests that it is difficult to endow strategy orderings that yield strategic complementarity. In fact, it turns out that this difficulty is not specific to the above particular ordering, but is due to a general impossibility result. Specifically, the following theorem establishes that Theorem 1 (i) cannot hold with any ordering unless overbooking is allowed.

**Theorem 2** Consider the game in which each college \( c \) is restricted to apply to at most \( q_c \) students. Then there exists a matching problem \((S,C,P)\) such that there exists no profile of partial orderings over colleges’ strategy spaces such that,

1. the ordering induces a lattice for each college’s strategy space,
2. there exists a best reply selection that is monotone increasing with respect to the ordering.\(^8\)

**Proof.** Consider a matching problem with three colleges \(c_1, c_2, c_3\), each with a quota of 1, and two students \(s_1, s_2\). Fix a preference profile of students such that \(c_1 \succ_s c_2 \succ_s c_3 \succ_s \emptyset\) for \(s = s_1, s_2\), and both students are acceptable to all colleges. By condition (1), each college has largest and smallest strategies. This, combined with the fact that there are only three strategies (applying to \(s_1, s_2\), and making offers to no student), implies that \(s_1 \succ_{(c)} s_2\) or \(s_2 \succ_{(c)} s_1\) according to the partial order \(\succ_{(c)}\) over strategies for each college \(c\). Since we have three colleges, there are at least two colleges \(c_i\) and \(c_j\) with \(i, j \in \{1, 2, 3\}, i < j\) such that the ordering \(\succ\) on their strategies over the two students coincide, say \(s_1 \succ s_2\) without loss of generality. Consider strategy profiles where \(c_k\) with \(k \neq i, j\) makes an offer to no student. When \(c_i\) increases its strategy from \(s_2\) to \(s_1\), it is uniquely optimal for \(c_j\) to decrease its strategy strictly from \(s_1\) to \(s_2\). Thus condition (2) is violated, completing the proof. \(\blacksquare\)

### 3.2 Deferred Acceptance Algorithm as a Learning Process

In this section, we investigate the relationship between our non-co-operative game and the celebrated **deferred acceptance algorithm** (Gale and Shapley, 1962). The student-proposing version of the deferred acceptance algorithm is defined as follows.

**Iteration:** At each step \(t \in \{1, 2, \ldots\}\), every student makes offers to her most preferred subset of colleges that have never rejected her at any previous step. Each college tentatively accepts its most preferred subset of students who are applying to it and rejects every other student.

**Termination:** The algorithm terminates in a step at which no rejection occurs. The tentative matches at that step becomes finalized, producing a matching.

The college-proposing deferred acceptance algorithm is defined similarly, by switching the roles of students and colleges.

Note that we have not shown that the deferred acceptance algorithm terminates in a finite number of steps. We will establish that it terminates as a corollary of Theorem 3 below.

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\(^8\)Echenique (2004) adopts a weaker definition of strategic complementarity than ours and shows that many games can be endowed with a partial order on strategy profiles such that the game has strategic complementarity. His result does not have a logical relationship with ours because his definition of strategic complementarity is different from ours.
We will relate the deferred acceptance algorithm to a learning process on our students’ final offers game. For that purpose, we first define the following learning process, called the best response dynamics.

Initialization: Let \( C^1_s \subseteq C \) be given for each \( s \in S \).

Iteration: At each period \( t \geq 1 \), each student \( s \) updates her action to \( C_{s}^{t+1} = \text{br}_s(C^t_S) \).

Termination: The algorithm terminates in the smallest step \( T \) such that \( \text{br}(C^T_S) = C^T_S \).

If there is a finite \( T \) such that the best response dynamics terminates at step \( T \), we say that the dynamics converges in a finite step. Clearly, the dynamics converges at the action profile \( C^T_S \) if and only if it is a Nash equilibrium in largest strategies.

The student-proposing deferred acceptance algorithm is based on an intuitive applications and rejections process, and the adaptive nature of the algorithm – at each step, each student makes the most preferred offers given which colleges have already rejected her – appears to be analogous to the best response dynamics in our students’ final offers game. Thus, one conjecture may be that the student-proposing deferred acceptance algorithm is identical to the best response dynamics with initial state \( C^1_s = \emptyset \) for each \( s \in S \). However, it turns out that these processes do not exactly coincide, as the following example shows.

Example 2 Let \( S = \{s_1, s_2, s_3\} \) and \( C = \{c_1, c_2, c_3\} \). Every player has a responsive preference with quota one, and preferences are

\[
\begin{align*}
&c_1 \succ s_1, c_2 \succ s_1, c_3 \succ s_1, \emptyset, \quad c_1 \succ s_2, c_2 \succ s_2, c_3 \succ s_2, \emptyset, \quad c_2 \succ s_3, c_1 \succ s_3, c_3 \succ s_3, \emptyset, \\
&s_3 \succ c_1, s_1 \succ c_1, s_2 \succ c_1, \emptyset, \quad s_1 \succ c_3, s_3 \succ c_3, s_2 \succ c_3, \emptyset, \quad s_1 \succ c_3, s_3 \succ c_3, s_2 \succ c_3, \emptyset.
\end{align*}
\]

Consider the student-proposing deferred acceptance algorithm. At Step 1, \( s_1 \) and \( s_2 \) apply to \( c_1 \) and \( s_3 \) applies to \( c_2 \), and \( s_2 \) is rejected. At Step 2, \( s_2 \) applies to \( c_2 \), but she is rejected again because \( c_2 \) prefers \( s_3 \) to her. Finally at Step 3, \( s_2 \) applies to \( c_3 \) and is accepted, terminating the algorithm. In the resulting matching, \( s_1, s_2 \) and \( s_3 \) are matched to \( c_1, c_3 \) and \( c_2 \), respectively.

Next let us consider the best response dynamics. At Step 1, \( s_1 \) and \( s_2 \) apply to \( c_1 \) and \( s_3 \) applies to \( c_2 \), and \( s_2 \) is rejected, just as in the deferred acceptance algorithm. In the second step, however, \( s_2 \) makes an offer to \( c_1, c_2 \) and \( c_3 \), and she is rejected by \( c_1 \) and \( c_2 \) while being accepted by \( c_3 \). Thus the application profiles of students are different between the deferred acceptance algorithm and the best response dynamics. Still, the resulting matchings are identical under these processes, matching \( s_1, s_2 \) and \( s_3 \) to \( c_1, c_3 \) and \( c_2 \), respectively.
A close look at the above example reveals several differences between the deferred acceptance algorithm and the best response dynamics. First, students apply to colleges that have already rejected them under the best response dynamics while they do not under the deferred acceptance algorithm. In the above example, student $s_2$ applies to $c_1$ in Step 2 of the best response dynamics but not in the deferred acceptance algorithm.

Second, a student “skips” some applications and immediately applies to a college that accepts her in the best response dynamics, but not in the deferred acceptance algorithm. In the above example, in Step 2 of the best response dynamics student $s_2$ takes into account the fact that her second choice $c_2$ will reject her in favor of $s_3$, so makes an offer to her third choice $c_3$. By contrast, $s_2$ applies only to $c_2$ and gets rejected in Step 2 of the deferred acceptance algorithm, and it is only at Step 3 that she applies to, and gets accepted by, $c_3$.

Despite these differences, note that the two processes result in the same matching. Indeed, this is not a coincidence. Appendix D presents an equivalence between a certain version of the deferred acceptance algorithm and the best response dynamics, which addresses the issues illustrated above.

However, an exact equivalence can be established between the best response dynamics in a students’ final offers game and the college-proposing deferred acceptance algorithm. Consider the best response dynamics with initial state $C_1^s = C$ for every $s \in S$ and focus on the acceptance or rejection choices by colleges. More specifically, let $x_t^c$ be the set of students whose offers are chosen by college $c$ in the students’ final offers game under strategy profile $(C_t^s)_{s \in S}$ – the strategy profile realizing at Step $t$ in the best response dynamics with initial state $C_1^s = C$ for every $s \in S$ – and let $\mu_t^c$ be the set of students that college $c$ makes an offer to in the college-proposing deferred acceptance algorithm at Step $t$. The main result of this section is presented below.

**Theorem 3** $x_t^c = \mu_t^c$ for all $c \in C$ and $t \in \{1, 2, \ldots\}$.

**Proof.** (*** we need to write the proof ***)

**Corollary 3** The college-proposing deferred acceptance algorithm terminates in a finite number $T$ of steps. The resulting matching $\mu^T$ is the college-optimal stable matching.

**Proof.** Since the students’ final offers game is a game with a finite set of strategies and strategic complementarity, the best-response dynamics as described above converges in a finite time $T$ at the largest Nash equilibrium (note that the dynamics starts at the largest strategy profile possible). Since the largest Nash equilibrium results in the college-optimal stable matching (Proposition 2), by Theorem 3 the college-proposing deferred acceptance algorithm terminates in a finite number of steps $T$ and the resulting matching $\mu^T$ is the college-optimal stable matching. ■
3.3 Generalized Threshold Strategies

In this section we consider an alternative class of strategies. Given \( c \in C \) and \( x - c \), the generalized threshold best response of \( c \) to \( x - c \) is defined as

\[
\{ s \in S | s \in Ch_c(A_c(x - c) \cup s) \}.
\]

**Generalized threshold best response function** \( br : (2^S)^{|C|} \rightarrow (2^S)^{|C|} \) is a function such that, for each \( c \in C \), \( br_c(x_C) \) is the generalized best response of \( c \) to \( x - c \), that is,

\[
br_c(x_C) = \{ s \in S | s \in Ch_c(A_c(x - c) \cup s) \},
\]

for every \( c \in C \). The generalized threshold best response is a function under which \( c \) makes an offer to its most preferred available students plus those who the college wants to admit although they won’t accept the college’s offer.

A generalized threshold best response is clearly a best response selection. Thus by Lemma ** and Proposition 1, the set of stable matchings is equivalent to the set of Nash equilibrium outcomes in generalized threshold best responses. Further, the generalized threshold best response function is monotone increasing, as shown in the following proposition.

**Proposition 2** When students and colleges have substitutable preferences, the generalized threshold best response function is monotone increasing.

**Proof.** Suppose \( x'_c \succeq_c x_c \). Since students have substitutable preferences, we have \( A_c(x'_c) \subseteq A_c(x_c) \). By definition, the desired claim that \( br_c(x - c) \subset br_c(x'_c - c) \) is equivalent to the property that \( s \in Ch_c(A_c(x - c) \cup s) \Rightarrow s \in Ch_c(A_c(x'_c - c) \cup s) \). The latter property directly follows from substitutability of \( c \)'s preferences. (Note that \( s \) is available both in the larger set \( A_c(x - c) \cup s \) and the smaller set \( A_c(x'_c - c) \cup s \), and \( s \) is chosen in the larger set. Then \( s \) must also be chosen in the smaller set.)

An important feature of the generalized best response is that it is the smallest selection from best responses that are “always” monotonic in the sense formalized below. Our “always” requirement is important because in a specific game seen in isolation, there can be a monotone selection of best replies that is smaller than the generalized threshold best response. For example, suppose that (i) there are two students \( S = \{ s, s' \} \), (ii) no college is acceptable to \( s' \), and (iii) college \( c \) is the only acceptable college to \( s \). Assume further that \( s'P_c s \) and \( sP_c \emptyset \). Then, a function \( br_c \) defined by \( br_c(x - c) = \{ s \} \) for all \( x - c \) is a best response selection, monotone increasing (because it is a constant function), and this is smaller than the generalized threshold best response \( br_c(x - c) = \{ s, s' \} \) (again a constant function).


To formally define the smallest selection (in the sense of set inclusion) of best replies which is “always” monotonic, we look at a generally applicable selection criterion of a best reply which relies only on the essential features of the game. Denote by $A_c(x_{-c}; P_S)$ the set of students that would accept $c$, given other colleges’ offers $x_{-c}$ and student preferences $P_S$. The set of best responses of $c$ can be determined by $A_c(x_{-c}; P_S)$ and $c$’s own preferences $P_c$.\footnote{Note that $Ch_c(A_c(x_{-c}; P_S))$ is the optimal set of students for $c$, given other colleges’ offers $x_{-c}$ and student preferences $P_S$. Obviously this is a best reply. Since $c$ does not mind adding any offer that is going to be rejected to this optimal set, adding such students provides another best reply. The set of all best replies, or the best reply correspondence is given by $BR_c(x_{-c}) = \{Ch_c(A_c(x_{-c}; P_S)) \cup X \mid X \subset C \setminus A_c(x_{-c}; P_S)\}$.} We restrict our attention to a selection of best reply that only depends on those two essential data. For each matching problem $\Gamma = (S, C, P)$, a **best response selection** specifies a best reply (of the colleges’ final offer game) for each college $c$ as $br_c(x_{-c} \mid \Gamma)$. A best response selection is said to be **essential** if $br_c(x_{-c} \mid \Gamma)$ depends only on the essential aspects of the game, $A_c(x_{-c}; P_S)$ and $P_c$. In other words, a best response selection is essential if there is one function $\rho_c$ such that, for all matching problem $\Gamma$, $br_c(x_{-c} \mid \Gamma) = \rho_c(A_c(x_{-c}; P_S), P_c)$. The generalized threshold best response function corresponds to essential best response selection $\rho_c(A_c(x_{-c}; P_S), P_c) \equiv \{s \in S \mid s \in Ch_c(A_c(x_{-c}; P_S) \cup s)\}$.

The next proposition demonstrates that the generalized threshold best response function is the smallest essential best response selection among those that are monotone increasing.

**Theorem 4** Assume $|C| \geq 2$. The generalized threshold best response function is the smallest essential best response selection that is monotone increasing in all two-stage games where all agents have substitutable preferences. That is, if an essential best response selection $\rho_c(A_c(x_{-c}; P_S), P_c)$ is monotone increasing, then $\rho_c(A_c(x_{-c}; P_S), P_c) \subseteq \rho_c(A_c(x_{-c}; P_S), P_c)$ for every $S, C, x_{-c}$, and $P$ with $P_i$ substitutable for all $i \in S \cup C$.

**Proof.** The generalized threshold best response function is associated with essential best response selection $\rho_c(A_c(x_{-c}; P_S), P_c) \equiv \{s \in S \mid s \in Ch_c(A_c(x_{-c}; P_S) \cup s)\}$, and Proposition 2 establishes that it specifies a monotone increasing best reply. Hence we only need to show that, for any essential best response selection $\rho_c(A_c(x_{-c}; P_S), P_c)$ that is monotone increasing,

\[
\rho_c(A_c(x_{-c}; P_S), P_c) \subseteq \rho_c(A_c(x_{-c}; P_S), P_c)
\]

for every $S, C, P$ with $P_i$ substitutable for all $i \in S \cup C$.

Suppose this condition is violated so that there are $x_{-c}$, $P_c$, $P_S$ and $s' \in C$ such that $s' \in Ch_c(A_c(x_{-c}; P_S) \cup s')$ and $s' \notin \rho_c(A_c(x_{-c}; P_S), P_c)$. Note that $s'$ cannot be the choice of $c$ in the available students to it in $A_c$, that is, $s' \notin Ch_c(A_c(x_{-c}; P_S))$. This is because any best response (thus $\rho_c(A_c(x_{-c}; P_S), P_c)$ in particular) must contain the best attainable
students for $c$, $Ch_c(A_c(x_{-c}; P_S))$. Also note that, as already mentioned, any best response should be a subset of $Ch_c(A_c(x_{-c}; P_S)) \cup (C \backslash A_c(x_{-c}; P_S))$. Hence $s'$ must be a student who rejects $c$, that is, $s' \notin A_c(x_{-c}; P_S)$. Then we show that the essential best response selection $\rho_c$ does not specify a monotone increasing best reply in some game.

**Case 1:** Assume $c \in Ch_s({c})$. In that case consider $x'_{-c}$ defined by $x'_{-c} = x_{-c} \backslash \{s'\}$ for every $c' \neq c$. Then $x_{-c} \succeq_{-c} x'_{-c}$ and $A_c(x'_{-c}; P_S) = A_c(x_{-c}; P_S) \cup s'$. Our premise that $s'$ is in the generalized threshold best reply, namely $s' \in Ch_c(A_c(x_{-c}; P_S) \cup s')$, implies $s' \in Ch_c(A_c(x'_c; P_S))$. Hence $s'$ must be in any best response against $x'_{-c}$, and in particular $s' \in \rho_c(A_c(x'_c; P_S), P_c)$. This contradicts our premise that $s' \notin \rho_c(A_c(x_{-c}; P_S), P_c)$ and monotonicity of the best reply.

**Case 2:** Assume $c \notin Ch_s({c})$. Let $P'_s$ be a responsive preference relation with quota one such that $c'P'_sc$ for every $c' \neq c$ and $cP'_s\emptyset$, and let $P''_S = (P'_s, P_S \setminus \{s'\})$. Also define $x'_{-c}$ by $x'_{-c} = x_{-c} \cup \{s'\}$ for every $c' \neq c$. Then $A_c(x_{-s}; P_S) = A_c(x'_{-c}; P'_S)$, so $s' \notin \rho_c(A_c(x'_{-c}; P'_S), P_c) = \rho_c(A_c(x_{-c}; P_S), P_c)$. Now define $x''_{-c}$ by $x''_{-c} = x'_{-c} \backslash \{s'\}$ for every $c' \neq c$. Then $x'_{-c} \succeq_{-c} x''_{-c}$ and $A_c(x''_{-c}; P''_S) = A_c(x'_{-c}; P'_S) \cup s'$. Our premise that $s'$ is in the generalized threshold best reply, namely $s' \in Ch_c(A_c(x'_{-c}; P'_S) \cup s')$, implies $s' \in Ch_c(A_c(x''_{-c}; P''_S))$. Hence $s'$ must be in any best response against $x''_{-c}$, and in particular $s' \in \rho_c(A_c(x''_{-c}; P''_S), P'')$. This contradicts $s' \notin \rho_c(A_c(x'_{-c}; P'_S), P_c)$ and monotonicity of the best reply. ■

We also consider the smallest best response selection when we fix a game, that is, players and their preferences. Let $S(c)$ be the set of students who never accept college $c$ under any strategy profile, that is,

$$S(c) = \{s \in S | c \notin Ch_s(C'), \forall C' \subseteq C\}.$$

For each $c \in C$, the **modified threshold best response function** $br^*_c(\cdot)$ is defined by

$$br^*_c(x_c) = br_c(x_c) \setminus S(c).$$

The modified threshold best response simply deletes from the generalized best response all students that do not choose $c$ in any instance. When a student has substitutable preferences (as assumed in this paper), $c$ is never chosen from any set of available colleges if and only if $c \notin Ch_s({c})$. Therefore,

$$br^*_c(x_c) = br_c(x_c) \setminus \{s \in S | c \notin Ch_s({c})\}.$$

In other words, the modified threshold best response function is associated with the essential best response selection $\rho_c$ defined by $\rho^*_c(A_c, P_c) \equiv \{s \in S | s \in Ch_c(A_c \cup s) \setminus S(c) \notin \}$.
$\text{Ch}_s(\{c\})$ for all $A_c$ and $P_c$.

**Theorem 5** Fix a game $\Gamma = (S, C, P)$. The modified threshold best response function is the smallest essential best response selection that is monotone increasing. That is, if an essential best response selection $\rho_c(A_c(x_{-c}; P_S), P_c)$ is monotone increasing, then $\rho_c^*(A_c(x_{-c}; P_S), P_c) \subseteq \rho_c(A_c(x_{-c}; P_S), P_c)$ for every $x_{-c}$.

**Proof.** The modified threshold best response function is associated with the essential best response selection $\rho_c^*(A_c, P_c) \equiv \{s \in S|s \in \text{Ch}_c(A_c \cup s)\} \setminus \{s \in S|c \notin \text{Ch}_s(\{c\})\}$, and Proposition 2 establishes that it specifies a monotone increasing best reply. Hence we only need to show that, if an essential best response selection $\rho_c$ specifies a monotone increasing best reply, then it is larger than the modified threshold best response selection: $\{s \in S|s \in \text{Ch}_c(A_c(x_{-c}; P_S) \cup s)\} \setminus \{s \in S|c \notin \text{Ch}_s(\{c\})\} \subseteq \rho_c(A_c(x_{-c}; P_S), P_c)$ for all $x_{-c}$. Suppose for contradiction that there exist $s' \in S$ and $x_{-c}$ such that $s' \in \text{Ch}_c(A_c(x_{-c}; P_S) \cup s')$ and $c \in \text{Ch}_s(\{c\})$ while $s' \notin \rho_c(A_c(x_{-c}; P_S), P_c)$. Note that $s'$ cannot be in the choice by $c$ from the available students $A_c(x_{-c}; P_S)$: $s' \notin \text{Ch}_c(A_c(x_{-c}; P_S))$. This is because any best response (and hence $\rho_c(A_c(x_{-c}; P_S), P_c)$ in particular) must contain the best attainable students for $c$, $\text{Ch}_c(A_c(x_{-c}; P_S))$. Hence $s'$ must be a student who rejects $c$, that is, $s' \notin A_c(x_{-c}; P_S)$. Then we show that the essential best response selection $\rho_c$ does not provide a monotone increasing best reply. Recall that, when colleges have substitutable preferences, $A_c(x_{-c}; P_S)$ is monotone decreasing. Since $c \in \text{Ch}_s(\{c\})$ by assumption, we can find $x_{-c} \succeq x'_{-c}$ such that $A_c(x'_{-c}; P_S) = A_c(x_{-c}; P_S) \cup s'$. Our premise that $s'$ is in the modified threshold best reply, namely $s' \in \text{Ch}_c(A_c(x_{-c}; P_S) \cup s')$, implies $s' \in \text{Ch}_c(A_c(x'_{-c}; P_S))$. Hence $s'$ must be in any best response against $x'_{-c}$, and in particular $s' \in \rho_c(A_c(x'_{-c}; P_S), P_c)$. This contradicts our premise that $s' \notin \rho_c(A_c(x_{-c}; P_S), P_c)$, which completes the proof. ■

More generally we present, without a proof, a characterization of the set of essential best response selections in a fixed game. An essential best response selection is monotone if and only if

$$\text{br}_c^*(x_C) \cup D(x_{-c}),$$

where $D(x_{-c})$ is a monotone increasing selection (with respect to $x_{-c}$) from $S(x_C)$. As the largest best response and the generalized (modified) threshold best response are the largest and smallest best responses that are monotone increasing, respectively, these results characterize the set of monotone increasing best responses.

All previous conclusions, such as Theorems 1, 2 and 3, can be shown based on generalized threshold best responses instead of largest best responses. The proofs are omitted because they are almost identical to those with largest best responses.
One potential advantage of generalized threshold best responses is its simplicity. Suppose that college $c$ has responsive preferences. Then any generalized threshold strategy is of the form $\{s \in S | sR_c s\}$ for some “threshold student” $s \in S$ – this is why we use the term “generalized threshold best response.” Moreover, any pair of a college’s generalized threshold best responses can be ordered since

$$\{s \in S | sR_c s\} \succeq \{s \in S | sR_c s'\} \iff s' R_c s,$$

for any $s, s' \in S \cup \{\emptyset\}$. Thus when colleges have responsive preferences, one can restrict attention to one dimensional strategies of (generalized) threshold best responses. Having a one-dimensional best responses for each player makes the analysis simple, and enables us to show further results. For example, adopting the proof by Takahashi and Yamamori (2008), with one-dimensional best responses and increasing best response, one can show that the best response dynamics with any initial strategy profile converges in finite time to a Nash equilibrium.

4 Monotone Methods for Stable Matching

To our knowledge, the current paper is the first work that establishes the connection between two-sided matching and games with strategic complementarity. However, mathematical structures similar to ours have been recognized and investigated in recent works. A pioneering work by Adachi (2000) investigates one-to-one matching markets and finds that the space of “prematchings” allows for an increasing function, and that the set of fixed points of that increasing function corresponds to the set of stable matchings in his domain. His method has been extended to many-to-one matching (Echenique and Oviedo, 2004), many-to-many matching (Fleiner, 2003; Echenique and Oviedo, 2006), matching with contracts (Hatfield and Milgrom, 2005; Hatfield and Kominers, 2009), and supply-chain networks (Ostrovsky, 2008; Hatfield and Kominers, 2010). This section studies the relationship between our method and these algorithmic approaches in the literature.

To study the connection with these alternative approaches in more detail, some mathematical preliminary is in order. For arbitrary sets $X_1$ and $X_2$, consider the solutions to a system of two equations, $G_1: X_2 \to X_1$ and $G_2: X_1 \to X_2$:

$$\begin{cases} x_1 = G_1(x_2) \\ x_2 = G_2(x_1) \end{cases},$$

The published version by Kukushkin, Takahashi, and Yamamori (2005) uses a different proof approach and hence cannot be invoked here.
or equivalently, the fixed points of mapping $G(x_1, x_2) := (G_1(x_2), G_2(x_1))$. The next (trivial) lemma shows that we can focus on one of the elements, $x_1$ or $x_2$, to find the fixed points.

**Lemma 2** (Composition Lemma): Define a composite mapping associated with $G$ by $H_i(x_i) \equiv G_i(G_j(x_i)), i \neq j$. The fixed points of $H_i$ is isomorphic to the fixed points of $G$, for $i = 1, 2$. More precisely, for $i \in \{1, 2\}$,

$$(x^*_1, x^*_2) = G(x^*_1, x^*_2) \iff x^*_i = H_i(x^*_i) \text{ and } x^*_j = G_j(x^*_i).$$

**Proof.** Consider the case $i = 1$. Obviously,

$$\begin{cases}
    x^*_1 = G_1(x^*_2) \\
    x^*_2 = G_2(x^*_1)
\end{cases} \iff \begin{cases}
    x^*_1 = G_1(G_2(x^*_1)) = H_1(x^*_1) \\
    x^*_2 = G_2(x^*_1)
\end{cases}.$$

The case for $i = 2$ follows from a symmetric argument. ■

Now we present the mappings employed by the existing literature, whose fixed points coincide with stable matchings. We consider many-to-many matching (without contract). We call one side students and the other side colleges as before, and denote the set of students and colleges by $S$ and $C$, respectively. The literature considers an object called a **prematching**, which is a profile $(x_i)_{i \in S \cup C}$ where $x_s \subset C$ for each $s \in S$ and $x_c \subset S$ for each $c \in S$. Following the literature cited above, we consider the following mappings.

1. Hatfield-Milgrom mapping HM:

$$\begin{cases}
    H_{M_s}(x_C) \equiv C \setminus r_s(x_C) \text{ for all } s \in S, \\
    H_{M_c}(x_S) \equiv S \setminus r_c(x_S) \text{ for all } c \in C,
\end{cases}$$

where $r_s(x_C)$ is the set of rejected offers by student $s$. More precisely, let $AC_s(x_C) := \{c \in C | s \in x_c\}$ be the set of colleges which are making an offer to $s$ at colleges’ offer profile $x_C$. Let $R_s(\cdot)$ denote the rejected set function of student $s$ ($R_s(Y) = Y \setminus Ch_s(Y)$, where $Ch_s$ is student $s$’s chosen set function). Then, $r_s(x_C) \equiv R_s(AC_s(x_C))$. A similar definition applies to $r_c$. Let $H_{M_s}(x_C) = (HM_{s_s}(x_C))_{s \in S}$ and $H_{M_c}(x_S) = (HM_{c_s}(x_S))_{c \in C}$ and $H_M = (HM_s, HM_c)$.\(^{11}\)

\(^{11}\)To be precise, their model is different in two ways from ours: first, they allow for more than one possible contract term for each doctor-hospital pair. Second, they assume that each doctor can sign at most one contract. Since the adaptation of their model to our situation seems to be straightforward, however, we ignore these differences and refer to the Hatfield-Milgrom model as the adaptation of their model to many-to-many matching without contracts. Also, Hatfield and Milgrom (2005) consider mapping
2. The mapping $T$ (Ostrovsky, 2008; Adachi, 2000; Echenique and Oviedo, 2004, 2006)\(^{12}\)

\[
\begin{align*}
T_s(x_C) &\equiv \{c \in C | c \in Ch_s(AC_s(x_C) \cup c)\} \text{ for all } s \in S, \\
T_c(x_S) &\equiv \{s \in S | s \in Ch_c(AC_c(x_S) \cup s)\} \text{ for all } c \in C,
\end{align*}
\]

where $AC_i$ is defined in item 1 above. Let $T_S(x_C) = (T_s(x_C))_{s \in S}$, $T_C(x_S) = (T_c(x_S))_{c \in C}$, and $T = (T_S, T_C)$.

Hatfield and Milgrom (2005) and Echenique and Oviedo (2006) showed that the set of all fixed points of those mappings coincide with the set of all stable matchings. More precisely,

1. If $(x_s^*, x_C^*)$ is a fixed point of $HM$ or $T$, then a stable matching is obtained by matching every pair of a student and a college who are “making an offer to each other” at $(x_s^*, x_C^*)$, that is, the matching where $s$ and $c$ are matched with each other if and only if $s \in x_C^*$ and $c \in x_s^*$ is stable.

2. Conversely, any stable matching can be obtained by a fixed point of $HM$ or $T$.\(^{13}\) That is, for any stable matching $\mu$, there exists a fixed point $(x_S^*, x_C^*)$ of their mappings such that $\mu$ results from the procedure described in item 1.

Now, we show that those mappings can be interpreted as the best reply functions in the games we have constructed. We first consider our simultaneous demand games.

**Theorem 6** Hatfield-Milgrom mapping $HM$ coincides with the best reply function of the general offer demand game. When preferences are substitutable, mapping $T$ is the best reply mapping of the threshold demand game, that is, $T_s(x_C) = br_s(x)$ for every $s \in S$ and $T_c(x_S) = br_c(x)$ for every $c \in C$ where $br_i$ is the best reply function (in terms of the offers made) of agent $i$ in the threshold demand game.

**Proof.** Consider the general offer demand game and let $x = (x_S, x_C)$ be a given strategy profile. Then, by the lexicographic preferences as imposed in the definition of the game, a best response of an arbitrary student $s \in S$ is

\[
Ch_s(AC_s(x_C)) \cup [C \setminus AC_s(x_C)] = C \setminus r_s(x_C).
\]

\(^{(HM_C(x_S), HMS(HM_C(x_S)))}\) (which corresponds to their mapping $F(X_D, X_H)$), but this has the same set of fixed points as $HM$, as the composition lemma shows.

\(^{12}\)Ostrovsky (2008) considers a model in which there is a finite partially ordered set of agents and contracts are incorporated. His function reduces to the mapping $T$ here when we focus on many-to-many matching without contracts.

\(^{13}\)The fixed points of $HM$ and $T$ are generally different, but the matchings obtained by the fixed point offer profiles are the same for $HM$ and $T$. 

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Similarly, the best response of an arbitrary college \(c \in C\) is

\[
Ch_c(AC_c(x_S)) \cup [S \setminus AC_c(x_S)] = S \setminus r_c(x_S).
\]

Therefore the best reply mapping of the general offer game is identical to mapping \(HM\).

Next, consider the threshold demand game and let \(x = (x_S, x_C)\) be a given strategy profile. Then, by the lexicographic preferences as imposed in the definition of the game, a best response of an arbitrary student \(s \in S\) is \(br_s(x) = Ch_s(AC_s(x_c))\). Therefore

\[
br_s(x) = \{c \in C | c \in Ch_s(AC_s(x_C)) \cup c\}.
\]

Since preferences of \(s\) is substitutable by assumption, \(Ch_s(Ch_s(AC_s(x_c)) \cup c) = Ch_s(AC_s(x_C) \cup c)\) for any \(c \in C\). Therefore we obtain

\[
br_s(x) = \{c \in C | c \in Ch_s(AC_s(x_C) \cup c)\} = T_s(x_C).
\]

A symmetric argument establishes

\[
br_c(x) = T_c(x_S),
\]

establishing that mapping \(T\) coincides with the best response mapping in the threshold demand game. ■

Next we consider the students’ final offers game, where students make offers first and then colleges respond. Recall that \(br\) is the largest best reply function in the students’ final offers game.

**Theorem 7** The largest best reply function of a students’ final offers game is the composite mapping associated with a mixture of Hatfield-Milgrom and \(T\) mappings. More precisely,

\[
\overline{br}(x_S) = HM_S(T_C(x_S)).
\]

By Theorem 7, together with Proposition 1 before, we obtain the following corollary.

**Corollary 4** The set of matchings obtained by the fixed points of the mapping

\[
\left\{ \begin{array}{l}
x'_s = HM_S(x_C) \\
x'_C = T_C(x_S)
\end{array} \right.,
\]

(1)
coincides with the set of stable matchings. More precisely, if \((x^*_S, x^*_C)\) is a fixed point of mapping (2), then a matching where \(s\) and \(c\) are matched with each other if and only if \(s \in x^*_c\) and \(c \in x^*_s\) is stable, and any stable matching is obtained by this procedure from a fixed point of mapping (2).

Remark 2 By the composition lemma, the fixed points of the largest best reply mapping of a students’ final offers game corresponds to the set of fixed points of the mixed mapping (1). Intuitively, in this game, student \(s\)’s largest best reply to other students’ offers are calculated in two steps. The first step is to determine which college accepts \(s\), given other students’ offers. A college would accept \(s\) if and only if \(s\) is a welcome addition to the given offers by other students. Hence this part is quite similar to mapping \(T_C\). (There is, however, a minor subtlety here. See the proof below.) The second step is to calculate the largest best reply. The largest best reply of student \(s\) consists of (i) the best subset of colleges that accept \(s\) and (ii) all colleges that reject \(s\). This part is exactly equal to mapping \(HM_S\). The proof below makes this statement in a rigorous way.

Proof of Theorem 7. Recall that \(\overline{br}_s(x_{-s}) = C \setminus R_s(A_s(x_{-s}))\), where \(R_s\) is the rejected set function of student \(s\) and \(A_s(x_{-s})\) is the set of colleges which would accept \(s\), given other students’ offers \(x_{-s}\) in the students’ final offers game. Since \(HM_s(T_C(x_S)) = C \setminus r_s(T_C(x_S)) = C \setminus R_s(AC_s(T_C(x_S)))\), we need to show

\[
A_s(x_{-s}) = AC_s(T_C(x_S)). \tag{2}
\]

By the definition of \(A_s(x_{-s})\), college \(c\) is in \(A_s(x_{-s})\) if and only if

\[
s \in Ch_c(AC_c(\emptyset, x_{-s}) \cup s), \tag{3}
\]

where \((\emptyset, x_{-s})\) denote a profile where student \(s\) is making an empty set offer and others are making offers \(x_{-s}\). Note that \(AC_c(\emptyset, x_{-s})\) is simply the set of students who are making an offer to \(c\) under \(x_{-s}\).

In contrast, the right hand side of the desired equality (2) is the set of colleges which are making offers under \(T_C(x_S)\). By definition, college \(c\) is making an offer to \(s\) under \(T_C(x_S)\) if and only if

\[
s \in Ch_c(AC_c(x_S) \cup s). \tag{4}
\]

14The same claim also applies to the following mapping,

\[
\begin{align*}
x'_S &= T_S(x_C) \\
x'_C &= HM_C(x_S)
\end{align*}
\]

by symmetry.
Hence, we are done if we show (3) and (4) are equivalent. When $s \in AC_c(x_S)$, the right hand sides of (3) and (4) are both equal to $Ch_c(AC_c(x_S))$. When $s \notin AC_c(x_S)$, the right hand sides of (3) and (4) are both equal to $Ch_c(AC_c(x_S) \cup s)$. Hence, (3) and (4) are equivalent and the proof is complete. ■

We now examine the threshold best reply mapping in the students’ final offers game. This corresponds to mapping $T$, as in the threshold demand game.

**Theorem 8** The threshold best reply functions of a students’ final offers game is the composite mapping associated with mapping $T$. More precisely: Let $br(x_S) = (br_s(x_{-s}))_{s \in S}$ be the threshold best reply function (in terms of the actual offers made) in the students’ final offers game. Then,

$$br(x_S) = T_S(T_C(x_S)).$$

The proof is similar to the previous one and therefore we only provide a sketch. The first step to determine the best reply is the same as in the previous proof. That is, the set of colleges which accept $s$ given other students’ offers is given by mapping $T_C$. Then students apply threshold best reply $T_S$, and the end result is the threshold best reply in the students’ final offers game.

This section has shown that our two-stage game shares a mathematical structure very similar to those in studies in monotone methods in matching. However, the connection is somewhat indirect because best response selections correspond to a composition of the monotone mappings in those studies. This motivates our analysis in the next section, where we present an alternative game in which there is a direct connection between best response selections and the monotone functions in existing studies.

### 5 Simultaneous “Demand” Games

This section introduces a simultaneous game. There are two main motivations for studying a simultaneous game rather than a two-stage game. First, by considering a simultaneous game, we can make the connection between the existing mappings in the matching literature and noncooperative games precise. More specifically, we can interpret mapping HM (Hatfield and Milgrom, 2005) and the so-called T-mapping (Echenique and Oviedo, 2006) as specific selections from the best reply correspondence. Thus we can obtain a clearer noncooperative interpretation of the HM-mapping or the T-mapping. Second, it turns out that a connection between the “matching with contracts” model and a noncooperative game can be established for the simultaneous game, but not for the two-step final offers game.
To make these points precise, we begin this section by introducing a matching model with contracts in the next subsection, and then analyze our simultaneous game in subsequent subsections.

5.1 A Model of Matching with Contracts

A matching problem with contracts is described as a tuple \((S, C, Z, P)\). Finite sets \(S\), \(C\), and \(Z\) are the sets of students, colleges, and contracts, respectively. Each contract \(z \in Z\) is associated with one student and one college, denoted \(z_s \in S\) and \(z_c \in C\) respectively.

An allocation is a subset of contracts \(Z' \subseteq Z\).

We allow each agent to sign multiple contracts and assume that each agent \(i \in S \cup C\) has a strict preference relation \(P_i\) on the set of subsets of contracts involving it. For any \(Z' \subseteq Z\), and \(s \in S\), define \(\text{Ch}_s(Z')\) as a set \(Z''\) such that (i) \(Z'' \subseteq Z'\), (ii) \(z \in Z'' \Rightarrow z_s = s\), and (iii) \(Z''P_sZ''\) for any \(Z'' \neq Z''\) that satisfies (i) and (ii). Let \(\text{Ch}_s(Z') = \bigcup_{z \in Z'} \text{Ch}_z(Z')\) be the set of contracts chosen from \(Z'\) by some student. Symmetric notations are defined for colleges. The rejection function for agent \(i\) is defined by \(r_i(Z') = Z' \setminus \text{Ch}_i(Z')\).

We write \(P_S = (P_s)_{s \in S}\) to denote a preference profile of students. We also write \(P_{-s}\) to denote \((P_s')_{s \in S \setminus \{s\}}\) for \(s \in S\), and \(P_{S'}\) to denote \((P_s)_{s \in S'}\) and \(P_{-S'}\) to denote \((P_s)_{s \in S \setminus S'}\) for \(S' \subset S\). Preference relations are extended to allocations in a natural way. For example, for two allocations \(Z', Z'' \subset Z\), we write \(Z' >_s Z''\) to mean \(\{z \in Z'|z_s = s\} >_s \{z \in Z''|z_s = s\}\). Similar notation will be used for colleges as long as there is no confusion. For a set of contracts \(Z'\), we denote \(Z'_s = \bigcup_{z \in Z'} \{z_s\}\) and \(Z_c = \bigcup_{z \in Z'} \{z_c\}\).

**Definition 1** A set of contracts \(Z' \subseteq Z\) is **stable** if

1. \(\text{Ch}_s(Z') = \text{Ch}_c(Z') = Z'\), and
2. there exists no set of contracts \(Z'' \not\subseteq Z'\) such that \(Z'' \subseteq \text{Ch}_s(Z' \cup Z'') \cap \text{Ch}_c(Z' \cup Z'')\).

When condition (2) is violated by some \(Z''\), we say that \(Z''\) blocks \(Z'\) or \(Z''\) is a block of \(Z'\).

A natural extension of substitutability from the standard matching model of Gale and Shapley (1962) to matching with contracts is to simply let hospital preferences be substitutable over contracts instead of over doctors. This is the approach employed by Hatfield and Milgrom (2005).

**Definition 2** Contracts are **substitutes** for \(i\) if there does not exist contracts \(z, z' \in Z\) and a set of contracts \(Z' \subseteq Z\) such that \(z \notin \text{Ch}_i(Z' \cup \{z\})\) and \(z \in \text{Ch}_i(Z' \cup \{z, z'\})\).
In other words, contracts are substitutes if the addition of a contract to the choice set never induces a hospital to take a contract it previously rejected. Hatfield and Milgrom (2005) show that there exists a stable allocation when contracts are substitutes for every hospital for many-to-one matching with contracts, and the result is extended to the current setting of many-to-many matching with contracts by Hatfield and Kominers (2009).

The model of matching with contracts subsumes a wide variety of special cases. The standard matching model a la Gale and Shapley (1962) is a case in which there is a fixed term of contract for each doctor-hospital pair, that is, \( z_D = z'_D \) and \( z_H = z'_H \) imply \( z = z' \). In addition, it includes (a discretized version of) the labor matching model with wages by Kelso and Crawford (1982) and combinatorial auction by Ausubel and Milgrom (2002) as special cases. Exchange economy with indivisible objects (Gul and Stachetti 1999) is also a special case of matching with contracts (again, with appropriate discretization). The model has also found a number of applications in practical market design problems, such as the allocation of military personnel (Sonmez and Switzer 2012, Sonmez 2012) and the allocation of residents with distributional constraints (Kamada and Kojima 2011).

Below is a particularly simple example of matching with contracts involving a bargaining between two players.

**Example 3** Let there be one student \( s \) and one college \( c \). There are two contracts \( z^H \) and \( z^L \) both involving \( s \) and \( c \) (that is, \( z^H_S = z^L_S = s, z^H_C = z^L_C = c \)). Each agent wants to sign at most one contract, and preferences are given by \( P_s : z^L, z^H \) and \( P_c : z^H, z^L \). One interpretation in the college admission context is that \( z^L \) is a contract in which \( s \) is admitted to \( c \) with low tuition while \( z^H \) represents admission with high tuition. Note that contracts are clearly substitutes because each agent wishes to sign at most one contract. In this problem, clearly there are two stable allocations, \( \{ z^L \} \) and \( \{ z^H \} \).

As simple this example is, it highlights a number of features that holds more generally. First, there exist stable allocations; in the current example, both high and low tuitions correspond to stable outcomes. Furthermore, the set of these prices has a lattice structure; in particular, there exist lowest and highest tuitions, corresponding to the college-optimal and student-optimal stable allocations. Such structures have been found in various situations, such as the standard two-sided matching (Gale and Shapley 1962), a labor market with heterogeneous workers and wages (Kelso and Crawford 1982), and an exchange economy with indivisible objects (Gul and Stacchetti 1999). In fact, the set of stable allocations forms a nonempty lattice more generally, for any matching model with contracts in which contracts are substitutes. This fact motivates our investigation to provide an interpretation of the matching with contracts model in terms of a non-cooperative game with strategic complementarity.
5.2 Best Reply Selection and Matching with Contracts

The goal of this study is to understand stable matchings as Nash equilibria of a game with increasing best reply, with the help of an appropriate best reply selection. The two-step game presented in the previous sections served this purpose for the matching model with fixed terms of contracts as in Gale and Shapley (1962), as we have seen before. The goal of this section is to provide a one-step, simultaneous game which does the same for the matching model with contracts.

One motivation to consider one-step games is that considering contracts poses a difficulty in our search of a game that implements stable allocations. More specifically, the set of equilibrium outcomes in the two-stage game no longer coincides with the set of stable allocations, as shown in the following example.

**Example 4** Consider the market of Example 3, where there are one student $s$, one college $c$, and two contracts $z^H$ and $z^L$ corresponding to admissions with high and low tuitions. In that problem, recall that there are two stable allocations, $\{z^L\}$ and $\{z^H\}$. However, in the unique equilibrium of the two-step noncooperative game, college $c$ offers $z^H$ and student $s$ accepts it. Thus the set of equilibrium outcomes is a proper subset of the set of stable allocations, showing that the equivalence result of Proposition *** no longer holds for matching with contracts.

Note that the noncooperative game in this example is a simplified version of the ultimatum game. The generality of matching with contracts model allows one to treat such bilateral bargaining problems in the unified framework that also generalizes two-sided matching, but this also introduces difficulty; the kind of strategic consideration related to the timing of moves found in situations like the ultimatum game emerges, making the non-cooperative interpretation of the model in terms of the two-step game impossible.

To recover a connection between stability and non-cooperative equilibrium, we consider a simultaneous-move, one-step game different from the two-step game of the previous sections. In the one-step game, the set of players is $S \cup C$. Every player $i$ simultaneously announces a set $x_i$ of contracts involving that player, that is, $x_i \subseteq \{z \in Z | z_S = i\}$ if $i \in S$ and $x_i \subseteq \{z \in Z | z_C = i\}$ if $i \in C$. A contract $z$ is signed as a result of this game if and only if $z \in x_s \cap x_c$, where $s = z_S$ and $c = z_C$. The utility of each player in this game is given by the utility of the set of contracts that she signs as a result of this game.

Note that as in the two-step game, our one-shot demand game allows players to engage in “overbooking,” that is, offering more contracts than they actually want to sign. Recall that, for two-step games, we showed that there is a sense in which overbooking is necessary for us to connect stable matching with a non-cooperative game with strategic complementarity.
In a similar spirit to that claim, we will now show that overbooking is necessary in the one-step games of this section.

To illustrate this point most clearly, let us focus on a model in which every agent $i$ has responsive preferences with a quota $q_i$. Consider a game in which, in contrast to our one-stage games, each agent $i$ offers at most $q_i$ contracts, that is, overbooking is prohibited.

**Theorem 9** Consider the one-step game in which each agent $i$ is restricted to make at most $q_i$ contracts. Then there exists a matching problem $(S, C, Z, P)$ such that there exists no profile of partial orderings over agents’ strategy spaces such that,

1. the ordering induces a lattice for each agent’s strategy space,
2. there exists a best reply selection $br$ such that

   (a) the best reply selection $br$ is monotone increasing with respect to the ordering, and
   (b) the set of the fixed points of the best reply selection $br$ coincides with the set of stable allocations.

**Proof.** Consider the following problem: There are two colleges $c_1$ and $c_2$, two students $s_1$ and $s_2$, and four contracts $z^{11}, z^{12}, z^{21}, z^{22}$, where $z_{S}^{ij} = s_i, z_{C}^{ij} = c_j$ for each $i, j \in \{1, 2\}$. Each agent prefers signing no contract to signing two contracts (in other words, each agent has single-unit demand), and agent preferences are given by:

\[
\begin{align*}
P_{s_1} &: z^{11}, z^{12}, \\
P_{s_2} &: z^{21}, z^{22}, \\
P_{c_1} &: z^{11}, z^{21}, \\
P_{c_2} &: z^{21}, z^{22}.
\end{align*}
\]

Then clearly $\{z^{11}, z^{22}\}$ is the unique stable allocation.

Consider the following strategy profile:

\[
\begin{align*}
x_{s_1} &= z^{12}, \\
x_{s_2} &= z^{21}, \\
x_{c_1} &= z^{21}, \\
x_{c_2} &= z^{12}.
\end{align*}
\]

It is easy to see by inspection that this strategy profile is a fixed point of any best reply selection (for any agent, offering a different contract or offering no contract results in signing
no contract, which is a strictly less preferred outcome). And the outcome of this strategy profile is \( \{z_{12}, z_{21}\} \), which is unstable (agents \( s_1 \) and \( c_1 \) can block it via contract \( z_{11} \)). This completes the proof.

The above result shows that overbooking is needed to obtain the desired connection between stable matching and games with increasing best replies. This result motivates us to allow for overbooking, but it also leads to a feature that needs some care, as overbooking typically leads to indifferences in players’ preferences: A player is indifferent between any pair of strategies that differ only in offers that are rejected.

With such a wide range of indifferences, a player typically has multiple best replies. Faced with multiple best replies, in the following we will consider best reply selections as we have done in the previous sections (however, we will later point out important differences from the case of two-step games). We begin with the largest best reply in the spirit of the previous sections. That is, we say that a player’s strategy is a largest best reply if it is a best reply and it is a (weak) superset of any best reply. It is easy to see by inspection that the largest best reply \( \text{br} \) exists and is given by \( \text{br}_s(x_C) \equiv Z_s \setminus r_s(x_C) \) for all \( s \in S \) and \( \text{br}_c(x_S) \equiv Z_c \setminus r_c(x_S) \) for all \( c \in C \), where \( Z_s = \{ z \in Z | z_S = s \}, Z_c = \{ z \in Z | z_C = c \}, \) and \( r_s(x_C) \) is the set of rejected offers by student \( s \). More precisely, let \( AC_s(x_C) := \{ z \in \bigcup_{c \in C} x_c | z_S = s \} \) be the set of contracts offered to \( s \) at colleges’ offer profile \( x_C \). Let \( R_s(\cdot) \) denote the rejected set function of student \( s \) (\( R_s(Z') = Z' \setminus Ch_s(Z') \)). Then, \( r_s(x_C) \equiv R_s(AC_s(x_C)) \). A similar definition applies to \( r_c \).

The largest best replies are formally identical to the Hatfield-Milgrom mapping, as generalized by Hatfield and Kominers (2009) to many-to-many matching with contracts, whose fixed points correspond to stable allocations. Since the fixed points of best replies are equivalent to Nash equilibria of the noncooperative game, we obtain the following characterization of stable matchings in terms of equilibrium outcomes.

**Proposition 3** If contracts are substitutes for every agent, then the set of outcomes achieved by the pure strategy Nash equilibria in largest best replies is identical to the set of stable allocations.

**Remark 3** The conclusion of Proposition *** is false if contracts are not substitutes. To see this point, consider a many-to-many situation with two students \( s_1, s_2 \) and two colleges \( c_1, c_2 \). There is a fixed term of contract for any given student-college pair, so there are four contracts \( z_{11}, z_{12}, z_{21}, z_{22} \), where \( z_{ij} \) is the unique contract such that \( z_{ij}^S = s_i \) and \( z_{ij}^C = c_j \). For each player, her most preferred outcome is the one in which both of the contracts involving her are signed. If a player signs only one contract, she is worse off than being unmatched (note that there are complementarities among the potential partners, thus
the contracts are not substitutes). Consider the following strategy profile in general offer demand game: \( x_{si} = \{z^{ii}\} \) for \( i = 1, 2 \), \( x_{ci} = \{z^{ji}\} \) where \( j \neq i \). These “criss-cross” offers result in an allocation \( i \) which no contract is signed. If \( s_i \) changes his offer to either \( \{z^{i1}, z^{i2}\} \) or \( \{z^{ji}\}, j \neq i \), he is matched only to \( z^{ji} \), and hence he is worse off. Symmetric argument applies to all other players, and hence this is a Nash equilibrium. Obviously, however, this outcome is not stable because \( \{z^{11}, z^{12}, z^{21}, z^{22}\} \) blocks it.

The above proposition allows us to interpret stable allocations as the outcomes of Nash equilibria in largest best replies in a one-shot noncooperative game, because the set of the fixed points of the HM-mapping is equivalent to the set of stable allocations (Hatfield and Milgrom 2005 and Hatfield and Kominers 2009).

The next task is to connect the largest best reply with strategic complementarity. To do so, we introduce a partial order on strategy profiles by

\[
x' \succeq x \iff x_s \supseteq x'_s, \forall s \in S, x_c \subseteq x'_c, \forall c \in C.
\]

Note that we “reverse” the set inclusion ordering for students and colleges. With this ordering, and by the construction of the largest best reply and substitutability, it is clear that the largest best reply is a weakly increasing function (the argument is analogous to the ones for the two-step game and thus omitted). This latter fact implies that the set of Nash equilibria in largest strategies forms a nonempty lattice. Thus the existence and the lattice structure of stable allocations follow.

The two-step game of the previous sections has the “selection invariance” property, in the sense that the set of fixed points of a best reply selection does not vary with how the selection is made (Lemma **). This property does not hold for the one-step game of this section. To see this point, note that it is always a Nash equilibrium (a fixed point of a best reply correspondence) for every agent to report that no contract is acceptable. The outcome of such an equilibrium is an allocation in which no contract is made, which is not stable in general. By contrast, Proposition *** above shows that any fixed point of largest best replies is always stable, showing that the selection invariance property does not hold in the one-step game.

The failure of the selection invariance property motivates our use of a certain best response selection. The largest best reply is one selection that works for our purposes. Let us emphasize that it is not our intention to claim that our one-step game is a realistic description of the real matching markets. Nor is it our intention to claim that the largest best replies are a particularly realistic selection from the best response correspondence. On the contrary, our goal is to further our understanding of the structure of stable matching by connecting it to a certain non-cooperative game with strategic complementarity. More
specifically, we provide the mapping, which we noted is equivalent to the mapping of Hatfield and Milgrom (2005), with a particular interpretation as a best reply in a game with strategic complementarity.

**Remark 4** This largest best reply can be derived as a unique best reply if we assume that each agent has certain lexicographic preferences. Specifically, assume that (i) an agent primarily prefers to sign a preferred set of contracts, and (ii) if player $i$ has two strategies (sets of offers) that yield the same set of contracts for him and one is larger than the other (in the sense of set inclusion), then $i$ prefers the larger strategy. Alternatively, we can design the rules of the game in such a way that players end up having the lexicographic payoffs. For example, we can pay a small amount of money (say 1 cent) to a player for any additional offer of a contract. If the utility from money is lower than the marginal disutility of any change in the realized match, then we get the lexicographic payoffs.

### 5.3 Threshold Best Replies

As pointed out above, the one-step game does not have the selection invariance property, and the largest best reply is merely one best reply selection that successfully connects stable allocations and the non-cooperative game. Given this observation, a question of interest is what, if any, other selection performs this task.

For this purpose, we consider the generalized threshold best response, which extends the notion with the same name of the previous sections to matching with contracts. More specifically, define

$$\text{br}_s(x_C) \equiv \{z \in Z | z \in Ch_s(AC_s(x_C) \cup z)\}$$ for all $s \in S$,

$$\text{br}_c(x_S) \equiv \{z \in Z | z \in Ch_c(AC_c(x_S) \cup z)\}$$ for all $c \in C$,

where $AC_i$ is as defined in the last subsection.

The generalized threshold best replies are formally identical to the T-mapping of Echenique and Oviedo (2006) and Ostrovsky (2009),\(^\text{15}\) whose fixed points correspond to stable allocations. Since the fixed points of best replies are equivalent to pure Nash equilibria of the noncooperative game, we obtain the following characterization of stable allocations in terms of equilibrium outcomes.

**Proposition 4** If contracts are substitutes for every agent, then the set of outcomes achieved by the pure strategy Nash equilibria in threshold best replies is identical to the set of stable allocations.

\(^{15}\)Recall the caveats mentioned in footnote 11.
One of the reasons that generalized threshold strategies are of interest in our one-step game is that the threshold strategy admits a simple characterization. The next result demonstrates that the generalized threshold strategy is the smallest best response selection among those that are monotone increasing.

**Theorem 10** The generalized threshold best reply is the smallest best response selection that is monotone increasing. That is, if \( br_i(x) \) is a best response selection that is monotone increasing, then \( br_i(x) \subseteq br_i(x) \) for each agent \( i \) and strategy profile \( x \).

**Proof.** We prove the claim by contradiction. To do so, assume that there exists a problem \((S, C, Z, P)\), \( i \in S \cup C \), a best reply selection \( br \) that is monotone increasing, and strategy profile \( x \) such that \( br_i(x) \not\subseteq br_i(x) \). Assume \( i \in S \) without loss of generality (the proof for the case with \( i \in C \) is symmetric). Denote \( i = s \). Then there exists \( z \) such that \( z_S = s \), \( z \in \{ z' \in Z | z' \in Ch_s(AC_s(x_C) \cup z') \} \), and \( z \not\in br_s(x) \). Since \( br_s(x) \) is a best reply to \( x \), \( z \) is not offered at \( x \).

Consider a strategy profile \( x' \) such that \( x'_c = x_c \cup \{ z \} \) for \( c = z_C \), and \( x'_{-c} = x_{-c} \). Then, by definition \( x' \succeq x \). Since \( br \) is a weakly increasing best reply selection, \( br_s(x') \subseteq br_s(x) \). This implies that \( z \not\in br_s(x') \). But this relation implies that \( br_s(x') \) is not a best response for \( s \) to \( x' \), a contradiction. ■

**Remark 5** The threshold best reply can be derived as a unique best reply if we assume that players have certain lexicographic preferences (different from those for largest best replies). Specifically, assume that (i) an agent primarily prefers to sign a preferred set of contracts, and (ii) if player \( i \) has two strategies (sets of offers) that yield the same set of contracts for him and one is larger than the other (in the sense of set inclusion), then \( i \) prefers to choose a “serious threshold” that is going to be realized.

As for the largest best replies, we do not argue that the lexicographic preference assumption is a particularly “realistic” assumption about agents’ preferences. Rather, our point is that this rather weak assumption helps us to obtain a better understanding of stable allocations in terms of a simple non-cooperative game. Alternatively, we can design the rules of the game in such a way that players end up having the lexicographic payoffs. For example, we can pay a small amount of money (say 1 cent) to a player when his threshold is equal to his realized match. If 1 cent is lower than the marginal disutility of any change in the realized match, then we do get the lexicographic payoffs.
6 Conclusion

A Proofs

Most of (or all) the proofs will be moved here...

B Examples

Example 5 Let $C = \{c\}, S = \{s_1, s_2\}$, $c \succ_s \emptyset$ for $s \in \{s_1, s_2\}$ and $\{s_1, s_2\} \succ_c \emptyset \succ_c \{s\}$ for $s \in \{s_1, s_2\}$. It is a Nash equilibrium for each student to apply to no college. On the other hand, it is not an equilibrium for $s_1$ to apply to $c$ while $s_2$ applies to no college, since if $s_2$ instead applies to $c$, $c$ accepts both $s_1$ and $s_2$, thus benefitting $s_2$.

Example 6 Let $C = \{c\}, S = \{s_1, s_2\}$, $c \succ_s \emptyset$ for $s \in \{s_1, s_2\}$ and $\{s_1, s_2\} \succ_c \emptyset \succ_c \{s\}$ for $s \in \{s_1, s_2\}$. This is the same market as the one in Example 5. The empty matching (a matching where every agent is unmatched) is a pairwise stable matching. However the following argument shows that the empty matching is not an outcome of any Nash equilibrium in largest strategies. First, $(C_{s_1}, C_{s_2}) = (\emptyset, \emptyset)$ is not a Nash equilibrium in largest strategies, since a largest best response of $s_1$ to $C_{s_2}$ is $\{c\}$. Second, $(\{c\}, \emptyset)$ is not a Nash equilibrium since $s_2$ can profitably deviate to $\{c\}$ and be matched to $c$ while she is unmatched under $(\{c\}, \emptyset)$. Third, $(\emptyset, \{c\})$ is not a Nash equilibrium since $s_1$ can profitably deviate to $\{c\}$ and be matched to $c$ while she is unmatched under $(\emptyset, \{c\})$. Finally, $(\{c\}, \{c\})$ is a Nash equilibrium in largest strategies with outcome $\phi(\{c\}, \{c\})(c) = \{s_1, s_2\}$.

C Understanding Properties of Stable Matchings

Recall a market is tuple $\Gamma = (S, C, P)$. Let $\Gamma^{-c} = (S, C \setminus c, P_{-c})$ be a market in which college $c$ is not present but otherwise the same as $\Gamma$. Formally, we define $\Gamma^{-c}$ from $\Gamma$ simply by changing preferences of $c$ in such a way that $Ch_c(S') = \emptyset$ for every $S' \in S$. This definition is convenient since we need to re-define neither preferences of students nor strategy sets in the new market $\Gamma^{-c}$ with this definition. Let $\overline{br}^{-c}$ be the largest best response function in $\Gamma^{-c}$. Since preferences are substitutable, $\overline{br}(C_S) \subseteq \overline{br}^{-c}(C_S)$ for any $C_S \subseteq C^S$. The following comparative statics is a many-to-many version of results due to Kelso and Crawford (1982), Gale and Sotomayor (1985b), and Crawford (1991).
Proposition 5 (Comparative Statics) Let \( \bar{C}_S \) and \( \bar{C}_S^c \) be the largest Nash equilibria in largest strategies in \( \Gamma \) and \( \Gamma^{-c} \), respectively. Then \( \phi(\bar{C}_S)R_s\phi(\bar{C}_S^c) \) for every \( s \in S \) and \( \phi(\bar{C}_S^c)R_{c'}\phi(\bar{C}_S) \) for every \( c' \in C \setminus c \). Let \( \underline{C}_S \) and \( \underline{C}_S^c \) be the smallest Nash equilibria in largest strategies in \( \Gamma \) and \( \Gamma^{-c} \), respectively. Then \( \phi(\underline{C}_S)R_s\phi(\underline{C}_S^c) \) for every \( s \in S \) and \( \phi(\underline{C}_S^c)R_{c'}\phi(\underline{C}_S) \) for every \( c' \in C \setminus c \).

**Proof.** Since the students’ final offers game has strategic complementarity, we have

\[
\bar{C}_S = \sup\{C_S \subseteq C^S | C_S \preceq \text{br}(C_S)\},
\]

\[
\bar{C}_S^c = \sup\{C_S \subseteq C^S | C_S \preceq \text{br}^{-c}(C_S)\}.
\]

Since \( \text{br}(C_S) \preceq \text{br}^{-c}(C_S) \), \( C_S \preceq \text{br}(C_S) \) implies \( C_S \preceq \text{br}^{-c}(C_S) \). Thus

\[
\bar{C}_S = \sup\{C_S \subseteq C^S | C_S \preceq \text{br}(C_S)\} \preceq \sup\{C_S \subseteq C^S | C_S \preceq \text{br}^{-c}(C_S)\} = \bar{C}_S^c.
\]

Moreover, \( c \) is not matched to any \( s \in S \) in \( \Gamma^{-c} \). Therefore \( \phi(\bar{C}_S)R_s\phi(\bar{C}_S^c) \) for every \( s \in S \) and \( \phi(\bar{C}_S^c)R_{c'}\phi(\bar{C}_S) \) for every \( c' \in C \setminus c \).

Similarly,

\[
\underline{C}_S = \inf\{C_S \subseteq C^S | C_S \succeq \text{br}(C_S)\} \succeq \inf\{C_S \subseteq C^S | C_S \succeq \text{br}^{-c}(C_S)\} = \underline{C}_S^c,
\]

and hence \( \phi(\underline{C}_S)R_s\phi(\underline{C}_S^c) \) for every \( s \in S \) and \( \phi(\underline{C}_S^c)R_{c'}\phi(\underline{C}_S) \) for every \( c' \in C \setminus c \), completing the proof. □

Let \( C_S^* \) be any Nash equilibrium in largest strategies in \( \Gamma \), and now assume \( c \) becomes unavailable to be matched to any student, and students try to make new offers and so on. This story suggests the following equilibration process, called the **vacancy chain dynamics associated with** \( C_S^* \) and \( c \) (Blum, Roth, and Rothblum, 1997).

**Initialization:** Let \( C_S(0) = C_S^* \).

**Iteration:** For each Step \( t \in \{1, 2, \ldots\} \), let \( C_S(t) = \text{br}^{-c}(C_S(t)) \).

We say that the vacancy chain dynamics terminates in finite steps if \( C_S(t+1) = C_S(t) \) for some \( t \in \{0, 1, \ldots\} \).

Proposition 6 (Vacancy Chain Dynamics) For any Nash equilibrium \( C_S^* \) in largest strategies in \( \Gamma \) and \( c \in C \), the vacancy chain dynamics associated with \( C_S^* \) and \( c \) terminates in finite steps at a Nash equilibrium \( C_S' \) in largest strategies in \( \Gamma^{-c} \), and \( \phi(C_S^*)R_s\phi(C_S') \) for every \( s \in S \) and \( \phi(C_S')R_{c'}\phi(C_S^*) \) for every \( c' \in C \setminus c \).
Proof. Recall that $\text{br}^{-c}(C_S) \succeq \text{br}(C_S)$ for any $C_S \subseteq C^S$, and $\text{br}(C^*_S) = C^*_S$ since $C^*_S$ is a Nash equilibrium in largest strategies in $\Gamma$. These properties imply that

$$C_S(1) = \text{br}^{-c}(C^*_S) \succeq \text{br}(C^*_S) = C^*_S. \quad (5)$$

By an inductive argument, property (5) implies $C_S(t) \supseteq C_S(t - 1)$ for all $t \in \{1, 2, \ldots \}$. Since the set of strategies is a finite set, this implies that the algorithm terminates to a Nash equilibrium $C^*_S$ in $\Gamma^{-c}$ satisfying $C^*_S \succeq C^*_S$. This set inclusion and the fact $c$ rejects every $s$ in $\Gamma^{-c}$ imply $\phi(C^*_S)R_s\phi(C^*_S)$ for every $s \in S$ and $\phi(C^*_S)R_c\phi(C^*_S)$ for every $c' \in C \setminus c$, completing the proof. 

Remark 6 For Theorems 5 and 6, we considered only situations in which one college $c \in C$ exits from the market for simplicity. It is tedious but easy to extend the analysis to the case in which a group of colleges or students enter or exit the market simultaneously.

Consider the following property due to Alkan (2002), Alkan and Gale (2003) and Hatfield and Milgrom (2005).

Definition 3 (The Law of Aggregate Demand, Size Monotonicity) Preference relation $P_c$ satisfies the law of aggregate demand (size monotonicity) if $|Ch_c(S')| \geq |Ch_c(S'')|$ for every $S'' \subseteq S' \subseteq S$. Similarly, preference relation $P_s$ satisfies the law of aggregate demand if $|Ch_s(C')| \geq |Ch_s(C'')|$ for every $C'' \subseteq C' \subseteq C$.

With substitutability and the law of aggregate demand, we show a version of the so-called “rural hospitals theorem.” Roth (1986) shows one version of the rural hospitals theorem for many-to-one matching with responsive preferences. More specifically, he shows that every hospital that has unfilled positions at some stable matching is assigned exactly the same doctors at every stable matching. Martínez, Massó, Neme, and Oviedo (2000) generalize the theorem for substitutable and $q$-separable preferences. Although there is no obvious notion of “unfilled positions” under substitutability, Theorem 7 below shows that a weaker version of the rural hospitals theorem still holds, as formulated by McVitie and Wilson (1970), Gale and Sotomayor (1985a,b), and Roth (1984). More specifically, every hospital signs exactly the same number of contracts at every stable allocation, although the doctors assigned and the terms of contract can vary.

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\(^{16}\)A discussion on this point is found in Hatfield and Kojima (2010), where a rural hospital theorem similar to Theorem 7 is shown in the context of many-to-one matching with contracts as in Hatfield and Milgrom (2005).
Proposition 7 (The Rural Hospital Theorem) If preference relations of all agents satisfy substitutability and the law of aggregate demand, then every student and college are matched with the same number of partners at every stable matching.

Proof. Let $\bar{C}_S$ and $C^*_S$ be the largest and smallest Nash equilibria in largest strategies. For any Nash equilibrium $C^*_S$ in largest strategies, by Theorem 1 we have $\bar{C}_S \succeq C^*_S \succeq C_S$. Since college preferences satisfy the law of aggregate demand, this implies

$$|\phi(\bar{C}_S)(c)| \geq |\phi(C^*_S)(c)| \geq |\phi(C_S)(c)| \quad \text{for every } c \in C. \quad (6)$$

On the other hand, since college preferences are substitutable, $\bar{C}_S \succeq C^*_S \succeq C_S$ implies that, for every $s \in S$, $A_s(\bar{C}_S) \subseteq A_s(C^*_S) \subseteq A_s(C_S)$. Since student preferences satisfy the law of aggregate demand, this implies

$$|\phi(C^*_S)(s)| \geq |\phi(C_S)(s)| \geq |\phi(\bar{C}_S)(s)| \quad \text{for every } s \in S. \quad (7)$$

Inequalities (6) and (7) can both hold if and only if all weak inequalities hold with equality, completing the proof. ■

Theorem 7 is applicable to the previous domains of McVitie and Wilson (1970), Gale and Sotomayor (1985a,b), Roth (1984, 1986) and Martínez, Massó, Neme, and Oviedo (2000). Hatfield and Milgrom (2005) show the theorem in the context of many-to-one matching with contracts under substitutes and the law of aggregate demand, and Hatfield and Kojima (2010) establish the same conclusion while relaxing the substitute to a condition called unilateral substitutes. Both the substitutes and unilateral substitutes conditions reduce to the substitutability condition when terms of contracts are exogenously given as in our paper.

D An Alternative Formulation of Best Response Dynamics

We consider a sequential version of the deferred acceptance algorithm originally formulated in one-to-one matching by McVitie and Wilson (1970) extended to the many-to-many setting. Let $S$ be ordered in an arbitrary manner, by $s_1, s_2, \ldots, s_{|S|}$.

Initialization: Let $t = 0$ and $\mu^0$ be the empty matching, that is, $\mu^0(s) = \emptyset$ for every $s \in S$. Let $A^0(s) = C$ for every $s \in S$.

Iteration: For each $t$, Step $t$ proceeds as follows.
1. If \( \mu^t(s) = Ch_s(A^t(s)) \) for all \( s \in S \), then \( \mu^{t+1} = \mu^t \) and \( A^{t+1}(s) = A^t(s) \) for all \( s \in S \).

2-1. Otherwise, let \( s^t \) be a student with the smallest index such that \( \mu^t(s) \neq Ch_s(A^t(s)) \). Let \( u = 0 \) and \( A^{t,0}(s^t) = A^t(s^t) \).

2-2. For each \( u \), Student \( s^t \) applies to the set of colleges \( Ch_s(A^{t,u}(s)) \). Each college \( c \in Ch_s(A^{t,u}(s)) \) chooses its most preferred group of students among those who applied, that is, \( Ch_c(\mu^t(c) \cup s^t) \). If no college rejects \( s^t \), then let the resulting matching be \( \mu^{t+1} \) and \( A^{t+1}(s^t) = A^{t,u}(s^t) \). If some college rejects \( s^t \), then delete the set of colleges that rejected \( s^t \) from \( A^{t,u}(s^t) \) and let it be \( A^{t,u+1}(s^t) \).

We say that the algorithm terminates in step \( T \) if \( \mu^T(s) = Ch_s(A^T(s)) \) for all \( s \in S \) and \( \mu^t(s) \neq Ch_s(A^t(s)) \) for some \( s \in S \) for all \( t < T \). Note that we have not shown that the deferred acceptance algorithm terminates at this point: We will derive termination as a consequence of Proposition 3 below.

We will relate the deferred acceptance algorithm to a learning process on our students’ final offers game. For that purpose, we first define the following learning process, called the best response dynamics.

**Initialization:** Let \( C^0_s = \emptyset \) for each \( s \in S \).

**Iteration:** For each \( t \): If \( C^t_s \) is a Nash equilibrium, then \( \mu^{t+1} = \mu^t \). If not, let \( s^t \) be the student with the smallest index whose current strategy is not a best response. Let \( s^t \) change her strategy to \( C^{t+1}_s = Br_s(C^t_s) \) while every other student \( s \) keeps taking \( C^{t+1}_s = C^t_s \).

We will say that the best response dynamics converges at a finite step \( T \) if \( C^T \) is a Nash equilibrium and \( C^t \) is not a Nash equilibrium for any \( t < T \).

**Proposition 8** \( \phi(C^t_s) = \mu^t \) for each \( t \in \{0,1,2,\ldots\} \).

**Proof.** We will show, by induction, that \( \phi(C^t_s) = \mu^t \) and \( A^t(s) \supseteq A_s(C^t_{-s}) \) for all \( s \in S \) and \( t = 1,2,\ldots \). For \( t = 0 \), \( \phi(C^0_s) = \mu^0 \) since both are the empty matching and \( A^0(s) \supseteq A_s(C^0_{-s}) \) since \( A^0(s) = C \) for all \( s \in S \) by definition.

Suppose \( \phi(C^t_s) = \mu^t \) and \( A^t(s) \supseteq A_s(C^t_{-s}) \) for all \( s \in S \). If \( \mu^t(s) = Ch_s(A^t(s)) \) for all \( s \in S \), then for each \( s \in S \), \( \mu^t(s) = \mu^{t+1}(s) \) and \( A^t(s) = A^{t+1}(s) \) by definition and since \( \mu^t \) is a stable matching, by Theorem 1, \( C^t \) is a Nash equilibrium and hence by definition of the best response dynamics, \( C^{t+1}_s = C^t_s \) so \( \phi(C^{t+1}_s) = \mu^{t+1} \) and \( A^{t+1}(s) = A^t(s) \supseteq A_s(C^{t+1}_{-s}) \). So suppose \( \mu^t(s) \neq Ch_s(A^t(s)) \) for some \( s \in S \) and let \( s^t \) be such a student with the smallest
index. Then, by construction of the deferred acceptance algorithm, \( A^t_u(s^t) \supseteq A_u'(C_{-s}^t) \) for all \( u \) and hence \( A^{t+1}(s^t) \supseteq A_u'(C_{-s}^{t+1}) \). Also by substitutability of preferences of colleges and \( s^t \), \( A^{t+1}(s^t) = A^t(s) \supseteq A_u'(C_{-s}^{t+1}) \) for all \( s \neq s^t \). Moreover \( \mu^{t+1}(s^t) = Ch_{s^t}(A_u'(\mu^t)) \) and for any \( s \neq s^t \), \( \mu^{t+1}(s) = \mu^t(s) \setminus \{ c \in \mu^{t+1}(s^t) | s \notin Ch_c(\mu^t(c) \cup s^t) \} \). In the best response dynamics, \( \phi(C_{S}^{t+1})(s^t) = Ch_{s^t}(A(C_{-s})) = Ch_{s^t}(A_u'(\mu^t)) = \mu^{t+1}(s^t) \), and for any \( s \neq s^t \), by substitutability, \( \phi(C_{S}^{t+1})(s) = \phi(C_{S}^{t+1})(s) \setminus \{ c \in \phi(C_{S}^{t+1})(s^t) | s \notin Ch_c(\mu^t(c) \cup s^t) \} = \mu^{t+1}(s) \). This completes the proof. ■

**Corollary 5** The student-proposing deferred acceptance algorithm terminates in a finite number \( T \) of steps. The resulting matching \( \mu^T \) is the student-optimal stable matching.

**Proof.** Since the students’ final offers game is a game with a finite set of strategies and strategic complementarity, the best-response dynamics as described above converges in a finite time \( T \) at the smallest Nash equilibrium (note that the dynamics starts at the smallest strategy profile). Since the smallest Nash equilibrium results in the student-optimal stable matching by Proposition 2, by Proposition 3 the student-proposing deferred acceptance algorithm terminates in a finite number of steps \( T \) and the resulting matching \( \mu^T \) is the student-optimal stable matching. ■

**E Structure of the set of stable matchings**

We can further obtain the upper bound and uniqueness conditions for stable matchings by restricting our attention to many-to-one cases. Suppose there are \( N \) students and \( L \) colleges, and each student can be matched with only one college. For simplicity, assume that all colleges are acceptable to all students.\(^{17}\) Then, let us assign an integer to each generalized threshold strategy for every student in the way that a strategy is assigned an integer \( k \) if its threshold college is the \( k \)’th best college for the student. That is, each student is endowed with (generalized threshold) strategies \( \{1, 2, \ldots, L\} \) such that strategy \( k \) is to apply to all the colleges weakly preferred to her \( k \)’th best college. Let \( x^{**} \) and \( x^* \) be the largest and smallest pure strategy Nash equilibria (\( x^{**} \geq x^* \)). Remember that each Nash equilibrium corresponds to a stable matching. Now we are ready to show the following results giving the upper bound of the size of stable matchings. Denote a best response function (of the form of a generalized threshold strategy) of player \( i \) by \( BR_i \).

\(^{17}\)It is easy to incorporate unacceptable colleges, which would not change any of the following results.
Theorem 11  (i) Suppose there is an integer $K$ such that

$$BR_i(x^*_i + (k, \ldots, k)) - BR_i(x^*_i) < k$$  \hspace{1cm} (8)$$

for all $k \geq K$, and for all player $i$, at the smallest Nash equilibrium $x^*$. Then, each student is matched with at most $K$ different colleges in the set of all stable matchings.

(ii) Suppose there is an integer $K$ such that,

$$\text{for all } x' \geq x \text{ with } \sum_{i=1}^{N} (BR_i(x'_i) - BR_i(x_i)) < \sum_{i=1}^{N} (x'_i - x_i).$$

Then $\sum_{i=1}^{N} (x'_i - x_i) < K$ must hold for any equilibria $x$, $x'$.

**Proof.** (i) It is sufficient to show that $x^*_i - x'_i < K$ for all student $i$. Let $n$ be a student where $x^*_n - x'_n$ is the largest, i.e., $x^*_n - x^*_i \geq x^*_n - x'_i$ for all $i$. Suppose that $x^*_n - x'_n \geq K$. Let us denote $k \equiv x^*_n - x'_n$. Then, we have a contradiction:

$$x^*_n - x'_n = BR_n(x^*_n) - BR_n(x'_n) \leq BR_n(x^*_n + (k, \ldots, k)) - BR_n(x^*_n) < k = x^*_n - x'_n.$$

The first equality comes from the fact that $x^*$ and $x^*$ are Nash equilibria. The weak inequality comes from monotone increasing best replies and $k \equiv x^*_n - x'_n \geq x^*_n - x'_i$ for all $i$ (the definition of player $n$). The strict inequality is the condition (8).

(ii) Suppose, contrary to our claim, we had $\sum_{i=1}^{N} (x'_i - x_i) \geq K$, for some equilibria $x$, $x'$. Then, the above condition would apply and $\sum_{i=1}^{N} (BR_i(x'_i) - BR_i(x_i)) < \sum_{i=1}^{N} (x'_i - x_i)$, which would contradict the equilibrium conditions $BR(x') = x'$ and $BR(x) = x$. \Halmos

The insight of the above two results comes from the well-known sufficient conditions for the unique equilibrium point. Theorem 11-(i) is a discrete analogue of the slope condition for the unique pure strategy equilibrium, when a strategy is one-dimensional continuous variable $x_i \in [0, x]$ and the best replies are increasing. The slope condition is

$$\text{for any } i \text{ and any } x_{-i}, \sum_{j \neq i} \frac{\partial BR_i}{\partial x_j}(x_{-i}) < 1.$$  

Another well-known condition for uniqueness is the contraction mapping property. Let $BR(x) = (BR_1(x_{-1}), \ldots, BR_N(x_{-N}))$. The contraction mapping condition is

$$\exists \beta \in (0, 1) \ \forall x, x' \ |BR(x') - BR(x)| \leq \beta |x' - x|.$$
where $| \cdot |$ can be any norm in $\mathbb{R}^N$, but consider the norm $|x| = |x_1| + \cdots + |x_N|$. Then, the contraction mapping condition is stated as

$$\exists \beta \in (0, 1) \ \forall x, x' \sum_{i=1}^{N} |BR_i(x'_{-i}) - BR_i(x_{-i})| \leq \beta \sum_{i=1}^{N} |x'_i - x_i|.$$ 

When best replies are monotone increasing, the following relaxation of this contraction mapping condition suffices for the uniqueness.

$$\forall x' \geq x, x \neq x' \sum_{i=1}^{N} (BR_i(x'_{-i}) - BR_i(x_{-i})) < \sum_{i=1}^{N} (x'_i - x_i).$$

Theorem 11-(ii) states the discrete analogue of this contraction condition.

The above theorem also seems to be similar to the small core theorem by Immorlica and Mahdian (2005) and Kojima and Pathak (2009), although the exact relation is yet to be found. A slightly weaker sufficient condition than Theorem 11-(ii) is obtained as follows.

**Theorem 12** Suppose there is an integer $K$ such that,

$$\forall x' \geq x \text{ with } \sum_{i=1}^{N} (x'_i - x_i) \geq K \ \exists n BR_n(x'_{-n}) - BR_n(x_{-n}) < x'_n - x_n.$$ 

Then $\sum_{i=1}^{N} (x'_i - x_i) < K$ must hold for any equilibria $x, x'$.

**Proof.** Suppose, contrary to our claim, we had $\sum_{i=1}^{N} (x'_i - x_i) \geq K$, for some equilibria $x, x'$. Then, the above would apply and $\exists n BR_n(x'_{-n}) - BR_n(x_{-n}) < x'_n - x_n$. This would contradict the equilibrium conditions $BR_n(x'_{-n}) = x'_n$ and $BR(x_{-n}) = x_n$. 

Substituting $K = 1$ into Theorem 12, we obtain the following uniqueness result.

**Corollary 6** There exists a unique Nash equilibrium if the following condition is satisfied.

$$\forall x' \geq x, x \neq x' \ \exists n BR_n(x'_{-n}) - BR_n(x_{-n}) < x'_n - x_n. \quad (9)$$

It is easy to verify that condition (9) is satisfied when all colleges have the same preferences over students. Thus, Corollary 6 gives us a weaker uniqueness condition than the well-known one that players on one side all have the identical preferences by Gusfield and Irving (1989).
References


